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CONTENTS

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9 772077 987003

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AN INVERSE PROBLEM FOR 1D FRACTIONAL INTEGRO-DIFFERENTIAL WAVE EQUATION WITH FRACTIONAL TIME DERIVATIVE

A.A. Rahmonov

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Key words: fractional integro-differential wave equation, Gerasimov-Caputo fractional derivative, Fourier method, Mittag-Leffler function, Bessel inequality.

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Abstract. This paper is devoted to obtaining a unique solution to an inverse problem for a one-dimensional time-fractional integro-differential equation. First, we consider the direct problem, and the unique existence of the weak solution is established, after that, the smoothness conditions for the solution are obtained. Secondly, we study the inverse problem of determining the unknown coefficient and kernel, and the well-posedness of this inverse problem is proved. The local existence and global uniqueness results are based on the Fourier method, fractional calculus, properties of the Mittag-Leffler function, and Banach fixed point theorem in a suitable Sobolev space.

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1 Introduction and setting up the problem

For studying objects or processes in the surrounding world, methods of mathematical modeling are extensively used. An efficient way to study processes by mathematical methods is by modeling these processes in the form of fractional differential equations.

Fractional differential equations have excited, in recent years, a considerable interest both in mathematics and in applications. They were used in the modeling of many physical and chemical processes and engineering (see, e.g., [2]-[4]). Other studies [7]-[6] demonstrate several interesting features of the fractional diffusion-wave equations, which represent a peculiar union of properties typical for second-order parabolic and wave differential equations. Fractional evolution inclusions are an important form of differential inclusions within nonlinear mathematical analysis. They are generalizations of the much more widely developed fractional evolution equations (such as time-fractional diffusion equations) seen through the lens of multivariate analysis. Compared with fractional evolution equations, research on the theory of fractional differential inclusions is however only in its initial stage of development. This is important because differential models with the fractional derivative provide an excellent instrument for the description of memory and hereditary properties, and have recently been proven valuable tools in the modeling of many physical phenomena (see, [20] and the references therein).

According to the fractional order α, the diffusion process can be specified as sub-diffusion ($\alpha \in (0,1)$) and super-diffusion ($\alpha \in (1,2)$), respectively. There is abundant literature on the studies of fractional equations on various aspects, such as physical backgrounds, weak solutions, maximum principle and numerical methods (see, [19] and the references therein).

Practical needs often lead to problems in determining the coefficients, kernel, or the right-hand side of a differential equation from certain known information about its solution. Such problems have received the name inverse problems of mathematical physics. Inverse problems arise in various domains of human activity, such as seismology, prospecting for mineral deposits, biology, medical visualization, computer-aided tomography, the remote sounding of the Earth, spectral analysis, nondestructive control, etc., (see [3], [10]-[17]). In this paper, we discuss an inverse problem of determining a coefficient and kernel depending on the time in a fractional-differential equation by the measurement data of time trace at a fixed point x_i.

Let $Q_0^T := (0,1) \times (0,T)$ for a given time $T > 0$. We consider the following fractional integro-differential equation with a fractional derivative in time t:

$$\partial_t^\alpha u(x,t) + \mathcal{L}u(x,t) = q(t)u_t(x,t) + k * u(x,\cdot) + f(x,t), \quad (x,t) \in Q_0^T, \tag{1.1}$$

where $1 < \alpha < 2$ and $\partial_t^\alpha u(x,t)$ is the left Gerasimov-Caputo fractional derivative with respect to t and is defined in [9] as

$$\partial_t^\alpha v(t) = \mathcal{K} * v'',$$

here the kernel function $\mathcal{K}$ is given by

$$\mathcal{K}(t) = \begin{cases} \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

$\Gamma(\cdot)$ is the Gamma function and $\mathcal{L}$ is the differential operator defined by

$$\mathcal{L}u \equiv -(\varrho(x)u')' + c(x)u,$$

where the coefficients belong to the set:

$$\Lambda := \left\{ (\varrho, c) \in C^1[0,1] \times C[0,1] : \, c(x) > 0, \; \varrho(x) \geq \varrho_0 > 0 \right\},$$

and $*$ denotes the Laplace convolution

$$f * g(t) = \int_0^t f(t-\tau)g(\tau)d\tau.$$

Note that if $\alpha = 1$ and $\alpha = 2$, then equation (1.1) represents a parabolic and a hyperbolic integro-differential equations, respectively. Since we are interested mainly in the fractional cases, we restrict the order α to $1 < \alpha < 2$.

We supplement the above fractional wave equation with the following initial conditions:

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad 0 < x < 1, \tag{1.2}$$

and the zero boundary condition:

$$u(0,t) = u(1,t) = 0, \quad 0 < t < T. \tag{1.3}$$

For convenience of the reader, we present here the necessary definitions from functional analysis and fractional calculus theory.

