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ON THE BOUNDEDNESS OF SOME CLASSES OF INTEGRAL OPERATORS IN WEIGHTED LEBESGUE SPACES

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Abstract. Some new Hardy-type inequalities for Hardy-Volterra integral operators are proved and discussed. The case $1 < q < p < \infty$ is considered and the involved kernels satisfy conditions, which are less restrictive than the usual Oinarov condition.

1 Introduction

Let $1 < p, q < \infty$, $\frac{1}{p}+\frac{1}{p'} = 1$, $-\infty \leq a < b \leq \infty$, ρ and w be nonnegative functions, such that the functions ρ^p, w^q, $\rho^{-p'}$ and $w^{-q'}$ are locally integrable on the interval (a, b). For a fixed parameter $1 \leq p < \infty$ and a weight function ρ we define the weighted Lebesgue space $L_{p,\rho}(a, b)$ as the set of all measurable functions f on (a, b) such that

$$\|f\|_{p,\rho} = \left(\int_a^b |f(x)|^p \rho^p(x) dx \right)^{\frac{1}{p}} < \infty.$$

In this paper we consider the problem of the boundedness from $L_{p,\rho}$ to $L_{q,w}$ of the integral operators:

$$\mathbf{K}f(x) = \int_a^x K(x, s)f(s)ds, \ a < x < b, \tag{1.1}$$

$$\mathbf{K}^*g(s) = \int_s^b K(x, s)g(x)dx, \ a < s < b, \tag{1.2}$$

with a nonnegative continuous kernel $K(x, s)$. This problem is equivalent to finding conditions under which the Hardy type inequality

$$\left(\int_a^b |Kf(x)|^q w^q(x) dx \right)^{\frac{1}{q}} \leq C \left(\int_a^b |f(x)|^p \rho^p(x) dx \right)^{\frac{1}{p}} \tag{1.3}$$

holds for all $f \in L_{p,\rho}$, with C which does not depend on the function f for fixed p, q.

Since the kernel $K(\cdot,\cdot)$ is non-negative then inequality (1.3) is equivalent to the inequality

$$\left(\int_a^b (Kf(x))^q w^q(x)dx\right)^{\frac{1}{q}} \leq C\left(\int_a^b f(x)^p \rho^p(x)dx\right)^{\frac{1}{p}} \text{ for } f \geq 0. \tag{1.4}$$

Hence, here and in the sequel we shall consider, without loss of generality, the case when the function f is non-negative.

In papers [5], [6] the class of kernels $K(x,s)$, satisfying the condition

$$d^{-1}(K(x,t)+K(t,s)) \leq K(x,s) \leq d(K(x,t)+K(t,s)), \tag{1.5}$$

for $a < s \leq t \leq x < b$ with a constant $d \geq 1$ independent of $x,\ t,\ s$, was introduced.

A classical example where such a kernel appears is the Riemann-Liouville operator with the kernel $K(x,s) = (x-s)^{\alpha-1}$ for $\alpha \geq 1$.

Later on R. Oinarov introduced less restrictive classes of kernels P_n and Q_n, $n \geq 0$, and in the case $1 < p \leq q < \infty$ he gave a criterion for (1.4) to hold (see [7]). The problem of the boundedness of operator (1.1) for $1 < q < p < \infty$ with kernels from the classes P_n or Q_n remains open.

In this paper we shall derive some criteria for the boundedness of integral operators (1.1) and (1.2) with kernels in the classes P_1 and Q_1 in the case $1 < q < p < \infty$. This means that we shall characterize Hardy-type inequalities of type (1.4) in cases, which are not known in the literature (see e.g. the books [2]-[4], [8] and the references given there).

Here and in the sequel we use the notation $p' = \frac{p}{p-1}$, $q' = \frac{q}{q-1}$ and $r = \frac{pq}{p-q}$. The symbol $A \ll B$ means that $A \leq cB$, where c is positive and depends only on unessential parameters. We write $A \approx B$ if $A \ll B \ll A$. Futhemore, $\chi_E(\cdot)$ stands for the characteristic function of a set $E \subset (a,b)$ and $\mathbb{Z}$ denotes the set of all integers.

The paper is organized as follows. Our main results (Theorems 1-4) are presented in Section 3. The proofs can be found in Section 4. In order not to disrupt our discussions later on, we present some definitions and other preliminaries in Section 2.

2 Preliminaries

We first define the classes P_1 and Q_1.