For integers m, we denote $H^m(0,1) = W^{m,2}(0,1)$ and $W^{k,1}(0,T)$ the usual Sobolev spaces defined for spatial and time variables respectively (see [1]), and $H_0^m(0,1)$ is the closure of $C_0^\infty(0,1)$ in the norm of space $H^m(0,1)$. For a given Banach space V on $(0,1)$, we use the notation $C^m([0,T];V)$ to denote the following space:

$$C^m([0,T];V) := \left\{ u : [0,T] \to V : \, \|\partial_t^j u(t)\|_V \text{ is continuous in } t \text{ on } [0,T] \text{ for all } 0 \leq j \leq m \right\}.$$

We endow $C^m([0,T];V)$ with the following norm, making it a Banach space:

$$\|u\|_{C^m([0,T];V)} = \sum_{j=0}^{m} \left(\max_{0\le t\le T} \|\partial_t^j u(t)\|_V \right).$$

In addition, we define the Banach space X_0^T by

$$X_0^T := C([0,T];\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}})) \cap C^1([0,T];\mathcal{D}(\mathcal{L}^{\gamma}))$$

with the norm

$$\|u\|_{X_0^T} := \|u\|_{C([0,T];\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}}))} + \|u\|_{C^1([0,T];\mathcal{D}(\mathcal{L}^{\gamma}))}.$$

The notation X_0^T indicates that zero represents the initial state of the time variable, i.e., $t = 0$. Furthermore, we set

$$Y_0^T = X_0^T \times C^1[0,T] \times C[0,T]$$

endowed with the norm

$$\|(u,q,k)\|_{Y_0^T} := \|u\|_{X_0^T} + \|q\|_{C^1[0,T]} + \|k\|_{C[0,T]}.$$

We denote the domain of $\mathcal{L}$ by $\mathcal{D}(\mathcal{L}) = H^2(0,1)\cap H_0^1(0,1)$. It is well known that, if the coefficients ϱ and c of the operator $\mathcal{L}$ are in the set Λ, then the operator $\mathcal{L}$ has only real and simple eigenvalues λ_n, and with suitable numbering, we have $0 < \lambda_1 \le \lambda_2 \le \cdots$, $\lim_{k\to\infty} \lambda_k = \infty$. By e_k, we denote the eigenfunction corresponding to λ_k, which satisfies $\|e_k\|^2_{L^2(0,1)} = (e_k, e_k) = 1$, where $(\cdot,\cdot)$ denotes the inner product in the Hilbert space $L^2(0,1)$ and λ_k, e_k satisfy $\mathcal{L}e_k = \lambda_k e_k$, $e_k(0) = e_k(1) = 0$, $\{e_k\} \subset H^2(0,1) \cap H_0^1(0,1)$ is an orthonormal basis of $L^2(0,1)$.

Now we define the fractional power operator $\mathcal{L}^\gamma$ for $\gamma \in \mathbb{R}$ (e.g. [13]) and the Hilbert space $\mathcal{D}(\mathcal{L}^\gamma)$ by

$$\mathcal{D}(\mathcal{L}^\gamma) := \left\{ u \in L^2(0,1) : \sum_{k=1}^{\infty} \lambda_k^{2\gamma} |(u,e_k)|^2 < \infty \right\}, \quad \mathcal{L}^\gamma u = \sum_{k=1}^{\infty} \lambda_k^{\gamma} (u,e_k) e_k$$

with the inner product $(u,v)_{\mathcal{D}(\mathcal{L}^\gamma)} = (\mathcal{L}^\gamma u, \mathcal{L}^\gamma v)_{L^2(0,1)}$ and, respectively, the norm

$$\|u\|_{\mathcal{D}(\mathcal{L}^\gamma)} = \|\mathcal{L}^\gamma u\| = \left(\sum_{k=1}^{\infty} \lambda_k^{2\gamma} |(u,e_k)|^2 \right)^{1/2}.$$

Moreover, we shall use the Mittag-Leffler function (see [9]):

$$E_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\rho k + \mu)}, \quad z \in \mathbb{C}$$

with $\mathrm{Re}(\rho) > 0$ and $\mu \in \mathbb{C}$. It is known that $E_{\rho,\mu}(z)$ is an entire function in $z \in \mathbb{C}$.

Lemma 1.1. *Let $0 < \rho < 2$, $\mu \in \mathbb{R}$ be arbitrary and θ satisfy $\frac{\pi\rho}{2} < \theta < \min\{\pi, \pi\rho\}$. Then there exists a constant $c = c(\rho,\mu,\theta) > 0$ such that*

$$|E_{\rho,\mu}(z)| \le \frac{c}{1+|z|}, \qquad \theta \le |arg(z)| \le \pi,$$

and the asymptotic behavior of $E_{\rho,\mu}(z)$ at infinity is as follows: for any $N \in \mathbb{N}$

$$E_{\rho,\mu}(z) = -\sum_{n=1}^{N} \frac{z^{-n}}{\Gamma(\mu - \rho n)} + O(z^{-N-1}).$$

For the proof, we refer, for example, to [5].

Remark 1. In the paper, the Mittag-Leffler function is used only for real negative z, in which case the constant c depends only on ρ and μ.

Proposition 1.1. *(see [9]) For $\lambda > 0$, $\alpha > 0, \beta \in \mathbb{C}$ and positive integer $m \in \mathbb{N}$, we have*

$$\frac{d^m}{dt^m} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-m} E_{\alpha,\alpha-m+1}(-\lambda t^\alpha), \quad t > 0,$$

$$\frac{d}{dt}\left(t^{\beta-1} E_{\alpha,\beta}(-\lambda t^\alpha)\right) = t^{\beta-2} E_{\alpha,\beta-1}(-\lambda t^\alpha), \quad t > 0$$

and

$$\partial_t^\alpha (E_{\alpha,1}(-\lambda t^\alpha)) = -\lambda E_{\alpha,1}(-\lambda t^\alpha), \quad t \geq 0.$$

Also, we shall use the following simple equality

$$\max_{y \geq 0} \frac{y^\theta}{1+y} = \theta^\theta (1-\theta)^{1-\theta} < 1 \quad \text{for} \quad 0 < \theta < 1. \tag{1.4}$$

If $q(t)$, $k(t)$, $f(x,t)$, $\varphi(x)$ and $\psi(x)$ are known, then problem (1.1)-(1.3) is called a direct problem. The inverse problem in this paper is to reconstruct $q(t)$ and $k(t)$ according to the additional data

$$u(x_i, t) = h_i(t), \quad t \in [0, T], \tag{1.5}$$

where $x_i \in (0,1)$, $i = 1, 2$ are fixed points, $h_i(t)$, $i = 1, 2$ are given functions.

We investigate the following inverse problem.

Inverse problem. Find $u \in X_0^T$, $q \in C^1[0,T]$ and $k \in C[0,T]$ to satisfy (1.1)-(1.3) and additional measurements (1.5), where $\mathcal{D}(\mathcal{L}^\gamma)$ is a Hilbert space with some positive constant γ satisfying inequality (1.6).

We now give a similar definition of a weak solution to (1.1)-(1.3), which is introduced in [15].

Definition 1. We call u a weak solution to (1.1)-(1.3) if (1.1) holds in $L^2(0,1)$ and $u(\cdot, t) \in H_0^1(0,1)$ for almost all $t \in (0,T)$, $u, \partial_t u \in C([0,T]; \mathcal{D}(\mathcal{L}^{-\gamma}))$ and

$$\lim_{t \to 0} \|u(\cdot,t) - \varphi\|_{\mathcal{D}(\mathcal{L}^{-\gamma})} = \lim_{t \to 0} \|u_t(\cdot,t) - \psi\|_{\mathcal{D}(\mathcal{L}^{-\gamma})} = 0$$

with some $\gamma > 0$.

Throughout this paper, we assume that $\frac{3}{2} < \gamma_0$ and

$$\frac{5}{4} < \gamma \leq \gamma_0. \tag{1.6}$$

We make the following assumptions:

(C1) $\partial_t^\alpha h_i \in C^1[0,T]$ $(i = 1, 2)$, $\varphi \in \mathcal{D}(\mathcal{L}^{\gamma_0 + \frac{1}{\alpha}})$, $\psi \in \mathcal{D}(\mathcal{L}^{\gamma_0})$, $f \in C^1([0,T]; \mathcal{D}(\mathcal{L}^\gamma))$;
(C2) $h_i'(0) q(0) = \partial_t^\alpha h_i(0) + \mathcal{L}\varphi(x_i) - \tilde{f}_i(0)$, where $\tilde{f}_i(t) = f(x_i, t)$, $(i = 1, 2)$;
(C3) $\varphi(x_i) = h_i(0)$, $\psi(x_i) = h_i'(0)$, $(i = 1, 2)$;
(C4) $p(t) = h_1'(t) h_2(0) - h_2'(t) h_1(0) \neq 0$ and $p \in C^1[0,T]$ satisfies the following inequality:

$$|p(t)| \geq \frac{1}{p_0},$$

where p_0 is a positive constant.

Remark 2. In (C1), $\partial_t^\alpha h \in C^1[0,T]$ implies $h_i \in W^{2,1}(0,T) \hookrightarrow H^1(0,T)$ (see [17]).

Remark 3. (C2)-(C3) are the consistency conditions for our problem (1.1)-(1.3), (1.5), which guarantees that inverse problem (1.1)-(1.3), (1.5) is equivalent to (2.36) and (2.38) (see Lemma 2.6).

Remark 4. In order to guarantee that $\partial_t^\alpha h \in C^1[0,T]$, we could give the usual regularity condition $h_i \in C^3[0,T]$, such that $h_i''(0) = 0$ (see [17]).

The main result of this paper is the following local existence and uniqueness result for an inverse problem.

Theorem 1.1. *Let the assumptions* (C1)-(C4) *hold. Then, the inverse problem has a unique solution* $(u, q, k) \in Y_0^T$ *for sufficiently small* $T > 0$.