Definition 1. Let $K(\cdot,\cdot)$ be continuous, non-negative and non-decreasing in the first argument, defined and measurable on the set $\{(x,s),\ a < s \leq x < b\}$. We say that the function $K(x,s)$ belongs to the class P_1 if there exist nonnegative measurable functions $V(\cdot)$ and $R(\cdot,\cdot)$ and a constant $d \geq 1$, such that for all $x,\ t,\ s:\ a < s \leq t \leq x < b$ the following inequalities hold:

$$d^{-1}(R(x,t)V(s)+K(t,s)) \leq K(x,s) \leq d(R(x,t)V(s)+K(t,s)). \tag{2.1}$$

Definition 2. Let $K(\cdot,\cdot)$ be continuous, non-negative and non-increasing in the second argument, defined and measurable on the set $\{(x,s),\ a<s\le x<b\}$. We say that the function $K(x,s)$ belongs to the class Q_1 if there exist non-negative measurable functions $U(\cdot)$ and $Q(\cdot,\cdot)$ and a constant $d\ge 1$, such that for all $x,\ t,\ s:\ a<s\le t\le x<b$ the following inequalities hold:

$$d^{-1}\left(K(x,t)+U(x)Q(t,s)\right)\le K(x,s)\le d\left(K(x,t)+U(x)Q(t,s)\right). \tag{2.2}$$

The classes P_1 and Q_1 are wider than the class of kernels satisfying (1.2). For example, the function $\tilde{K}(x,s)=(f(x)+g(s))^{\beta}$, where $a<s\le x<b$, $\beta>0$, $g(s)\ge 0$ and $f(x)$ is a non-negative increasing function, does not satisfy (1.2), but it belongs to P_1 since for $a<s\le t\le x<b$ the following two-sided estimate holds:

$$(f(x)+g(s))^{\beta}\approx(f(x)-f(t))^{\beta}+(f(t)+g(s))^{\beta}.$$

The function $\bar{K}(x,s)=(f(x)-g(s))^{\beta}$ where $a<s\le x<b$, $\beta>0$, $f(x)\ge 0$, $g(s)$ is a non-negative decreasing function, does not satisfy (1.2), but it belongs to Q_1 since

$$(f(x)+g(s))^{\beta}\approx(f(x)+g(t))^{\beta}+(g(s)-g(t))^{\beta}$$

for $a<s\le t\le x<b$.

Remark 1. Without loss of generality we can assume that $R(\cdot,\cdot)$ is non-decreasing with respect to the variable x and non-increasing with respect to the variable y. Otherwise we can replace the function $R(x,s)$ by $\widetilde{R}(x,s)=\inf\limits_{a<s<t}\frac{K(x,s)}{V(s)}$. Then the function $\widetilde{R}(x,s)$ has both these monotonicity properties and inequality (2.1) with $R(x,s)$ replaced by $\widetilde{R}(x,s)$ holds.

First we note that by (2.1) it follows that $R(x,t)V(s)\ll K(x,s)$. Hence, by taking infimum we have that $R(x,t)\le\widetilde{R}(x,t)$ and, by (2.1), the following estimate holds:

$$K(x,s)\ll\widetilde{R}(x,t)V(s)+K(t,s).$$

On the other hand, it follows from the definition of the function $\widetilde{R}$, that $\widetilde{R}(x,t)V(s)\le K(x,s)$. Since the function $K(x,s)$ is non-decreasing in the first argument, then $K(t,s)\le K(x,s)$ for $t\le x$. By combining the last two inequalities we obtain the reverse estimate

$$\widetilde{R}(x,t)V(s)+K(t,s)\ll K(x,s).$$

Hence, $K(x,s)\approx\widetilde{R}(x,t)V(s)+K(t,s)$, i.e. the corresponding estimates hold with $\widetilde{R}$ replacing R in (2.1).

In a similar way we can prove that the function $\widetilde{Q}(t,s)=\inf\limits_{t\le x<\infty}\frac{K(x,s)}{U(x)}$ satisfies $K(x,s)\approx K(x,t)+U(x)\widetilde{Q}(t,s)$. Hence we can assume that in (2.2) the function $Q(\cdot,\cdot)$ is non-decreasing in the first argument and non-increasing in the second one.

3 The main results

Our main results read as follows.