The outline of the paper is as follows. Section 2 presents preliminary results, including the existence and uniqueness of the direct problem (1.1)-(1.3), along with an equivalent problem. In Section 3, we establish the local existence and global uniqueness of the solution to the inverse problem (1.1)-(1.3), (1.5) using the Fourier method and the Banach fixed-point theorem. In Section 4, we provide examples of the inverse problem (1.1)-(1.3), (1.5).

2 Preliminary results

This section presents some preliminary results, including the well-posedness for a fractional differential equation, an equivalent lemma for our inverse problem, and a technical result, which will be used to prove our main results.

Let

$$\begin{cases} Z_1(t)\eta(x) = \sum\limits_{n=1}^{\infty} (\eta, e_n) E_{\alpha,1}(-\lambda_n t^\alpha) e_n(x), \\ Z_2(t)\eta(x) = \sum\limits_{n=1}^{\infty} (\eta, e_n) t E_{\alpha,2}(-\lambda_n t^\alpha) e_n(x), & (x,t) \in Q_0^T, \\ Z_3(t)\eta(x) = -\sum\limits_{n=1}^{\infty} \lambda_n (\eta, e_n) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) e_n(x). \end{cases}$$

for $\eta \in L^2(0,1)$.

We first consider the following initial and boundary value problem:

$$\begin{cases} \partial_t^\alpha u(x,t) + \mathcal{L}u(x,t) = F(x,t), & (x,t) \in Q_0^T, \\ u(0,t) = u(1,t) = 0, & 0 < t < T, \\ u(x,0) = \varphi(x), \ u_t(x,0) = \psi(x), & 0 < x < 1. \end{cases} \tag{2.1}$$

We split (2.1) into the following two initial and boundary value problems:

$$\begin{cases} \partial_t^\alpha v(x,t) + \mathcal{L}v(x,t) = 0, & (x,t) \in Q_0^T, \\ v(0,t) = v(1,t) = 0, & 0 < t < T, \\ v(x,0) = \varphi(x), \ v_t(x,0) = \psi(x), & 0 < x < 1, \end{cases} \tag{2.2}$$

and

$$\begin{cases} \partial_t^\alpha w(x,t) + \mathcal{L}w(x,t) = F(x,t), & (x,t) \in Q_0^T, \\ w(0,t) = w(1,t) = 0, & 0 < t < T, \\ w(x,0) = 0, \ w_t(x,0) = 0, & 0 < x < 1. \end{cases} \tag{2.3}$$

Similarly to Theorem 2.3 in [15], it is easy to obtain the following assertion:

Lemma 2.1. *Let $\varphi \in H^2(0,1) \cap H_0^1(0,1)$ and $\psi \in H_0^1(0,1)$. Let $\gamma > 0$. Then there exists a unique weak solution $v \in C([0,T]; H^2(0,1) \cap H_0^1(0,1)) \cap C^1([0,T]; \mathcal{D}(\mathcal{L}^{-\gamma}))$ to (2.2). Moreover, there exists a constant $c > 0$, depending only on α, γ and λ_1, such that*

$$\|v(\cdot,t)\|_{H^2(0,1)} + \|v_t(\cdot,t)\|_{\mathcal{D}(\mathcal{L}^{-\gamma})} \le c\left(\|\varphi\|_{H^2(0,1)} + \|\psi\|_{H^1(0,1)}\right), \quad t \in (0,T). \tag{2.4}$$

Furthermore, we have

$$\begin{cases} v(x,t) = Z_1(t)\varphi(x) + Z_2(t)\psi(x), & (x,t) \in Q_0^T, \\ v_t(x,t) = Z_3(t)\varphi(x) + Z_1(t)\psi(x), & (x,t) \in Q_0^T. \end{cases} \tag{2.5}$$

Proof. The uniqueness and existence of a weak solution are verified similarly to Theorem 2.1 in [15], but the statement about the smoothness for the function v given in the lemma differs from the statement about the smoothness given in [15]. Therefore, here we prove only inequality (2.4). Using Lemma 1.1, we have

$$\begin{aligned} \|v(\cdot,t)\|^2_{H^2(0,1)} &\le 2\sum_{n=1}^{\infty} \lambda_n^2 \left|(\varphi,e_n)E_{\alpha,1}(-\lambda_n t^\alpha)\right|^2 + 2\sum_{n=1}^{\infty} \lambda_n^2 \left|(\psi,e_n)tE_{\alpha,2}(-\lambda_n t^\alpha)\right|^2 \\ &\le 2c^2\|\varphi\|^2_{H^2(0,1)} + 2c^2\sum_{n=1}^{\infty}\lambda_n|(\psi,e_n)|^2\left(\frac{(\lambda_n t^\alpha)^{\frac{1}{\alpha}}}{1+\lambda_n t^\alpha}\right)^2 \lambda_n^{1-\frac{2}{\alpha}}, \end{aligned}$$

where $c > 0$, depending only on α, is given in Lemma 1.1. Since $\lambda_n^{1-\frac{2}{\alpha}} \le \lambda_1^{1-\frac{2}{\alpha}}, n = 1,2,...,$ by (1.4) we have

$$\|v(\cdot,t)\|^2_{H^2(0,1)} \le 2c^2\max\{1,\lambda_1^{1-\frac{2}{\alpha}}\}\left(\|\varphi\|^2_{H^2(0,1)} + \|\psi\|^2_{H^1(0,1)}\right). \tag{2.6}$$