Theorem 3.1. *Let $1 < q < p < \infty$ and assume that the kernel $K(x,s)$ of the operator $\mathbf{K}^*$ defined by (1.2) belongs to the class Q_1. Then the operator $\mathbf{K}^*$ is bounded from $L_{p,\rho}(a,b)$ to $L_{q,w}(a,b)$ if and only if the quantities*

$$M_1 = \left(\int_a^b \left(\int_t^b K^{p'}(x,t)\rho^{-p'}(x)dx \right)^{\frac{r}{p'}} \left(\int_a^t w^q(s)ds \right)^{\frac{r}{p}} w^q(t)dt \right)^{\frac{1}{r}},$$

$$M_2 = \left(\int_a^b \left(\int_a^t Q^q(t,s)w^q(s)ds \right)^{\frac{r}{q}} \left(\int_t^b U^{p'}(x)\rho^{-p'}(x)dx \right)^{\frac{r}{q'}} U^{p'}(t)\rho^{-p'}(t)dt \right)^{\frac{1}{r}}$$

are finite. Moreover, $\|\mathbf{K}^*\| \approx M_1 + M_2$, where $\|\mathbf{K}^*\|$ denotes the norm of the operator $\mathbf{K}^*$ from the space $L_{p,\rho}(a,b)$ to $L_{q,w}(a,b)$.

Our corresponding result for the operator $\mathbf{K}$ reads as follows.

Theorem 3.2. *Let $1 < q < p < \infty$ and assume that the kernel $K(x,s)$ of the operator $\mathbf{K}$ defined by (1.1) belongs to the class P_1. Then the operator $\mathbf{K}$ is bounded from $L_{p,\rho}(a,b)$ to $L_{q,w}(a,b)$ if and only if the quantities*

$$L_1 = \left(\int_a^b \left(\int_t^b R^q(x,t)w^q(x)dx \right)^{\frac{r}{q}} \left(\int_a^t V^{p'}(s)\rho^{-p'}(s)ds \right)^{\frac{r}{q'}} V^{p'}(t)\rho^{-p'}(t)dt \right)^{\frac{1}{r}},$$

$$L_2 = \left(\int_a^b \left(\int_a^t K^{p'}(t,s)\rho^{-p'}(s)ds \right)^{\frac{r}{p'}} \left(\int_t^b w^q(x)dx \right)^{\frac{r}{p}} w^q(t)dt \right)^{\frac{1}{r}}$$

are finite. Moreover, $\|\mathbf{K}\| \approx L_1 + L_2$, where $\|\mathbf{K}\|$ denotes the norm of the operator $\mathbf{K}$ from the space $L_{p,\rho}(a,b)$ to $L_{q,w}(a,b)$.

By using the well-known duality principle we can obtain the following equivalence:

$$\|\mathbf{K}^*g\|_{q,w} \le C\|g\|_{p,\rho} \ \ \forall g \in L_{p,\rho} \Leftrightarrow \|\mathbf{K}f\|_{p',\rho^{-1}} \le C\|f\|_{q',w^{-1}}, \ \ \forall f \in L_{q',w^{-1}}. \tag{3.1}$$

For a simple proof of this duality see e.g. the book [2, p. 13]. We can apply equivalence (3.1) to obtain by Theorem 3.1 and Theorem 3.2 the following results of independent interest.

Theorem 3.3. *Let $1 < q < p < \infty$ and assume that the kernel $K(x,s)$ of the operator $\mathbf{K}$ defined by (1.1) belongs to the class Q_1. Then the operator $\mathbf{K}$ is bounded from $L_{p,\rho}(a,b)$ to $L_{q,w}(a,b)$ if and only if the quantities*

$$M_1^* = \left(\int_a^b \left(\int_t^b K^q(x,t) w^q(x) dx \right)^{\frac{r}{q}} \left(\int_a^t \rho^{-p'}(s) ds \right)^{\frac{r}{q'}} \rho^{-p'}(t) dt \right)^{\frac{1}{r}},$$

$$M_2^* = \left(\int_a^b \left(\int_a^t Q^{p'}(t,s) \rho^{-p'}(s) ds \right)^{\frac{r}{p'}} \left(\int_t^b U^q(x) w^q(x) dx \right)^{\frac{r}{p}} U^q(t) w^q(t) dt \right)^{\frac{1}{r}}$$

are finite. Moreover, $\|\mathbf{K}\| \approx M_1^* + M_2^*$, where $\|\mathbf{K}\|$ denotes the norm of the operator $\mathbf{K}$ from the space $L_{p,\rho}(a,b)$ to $L_{q,w}(a,b)$.

Theorem 3.4. *Let $1 < q < p < \infty$ and assume that the kernel $K(x,s)$ of the operator $\mathbf{K}^*$ defined by (1.2) belongs to the class P_1. Then the operator $\mathbf{K}^*$ is bounded from $L_{p,\rho}(a,b)$ to $L_{q,w}(a,b)$ if and only if the quantities*

$$L_1^* = \left(\int_a^b \left(\int_a^t K^q(t,s) w^q(s) ds \right)^{\frac{r}{q}} \left(\int_t^b \rho^{-p'}(x) dx \right)^{\frac{r}{q'}} \rho^{-p'}(t) dt \right)^{\frac{1}{r}},$$

$$L_2^* = \left(\int_a^b \left(\int_t^b R^{p'}(x,t) \rho^{-p'}(x) dx \right)^{\frac{r}{p'}} \left(\int_a^t V^q(s) w^q(s) dx \right)^{\frac{r}{p}} V^q(t) w^q(t) dt \right)^{\frac{1}{r}}$$

are finite. Moreover, $\|\mathbf{K}^*\| \approx L_1^* + L_2^*$, where $\|\mathbf{K}^*\|$ denotes the norm of the operator $\mathbf{K}^*$ from the space $L_{p,\rho}(a,b)$ to $L_{q,w}(a,b)$.

Remark 2. According to the famous Ando result [1] in the case $1 < q < p < \infty$ any integral operator is bounded if and only if it is compact. Hence, for example, in Theorem 3.1 as an equivalent condition we can also add the condition "$\mathbf{K}^*$ defined by (1.2) is compact", i.e. Theorem 3.1 gives also a characterization of compact operators. In the same way, we can add the equivalent condition "$\mathbf{K}$ defined by (1.1) is compact" in Theorem 3.2.

4 Proofs

According to the duality principle discussed in Section 3 (see (3.1)) we only need to prove Theorems 3.1 and 3.2.

Proof of Theorem 3.1 *Necessity.*

Let the operator $\mathbf{K}^*$ defined by (1.2) be bounded from $L_{p,\rho}(a,b)$ to $L_{q,w}(a,b)$. This means that there exists a constant $C > 0$, such that for all functions $g \in L_{p,\rho}$ the inequality

$$\|\mathbf{K}^* g\|_{q,w} \leq C\|g\|_{p,\rho} \tag{4.1}$$

holds. We need to prove that $M_1 < \infty, M_2 < \infty$.

Notice also that the dual inequality

$$\|\mathbf{K} f\|_{p',\rho^{-1}} \leq C\|f\|_{q',w^{-1}}, \quad \forall f \in L_{q',w^{-1}} \tag{4.2}$$

holds.

For a fixed $z \in (a,b)$ we put $f(\cdot) = \chi_{(a,z)}(\cdot)w(\cdot)$. By substituting f into (4.2) we get that

$$Cz^{\frac{1}{q'}} \geq \|\mathbf{K} f\|_{p',\rho^{-1}} = \Bigg(\int_a^b \rho^{-p'}(x)\Bigg(\int_a^x K(x,s)f(s)ds\Bigg)^{p'} dx\Bigg)^{\frac{1}{p'}} \geq$$

$$\Bigg(\int_z^b \rho^{-p'}(x)\Bigg(\int_a^z K(x,s)w(s)ds\Bigg)^{p'} dx\Bigg)^{\frac{1}{p'}} \geq \frac{1}{d}\Bigg(\int_z^b U^{p'}(x)\rho^{-p'}(x)dx\Bigg)^{\frac{1}{p'}} \cdot \int_a^z Q(z,s)w(s)ds.$$

In the last estimate we used the estimate $K(x,s) \geq \frac{1}{d}U(x)Q(z,s)$, where $a < s \leq z \leq x < b$, which follows by (2.2).

Since z is arbitrary, it follows that $\int_z^b U^{p'}(x)\rho^{-p'}(x)dx$ is finite.

Next, by choosing in (4.1) $\varphi(\cdot) = \chi_{(z,b)}(\cdot)\rho^{-p'}(\cdot)U^{p'-1}(\cdot)$ as a test function, we have that

$$C\left(\int_z^b U^{p'}(x)\rho^{-p'}(x)dx\right)^{\frac{1}{p}} \geq \|\mathbf{K}^*\varphi\|_{q,w} \geq$$

$$\left(\int_a^z w^q(s)\left(\int_z^b K(x,s)\rho^{-p'}(x)U^{p'-1}(x)dx\right)^q ds\right)^{\frac{1}{q}} \geq$$

$$\int_z^b U^{p'}(x)\rho^{-p'}(x)dx \cdot \left(\int_a^z Q^q(z,s)w^q(s)ds\right)^{\frac{1}{q}}.$$

Now, by dividing both parts of the previous inequality by the expression $\left(\int_z^b U^{p'}(x)\rho^{-p'}(x)dx\right)^{\frac{1}{p}}$ we obtain

$$\left(\int_z^b U^{p'}(x)\rho^{-p'}(x)dx\right)^{\frac{1}{p'}}\left(\int_a^z Q^q(z,s)w^q(s)ds\right)^{\frac{1}{q}} \leq C < \infty.$$

Hence, $\int\limits_a^z Q^q(z,s)w^q(s)ds < \infty$.