Further, by the second formula of (2.5), we have

$$\begin{aligned} \|v_t(\cdot,t)\|^2_{\mathcal{D}(\mathcal{L}^{-\gamma})} &\le 2\sum_{n=1}^{\infty}\lambda_n^{-2\gamma}\left|\lambda_n t^{\alpha-1}(\varphi,e_n)E_{\alpha,\alpha}(-\lambda_n t^\alpha)\right|^2 \\ &\quad + 2\sum_{n=1}^{\infty}\lambda_n^{-2\gamma}\left|(\psi,e_n)E_{\alpha,1}(-\lambda_n t^\alpha)\right|^2 \\ &\le 2c^2\sum_{n=1}^{\infty}\lambda_n^2|(\varphi,e_n)|^2\left(\frac{(\lambda_n t^\alpha)^{\frac{\alpha-1}{\alpha}}}{1+\lambda_n t^\alpha}\right)^2\lambda_n^{-2(\gamma+1-\frac{1}{\alpha})} + 2c^2\sum_{n=1}^{\infty}\lambda_n|(\psi,e_n)|^2\lambda_n^{-2(\gamma+\frac{1}{2})}. \end{aligned} \tag{2.7}$$

Now, using Lemma 1.1 and (1.4), we have

$$\|v_t(\cdot,t)\|^2_{\mathcal{D}(\mathcal{L}^{-\gamma})} \le 2c^2\max\{\lambda_1^{-2(\gamma+1-\frac{1}{\alpha})},\lambda_1^{-2(\gamma+\frac{1}{2})}\}\left(\|\varphi\|^2_{H^2(0,1)} + \|\psi\|^2_{H^1(0,1)}\right). \tag{2.8}$$

□

We introduce the following auxiliary lemmas to obtain the main results.

Lemma 2.2. *Let $F \in C([0,T]; \mathcal{D}(\mathcal{L}^{1/\alpha}))$. Then there exists a unique weak solution $w \in C([0,T]; H^2(0,1) \cap H_0^1(0,1))$ to (2.3) with $\partial_t^\alpha w \in C([0,T]; L^2(0,1))$. Moreover, for any $\gamma > 0$, we have $w_t \in C([0,T]; \mathcal{D}(\mathcal{L}^{-\gamma}))$,*

$$\lim_{t\to 0}\|w(\cdot,t)\|_{H^2(0,1)} = \lim_{t\to 0}\|w_t(\cdot,t)\|_{\mathcal{D}(\mathcal{L}^{-\gamma})} = 0. \tag{2.9}$$

Furthermore, there exists a constant $c > 0$, depending only on α, γ and λ_1, such that

$$\|w(\cdot,t)\|_{H^2(0,1)} + \|w_t(\cdot,t)\|_{\mathcal{D}(\mathcal{L}^{-\gamma})} \le c\big(t + t^{\alpha-1}\big)\|F\|_{C([0,T];\mathcal{D}(\mathcal{L}^{1/\alpha}))} \tag{2.10}$$

and we have

$$\begin{cases} w(x,t) = -\int_0^t \mathcal{L}^{-1} Z_3(t-s)F(x,s)ds, & (x,t) \in Q_0^T, \\ w_t(x,t) = \int_0^t \mathcal{L}^{-1} Z_4(t-s)F(x,s)ds, & (x,t) \in Q_0^T, \end{cases} \tag{2.11}$$

where

$$Z_4(t)\eta(x) = \sum_{n=1}^{\infty} \lambda_n(\eta, e_n)t^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_n t^\alpha)e_n(x)$$

and the function w belongs to the space $C([0,T]; H^2(0,1) \cap H_0^1(0,1)) \cap C^1([0,T]; \mathcal{D}(\mathcal{L}^{-\gamma}))$.

Proof. By Theorem 2.2 in [15], for $F \in C([0,T]; \mathcal{D}(\mathcal{L}^{1/\alpha}))$, the unique solution $w \in C([0,T]; H^2(0,1) \cap H_0^1(0,1))$ to (2.3) can be expressed by (2.11). As above, the uniqueness and existence of the weak solution are verified similarly to Theorem 2.1 in [15]. Therefore, here we omitted it and we prove only equality (2.9) and inequality (2.10).