For α and β such that $a < \alpha < \beta < b$ we define the function

$$g(x) = \chi_{(\alpha,\beta)}(x)\left(\int\limits_\alpha^x Q^q(x,s)w^q(s)ds\right)^{\frac{1}{p-q}}\left(\int\limits_x^\beta U^{p'}(y)\rho^{-p'}(y)dy\right)^{\frac{q-1}{p-q}} U^{p'-1}(x)\rho^{-p'}(x).$$

It is easy to see that

$$\|g\|_{p,\rho} = \left(\int\limits_\alpha^\beta \left(\int\limits_\alpha^x Q^q(x,s)w^q(s)ds\right)^{\frac{r}{q}}\left(\int\limits_x^\beta U^{p'}(y)\rho^{-p'}(y)dy\right)^{\frac{r}{q'}} U^{p'}(x)\rho^{-p'}(x)dt\right)^{\frac{1}{p}}. \tag{4.3}$$

Next, we estimate $\|\mathbf{K}^*g\|^q_{q,w}$ from above as follows:

$$\|\mathbf{K}^*g\|^q_{q,w} = \int\limits_a^b \left(\int\limits_s^b K(x,s)g(x)dx\right)^q w^q(s)ds \geq \int\limits_\alpha^\beta \left(\int\limits_s^\beta K(x,s)g(x)dx\right)^q w^q(s)ds =$$

$$q\int\limits_\alpha^\beta w^q(s)\int\limits_s^\beta K(x,s)g(x)\left(\int\limits_x^\beta K(\tau,s)g(\tau)d\tau\right)^{q-1} dxds \gg$$

$$\int\limits_\alpha^\beta w^q(s)\int\limits_s^\beta U(x)Q(x,s)g(x)\left(\int\limits_x^\beta U(\tau)Q(x,s)g(\tau)d\tau\right)^{q-1} dxds =$$

$$\int\limits_\alpha^\beta w^q(s)\int\limits_s^\beta U(x)Q^q(x,s)g(x)\left(\int\limits_x^\beta U(\tau)g(\tau)d\tau\right)^{q-1} dxds =$$

$$\int\limits_\alpha^\beta U(x)g(x)\int\limits_\alpha^x Q^q(x,s)w^q(s)\left(\int\limits_x^\beta U(\tau)g(\tau)d\tau\right)^{q-1} dsdx. \tag{4.4}$$

Moreover, we estimate the expression $\int\limits_x^\beta U(\tau)g(\tau)d\tau$ in the following way:

$$\int\limits_x^\beta U(\tau)g(\tau)d\tau =$$

$$\int\limits_x^\beta \left(\int\limits_\alpha^\tau Q^q(\tau,s)w^q(s)ds\right)^{\frac{1}{p-q}}\left(\int\limits_\tau^\beta U^{p'}(y)\rho^{-p'}(y)dy\right)^{\frac{q-1}{p-q}} U^{p'}(\tau)\rho^{-p'}(\tau)d\tau \geq$$

$$\left(\int\limits_\alpha^x Q^q(x,s)w^q(s)ds\right)^{\frac{1}{p-q}}\left(\int\limits_x^\beta\int\limits_\tau^\beta U^{p'}(y)\rho^{-p'}(y)dy\right)^{\frac{q-1}{p-q}} U^{p'}(\tau)\rho^{-p'}(\tau)d\tau =$$

$$\left(\int_{\alpha}^{x} Q^q(x,s)w^q(s)ds\right)^{\frac{1}{p-q}} \times$$

$$\times \left(-\int_{x}^{\beta}\left(\int_{\tau}^{\beta} U^{p'}(y)\rho^{-p'}(y)dy\right)^{\frac{q-1}{p-q}} d\left(\int_{\tau}^{z} U^{p'}(y)\rho^{-p'}(y)dy\right)\right) =$$

$$\left(\frac{p-q}{p-1}\right)\left(\int_{\alpha}^{x} Q^q(x,s)w^q(s)ds\right)^{\frac{1}{p-q}}\left(\int_{x}^{\beta} U^{p'}(y)\rho^{-p'}(y)dy\right)^{\frac{(p-1)}{p-q}}.$$