We first have

$$\|w(\cdot,t)\|^2_{L^2(0,1)} = \sum_{n=1}^{\infty}\left|\int_0^t (F(\cdot,s),e_n)(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)ds\right|^2$$
$$\le c^2\sum_{n=1}^{\infty}\left|\int_0^t \lambda_n^{\frac{1}{\alpha}}|(F(\cdot,s),e_n)|\frac{(\lambda_n(t-s)^\alpha)^{\frac{\alpha-1}{\alpha}}}{1+\lambda_n(t-s)^\alpha}\lambda_n^{-1}ds\right|^2,$$

or, by virtue of the generalized Minkowski inequality, we have

$$\|w(\cdot,t)\|^2_{L^2(0,1)} \le c^2\left|\int_0^t \left(\sum_{n=1}^{\infty}\lambda_n^{\frac{2}{\alpha}}|(F(\cdot,s),e_n)|^2\right)^{1/2}\lambda_1^{-1}ds\right|^2$$
$$\le c^2\lambda_1^{-2}\max_{0\le s\le t}\|F(\cdot,s)\|^2_{\mathcal{D}(\mathcal{L}^{1/\alpha})}\left|\int_0^t ds\right|^2 \le c^2\lambda_1^{-2}\|F\|^2_{C([0,T];\mathcal{D}(\mathcal{L}^{1/\alpha}))}t^2. \tag{2.12}$$

Furthermore, according to Lemma 2.2, for $F \in C([0,T]; \mathcal{D}(\mathcal{L}^{1/\alpha}))$ and by Lemma 1.1, we have

$$\|\omega(\cdot,t)\|^2_{H^2(0,1)}$$
$$\le \|\mathcal{L}\omega(\cdot,t)\|^2_{L^2(0,1)} = \sum_{n=1}^{\infty}\lambda_n^2\left|\int_0^t (F(\cdot,s),e_n)(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)ds\right|^2$$
$$\le c^2\left|\int_0^t \left(\sum_{n=1}^{\infty}\lambda_n^{\frac{2}{\alpha}}|(F(\cdot,s),e_n)|^2\right)^{1/2}\frac{(\lambda_n(t-s)^\alpha)^{\frac{\alpha-1}{\alpha}}}{1+\lambda_n(t-s)^\alpha}ds\right|^2$$
$$\le c^2\|F\|^2_{C([0,T];\mathcal{D}(\mathcal{L}^{1/\alpha}))}t^2. \tag{2.13}$$

By (2.3) and (2.13) we can estimate also $\|\partial_t^\alpha\omega(\cdot,t)\|_{C([0,T];L^2(0,1))}$ and we have $\lim\limits_{t\to 0}\|\omega(\cdot,t)\|_{H^2(0,1)} = 0$. Next, applying Lemma 1.1, Proposition 1, and the Cauchy-Schwarz inequality, for any $\gamma > 0$, we have

$$\begin{aligned}\|\omega_t(\cdot,t)\|^2_{\mathcal{D}(\mathcal{L}^{-\gamma})} &= \sum_{n=1}^{\infty}\lambda_n^{-2\gamma}\left|\int_0^t (F(\cdot,s),e_n)(t-s)^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_n(t-s)^{\alpha})ds\right|^2\\ &\le c^2\left|\int_0^t\left(\sum_{n=1}^{\infty}\lambda_n^{\frac{1}{\alpha}}|(F(\cdot,s),e_n)|^2\lambda_n^{-2\gamma-\frac{2}{\alpha}}\right)^{1/2}(t-s)^{\alpha-2}ds\right|^2\\ &\le \frac{c^2}{(\alpha-1)^2}\lambda_1^{-2\gamma-\frac{2}{\alpha}}\|F\|^2_{C([0,T];\mathcal{D}(\mathcal{L}^{1/\alpha}))}t^{2\alpha-2}. \end{aligned} \tag{2.14}$$

Therefore, $\lim_{t\to 0}\|\omega_t(\cdot,t)\|^2_{\mathcal{D}(\mathcal{L}^{-\gamma})}=0$. Inequality (2.10) follows from inequalities (2.13) and (2.14). □

By Lemma 2.1 and 2.2, we get the following assertion:

Lemma 2.3. *Let $\varphi\in H^2(0,1)\cap H_0^1(0,1)$, $\psi\in H_0^1(0,1)$ and $F(x,t)\in C([0,T];D(\mathcal{L}^{1/\alpha}))$. Then there exists a unique weak solution $u\in C([0,T];H^2(0,1)\cap H_0^1(0,1))\cap C^1([0,T];\mathcal{D}(\mathcal{L}^{-\gamma}))$ to (2.1), such that*

$$\begin{aligned}&\|u(\cdot,t)\|_{H^2(0,1)}+\|u_t(\cdot,t)\|_{\mathcal{D}(\mathcal{L}^{-\gamma})}\\ &\qquad\le c\left[\|\varphi\|_{H^2(0,1)}+\|\psi\|_{H^1(0,1)}+(t+t^{\alpha-1})\|F\|_{C([0,T];\mathcal{D}(\mathcal{L}^{1/\alpha}))}\right]\end{aligned} \tag{2.15}$$

for all $t\in[0,T]$, where the constant $c>0$ depends only on α,γ and λ_1, in particular, does not depend on T. Furthermore, for all $(x,t)\in Q_0^T$ we have

$$\begin{cases} u(x,t)=Z_1(t)\varphi(x)+Z_2(t)\psi(x)-\int_0^t\mathcal{L}^{-1}Z_3(t-s)F(x,s)ds,\\ u_t(x,t)=Z_3(t)\varphi(x)+Z_1(t)\psi(x)+\int_0^t\mathcal{L}^{-1}Z_4(t-s)F(x,s)ds,\end{cases} \tag{2.16}$$

where $Z_j(t)[\cdot](j=1,2,3,4)$ are defined above.