We put the last estimate in (4.4) and find that

$$\|\mathbf{K}^*\|_{q,w}^q \gg$$

$$\int_{\alpha}^{\beta}\left(\int_{\alpha}^{x} Q^q(x,s)w^q(s)ds\right)^{\frac{p-1}{p-q}}\left(\int_{x}^{\beta} U^{p'}(y)\rho^{-p'}(y)dy\right)^{\frac{(p-1)(q-1)}{p-q}} U(x)g(x)dx. \qquad (4.5)$$

Substituting the expression for the function $g(x)$ in (4.5) we obtain

$$C\|g\|_{p,\rho} \geq \|\mathbf{K}^* g\|_{q,w} \gg$$

$$\left(\int_{\alpha}^{\beta}\left(\int_{\alpha}^{x} Q^q(x,s)w^q(s)ds\right)^{\frac{r}{q}}\left(\int_{x}^{\beta} U^{p'}(y)\rho^{-p'}(y)dy\right)^{\frac{r}{q'}} U^{p'}(x)\rho^{-p'}(x)dx\right)^{\frac{1}{q}}. \qquad (4.6)$$

It follows by (4.6) and (4.3) that

$$\left(\int_{\alpha}^{\beta}\left(\int_{\alpha}^{x} Q^q(x,s)w^q(s)ds\right)^{\frac{r}{q}}\left(\int_{x}^{\beta} U^{p'}(y)\rho^{-p'}(y)dy\right)^{\frac{r}{q'}} U^{p'}(x)\rho^{-p'}(x)dx\right)^{\frac{1}{r}} \ll C.$$

If in the last estimate we pass to limits when $\alpha \to a$ and $\beta \to b$, then we get that $M_2 \ll C < \infty$.

In a similar way we prove that $M_1 < \infty$. To do this we use inequality (4.2) and the test function

$$f(s) = \chi_{(\alpha,\beta)}(s)\left(\int_{s}^{\beta} K^{p'}(y,s)\rho^{-p'}(y)dy\right)^{\frac{(q-1)(p-1)}{p-q}}\left(\int_{\alpha}^{s} w^q(\tau)d\tau\right)^{\frac{q-1}{p-q}} w^q(s)$$

where $\alpha,\ \beta:\ a < \alpha < \beta < b$.

Sufficiency. Let $M_1 < \infty$, $M_2 < \infty$.

First we consider the case when $g(t)$ is a non-negative function with compact support. In this case $\mathbf{K}^* g(t)$ is a non-increasing and bounded function on the interval (a,b).

Hence, there exist $m \in \mathbb{Z}$, such that

$$\mathbf{K}^* g(t) \le (d+1)^{-m}, \quad \forall t \in (a,b).$$

We put

$$m_0 = max\{m \in \mathbb{Z} : \ \mathbf{K}^* g(t) \le (d+1)^{-m}, \ \forall t \in (a,b)\};$$
$$t_{m_0} = a;$$
$$t_k = sup\{t : \ \mathbf{K}^* g(t) = (d+1)^{-k}\}, \ k > m_0.$$

It follows, by the continuity of the function $\mathbf{K}^* g(t)$, that $\mathbf{K}^* g(t_k) = (d+1)^{-k}$.

For any integer $k \ge m_0$ the inequality $t_k < t_{k+1}$ holds. Indeed, for all $k \ge m_0$ we obtain that

$$\mathbf{K}^* g(t_k) = (d+1)^{-k} > (d+1)^{-(k+1)} = \mathbf{K}^* g(t_{k+1}).$$

By using the monotonicity of $\mathbf{K}^* g(t)$, we conclude that $t_k < t_{k+1}$.

We have constructed the sequence $\{t_k\}_{k=m_0}^{\infty} \subset (a,b)$, such that $(a,b) = \bigcup\limits_{k=m_0+1}^{\infty} (t_{k-1}, t_k]$. Moreover, if $k \ne l$ then $(t_{k-1}, t_k] \bigcap (t_{l-1}, t_l] = \emptyset$.

Since the function $K(x,s)$ belongs to Q_1 for $k : m_0 \le k < \infty$, the estimate

$$(d+1)^{-(k+1)} = (d+1)^{-k} - d(d+1)^{-(k+1)} =$$

$$\int\limits_{t_k}^{b} K(x,t_k) g(x) dx - d \int\limits_{t_{k+1}}^{b} K(x,t_{k+1}) g(x) dx =$$

$$\int\limits_{t_k}^{t_{k+1}} K(x,t_k) g(x) dx + \int\limits_{t_{k+1}}^{b} K(x,t_k) g(x) dx - d \int\limits_{t_{k+1}}^{b} K(x,t_{k+1}) g(x) dx \le$$

$$\int\limits_{t_k}^{t_{k+1}} K(x,t_k) g(x) dx +$$

$$d \int\limits_{t_{k+1}}^{b} \left(K(x,t_{k+1}) + U(x) Q(t_{k+1},t_k) - K(x,t_{k+1})\right) g(x) dx =$$

$$\int\limits_{t_k}^{t_{k+1}} K(x,t_k) g(x) dx + d Q(t_{k+1},t_k) \int\limits_{t_{k+1}}^{b} U(x) g(x) dx \tag{4.7}$$

holds.

Since $(d+1)^{-(k+1)} \le Kg(t) \le (d+1)^{-(k-2)}$ for $t_{k-1} \le t \le t_k$, we have the following estimate:

$$\|\mathbf{K}^* g\|_{q,w}^q = \sum_{k=m_0+1}^{\infty} \int\limits_{t_{k-1}}^{t_k} \left(\mathbf{K}^* g(s) w(s)\right)^q ds \le \sum_{k=m_0+1}^{\infty} (d+1)^{-q(k-2)} \int\limits_{t_{k-1}}^{t_k} w^q(s) ds =$$

$$= (d+1)^q \sum_{k=m_0+1}^{\infty} (d+1)^{-q(k-1)} \int_{t_{k-1}}^{t_k} w^q(s)ds.$$

By using (2.1) we get the estimate

$$\|\mathbf{K}^* g\|_{q,w}^q \ll I_1 + I_2, \tag{4.8}$$

where

$$I_1 = \sum_{k=m_0+1}^{\infty} \left(\int_{t_k}^{t_{k+1}} K(x,t_k) g(x) dx \right)^q \int_{t_{k-1}}^{t_k} w^q(s)ds,$$

$$I_2 = \sum_{k=m_0+1}^{\infty} Q^q(t_{k+1},t_k) \left(\int_{t_{k+1}}^{b} U(x) g(x) dx \right)^q \int_{t_{k-1}}^{t_k} w^q(s)ds.$$

Now we estimate each term in (2.2) separately.
By using Hölder's inequality twice, we find that

$$I_1 = \sum_{k=m_0+1}^{\infty} \left(\int_{t_k}^{t_{k+1}} K(x,t_k) g(x) dx \right)^q \int_{t_{k-1}}^{t_k} w^q(s)ds \le$$

$$\sum_{k=m_0+1}^{\infty} \left(\int_{t_k}^{t_{k+1}} K^{p'}(x,t_k) \rho^{-p'}(x) dx \right)^{\frac{q}{p'}} \left(\int_{t_k}^{t_{k+1}} g^p(x) \rho^p(x) dx \right)^{\frac{q}{p}} \int_{t_{k-1}}^{t_k} w^q(s)ds \le$$

$$\left(\sum_{k=m_0+1}^{\infty} \left(\int_{t_k}^{t_{k+1}} K^{p'}(x,t_k) \rho^{-p'}(x) dx \right)^{\frac{r}{p'}} \left(\int_{t_{k-1}}^{t_k} w^q(s)ds \right)^{\frac{r}{q}} \right)^{\frac{q}{r}} \times$$

$$\times \left(\sum_{k=m_0+1}^{\infty} \int_{t_k}^{t_{k+1}} g^p(x) \rho^p(x) dx \right)^{\frac{q}{p}}.$$

By applying the representation

$$\left(\int_{t_{k-1}}^{t_k} w^q(s)ds \right)^{\frac{p}{p-q}} = \frac{p}{p-q} \int_{t_{k-1}}^{t_k} w^q(t) \left(\int_{t_{k-1}}^{t} w^q(s)ds \right)^{\frac{q}{p-q}} dt$$

and using the estimate

$$\int_{t_k}^{t_{k+1}} K^{p'}(x,t_k) \rho^{-p'}(x) dx \le d^{p'} \int_{t_k}^{t_{k+1}} K^{p'}(x,t) \rho^{-p'}(x) dx \text{ for } t < t_k,$$

we obtain

$$I_1 \ll \left(\sum_{k=m_0+1}^{\infty} \left(\int_{t_k}^{t_{k+1}} K^{p'}(x,t_k)\rho^{-p'}(x)dx \right)^{\frac{r}{p'}} \left(\int_{t_{k-1}}^{t_k} w^q(s)ds \right)^{\frac{r}{q}} \right)^{\frac{q}{r}} \|g\|_{p,\rho}^q \ll$$