The next two lemmas are regularity results of the solution u to problem (2.1).

Lemma 2.4. *Let $\varphi\in\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}})$, $\psi\in\mathcal{D}(\mathcal{L}^{\gamma})$ and $F\in C([0,T];\mathcal{D}(\mathcal{L}^{\gamma})$. Then the unique weak solution $u\in X_0^T$ to (2.1) is such that*

$$\|u(\cdot,t)\|_{\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}})}+\|u_t(\cdot,t)\|_{\mathcal{D}(\mathcal{L}^{\gamma})}\le c\Big(\|\varphi\|_{\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}})}+\|\psi\|_{\mathcal{D}(\mathcal{L}^{\gamma})}+t^{\alpha-1}\|F\|_{C([0,T];\mathcal{D}(\mathcal{L}^{\gamma}))}\Big), \tag{2.17}$$

where $c>0$ depends only on α.

Proof. Obviously, by Lemma 1.1 and the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
\|u(\cdot,t)\|^2_{D(\mathcal{L}^{\gamma+\frac{1}{\alpha}})} &\le 3\sum_{n=1}^{\infty}\lambda_n^{2\gamma+\frac{2}{\alpha}}|(\varphi,e_n)E_{\alpha,1}(-\lambda_n t^{\alpha})|^2 + 3\sum_{n=1}^{\infty}\lambda_n^{2\gamma+\frac{2}{\alpha}}t^2|(\psi,e_n)E_{\alpha,2}(-\lambda_n t^{\alpha})|^2 \\
&\quad + 3\sum_{n=1}^{\infty}\lambda_n^{2\gamma+\frac{2}{\alpha}}\left|\int_0^t (F(\cdot,s),e_n)(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^{\alpha})ds\right|^2 \\
&\le 3c^2\sum_{n=1}^{\infty}\lambda_n^{2\gamma+\frac{2}{\alpha}}|(\varphi,e_n)|^2 + 3c^2\sum_{n=1}^{\infty}\lambda_n^{2\gamma}(\psi,e_n)^2\left(\frac{(\lambda_n t^{\alpha})^{1/\alpha}}{1+\lambda_n t^{\alpha}}\right)^2 \\
&\quad + 3\left|\int_0^t\left(\sum_{n=1}^{\infty}\lambda_n^{2\gamma}|(F(\cdot,s),e_n)|^2\right)^{1/2}\lambda_n^{\frac{1}{\alpha}}(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^{\alpha})ds\right|^2 \\
&\le 3c^2\|\varphi\|^2_{\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}})} + 3c^2\|\psi\|^2_{\mathcal{D}(\mathcal{L}^{\gamma})} \\
&\quad + 3c^2\left|\int_0^t\left(\sum_{n=1}^{\infty}\lambda_n^{2\gamma}|(F(\cdot,s),e_n)|^2\right)^{1/2}\frac{(\lambda_n(t-s)^{\alpha})^{\frac{1}{\alpha}}}{1+\lambda_n(t-s)^{\alpha}}(t-s)^{\alpha-2}ds\right|^2 \\
&\le 3c^2\|\varphi\|^2_{\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}})} + 3c^2\|\psi\|^2_{\mathcal{D}(\mathcal{L}^{\gamma})} + 3c^2\|F\|^2_{C([0,T];\mathcal{D}(\mathcal{L}^{\gamma}))}\left|\int_0^t (t-s)^{\alpha-2}ds\right|^2. \qquad (2.18)
\end{aligned}
$$

As a result, we get

$$
\|u(\cdot,t)\|_{\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}})} \le c(\alpha)\left(\|\varphi\|_{\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}})} + \|\psi\|_{\mathcal{D}(\mathcal{L}^{\gamma})} + t^{\alpha-1}\|F\|_{C([0,T];\mathcal{D}(\mathcal{L}^{\gamma}))}\right), \qquad (2.19)
$$

where $c(\alpha) = \frac{3c^2}{(\alpha-1)^2}$. Furthermore, by Lemma 2.3, we have

$$
\begin{aligned}
u_t(x,t) &= \sum_{n=1}^{\infty}\left\{-\lambda_n t^{\alpha-1}(\varphi,e_n)E_{\alpha,\alpha}(-\lambda_n t^{\alpha}) + (\psi,e_n)E_{\alpha,1}(-\lambda_n t^{\alpha})\right\}e_n(x) \\
&\quad + \sum_{n=1}^{\infty}\left\{\int_0^t (F(\cdot,s),e_n)(t-s)^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_n(t-s)^{\alpha})ds\right\}e_n(x). \qquad (2.20)
\end{aligned}
$$