$$\left(\sum_{k=m_0+1}^{\infty} \int_{t_{k-1}}^{t_k} \left(\int_{t_k}^{t_{k+1}} K^{p'}(x,t)\rho^{-p'}(x)dx \right)^{\frac{r}{p'}} \left(\int_{t_{k-1}}^{t} w^q(s)ds \right)^{\frac{r}{p}} w^q(t)dt \right)^{\frac{q}{r}} \|g\|_{p,\rho}^q \ll$$

$$\left(\sum_{k=m_0+1}^{\infty} \int_{t_{k-1}}^{t_k} \left(\int_{t}^{b} K^{p'}(x,t)\rho^{-p'}(x)dx \right)^{\frac{r}{p'}} \left(\int_{a}^{t} w^q(s)ds \right)^{\frac{r}{p}} w^q(t)dt \right)^{\frac{q}{r}} \|g\|_{p,\rho}^q =$$

$$\left(\int_{a}^{b} \left(\int_{t}^{b} K^{p'}(x,t)\rho^{-p'}(x)dx \right)^{\frac{r}{p'}} \left(\int_{a}^{t} w^q(s)ds \right)^{\frac{r}{p}} w^q(t)dt \right)^{\frac{q}{r}} \|g\|_{p,\rho}^q.$$

Summing up, we have the following estimate for I_1:

$$I_1 \ll M_1^q \|g\|_{p,\rho}^q. \tag{4.9}$$

Next we estimate

$$I_2 = \sum_{k=m_0+1}^{\infty} Q^q(t_{k+1},t_k) \left(\int_{t_{k+1}}^{b} U(x)g(x)dx \right)^q \int_{t_{k-1}}^{t_k} w^q(s)ds.$$

Let $\delta_z(t)$ denote the delta-function at a point $z \in (a,b)$. The expression I_2 can be written in the following way:

$$I_2 = \int_a^b \left(\int_t^b U(x)g(x)dx \right)^q d\lambda(t) = \|H^*g\|_{q,\lambda}^q,$$

where

$$d\lambda(t) = \sum_{k=m_0+1}^{\infty} Q^q(t_{k+1},t_k) \left(\int_{t_{k-1}}^{t_k} w^q(s)ds \right) \delta_{t_{k+1}}(t)dt.$$

By using standard results from the theory of Hardy type inequalities (see e.g. [3]-[4]) we get that the inequality

$$\|H^*g\|_{q,\lambda} \le C\|g\|_{p,\rho}$$

holds if and only if the expression

$$M := \left(\int_a^b \left(\int_a^t d\lambda(s) \right)^{\frac{r}{q}} \left(\int_t^b U^{p'}(x)\rho^{-p'}(x)dx \right)^{\frac{r}{q'}} U^{p'}(t)\rho^{-p'}(t)dt \right)^{\frac{1}{r}}$$

is finite. Here

$$\int_a^t d\lambda(s) = \sum_{t \geq t_{k+1}} Q^q(t_{k+1}, t_k) \int\limits_{t_{k-1}}^{t_k} w^q(s)ds$$

Moreover, $M \approx \|H^*\|_{L_{p,\rho} \to L_{q,\lambda}}$.

The estimate

$$Q(t_{k+1}, t_k) \leq Q(t, s) \text{ for } s \leq t_k < t_{k+1} \leq t$$

follows from the fact that $Q(\cdot, \cdot)$ is non-decreasing in x and non-increasing in y.

Hence,

$$\int_a^t d\lambda(s) \ll \sum_{t \geq t_{k+1}} \int\limits_{t_{k-1}}^{t_k} Q^q(t, s) w^q(s) ds \leq \int\limits_a^t Q^q(t, s) w^q(s) ds. \tag{4.10}$$

By using (2.4) we get that $\|H^*\|_{L_{p,\rho} \to L_{q,\lambda}} \approx M \leq M_2$ and the inequality

$$I_2 \ll M_2^q \|g\|_{p,\rho}^q \tag{4.11}$$

holds.

It follows by (2.2), (2.3) and (2.5) that for any non-negative function $g \in L_{p,\rho}(a, b)$ with compact support the inequality

$$\|\mathbf{K}^* g\|_{q,w} \ll (M_1 + M_2) \|g\|_{p,\rho} < \infty \tag{4.12}$$

holds. This is equivalent to the fact that (2.6) holds for all functions in $L_{p,\rho}(a, b)$ with compact support. But the set of such functions is dense in $L_{p,\rho}(a, b)$. Therefore, we conclude that (2.6) holds for all $g \in L_{p,\rho}(a, b)$ and the proof is complete.

The proof of the Theorem 3.2 is completely analogous to the proof of Theorem 3.1 so we leave out the details.

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