Therefore, by applying (1.4), and Lemma 1.1 again, we have

$$
\begin{aligned}
&\|u_t(\cdot,t)\|^2_{\mathcal{D}(\mathcal{L}^{\gamma})} \\
&\le 3\sum_{n=1}^{\infty}\lambda_n^{2\gamma}\lambda_n^2|(\varphi,e_n)|^2|t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^{\alpha})|^2 + 3\sum_{n=1}^{\infty}\lambda_n^{2\gamma}|(\psi,e_n)|^2|E_{\alpha,1}(-\lambda_n t^{\alpha})|^2 \\
&\quad + 3\sum_{n=1}^{\infty}\lambda_n^{2\gamma}\left|\int_0^t (F(\cdot,s),e_n)(t-s)^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_n(t-s)^{\alpha})ds\right|^2 \\
&\le 3c^2\sum_{n=1}^{\infty}\lambda_n^{2\gamma+\frac{2}{\alpha}}(\varphi,e_n)^2\left(\frac{(\lambda_n t^{\alpha})^{\frac{\alpha-1}{\alpha}}}{1+\lambda_n t^{\alpha}}\right)^2 + 3c^2\sum_{n=1}^{\infty}\lambda_n^{2\gamma}(\psi,e_n)^2 \\
&\quad + 3c^2\left|\int_0^t\left(\sum_{n=1}^{\infty}\lambda_n^{2\gamma}|(F(\cdot,s),e_n)|^2\right)^{1/2}\frac{(t-s)^{\alpha-2}}{1+\lambda_1(t-s)^{\alpha}}ds\right|^2 \\
&\le 3c^2\|\varphi\|^2_{\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}})} + 3c^2\|\psi\|^2_{\mathcal{D}(\mathcal{L}^{\gamma})} + \frac{3c^2}{(\alpha-1)^2}t^{2(\alpha-1)}\|F\|^2_{C([0,T];\mathcal{D}(\mathcal{L}^{\gamma}))}. \qquad (2.21)
\end{aligned}
$$

Thus,

$$\|u_t(\cdot,t)\|_{\mathcal{D}(\mathcal{L}^{\gamma})} \le c_1\Big(\|\varphi\|_{\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}})} + \|\psi\|_{\mathcal{D}(\mathcal{L}^{\gamma})} + t^{\alpha-1}\|F\|_{C([0,T];\mathcal{D}(\mathcal{L}^{\gamma}))}\Big) \tag{2.22}$$

for all $t \in [0,T]$, where $c_1 > 0$ depends only on α. Then, we immediately obtain the desired estimate (2.17). □

It is easy to see that

$$\mathcal{L}u(x_i,t) = \mathcal{L}Z_1(t)\varphi(x_i) + \mathcal{L}Z_2(t)\psi(x_i) - \int_0^t Z_3(t-s)F(x_i,s)ds, \quad i=1,2. \tag{2.23}$$

The following lemma is valid.

Lemma 2.5. *Let $\varphi \in \mathcal{D}(\mathcal{L}^{\gamma_0+\frac{1}{\alpha}})$, $\psi \in \mathcal{D}(\mathcal{L}^{\gamma_0})$, $c_0 = \min_{x\in[0,1]} c(x) > 0$ and $F \in C([0,T];\mathcal{D}(\mathcal{L}^{\gamma}))$. Then there exists a positive constant $c > 0$, depending only on α, ρ_0, c_0, γ, γ_0, λ_1, such that*

$$\|\mathcal{L}u(x_i,\cdot)\|_{C[0,T]} \le c\Big(\|\varphi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0+\frac{1}{\alpha}})} + \|\psi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0})} + T\|F\|_{C([0,T];\mathcal{D}(\mathcal{L}^{\gamma}))}\Big), \quad i=1,2, \tag{2.24}$$

and

$$\|\mathcal{L}u_t(x_i,\cdot)\|_{C[0,T]} \le c\Big(\|\varphi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0+\frac{1}{\alpha}})} + \|\psi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0})} + T^{\alpha-1}\|F\|_{C([0,T];\mathcal{D}(\mathcal{L}^{\gamma}))}\Big), \quad i=1,2. \tag{2.25}$$

Proof. An inequality similar to the estimate in (2.24) was derived in [18]. However, the smoothness assumptions differ from those in [18], so we provide a detailed proof of inequalities (2.24) and (2.25).

Recall the following inequality for the fractional power $\mathcal{L}^{\beta}$ of $\mathcal{L}$ with $\beta \in \mathbb{R}$, $\beta > 0$:

$$\|u\|_{H^{2\beta}(0,1)} \le c_2\|\mathcal{L}^{\beta}u\|_{L^2(0,1)}$$

where constant $c_2 > 0$ depends only on β and λ_1 (see., [13], p. 208).

Let $\varepsilon_0 = \min\{\varepsilon_{01},\varepsilon_{02}\}$ with $2\varepsilon_{01} = \gamma_0 + \frac{1}{\alpha} - \frac{3}{2} > 0$ and $2\varepsilon_{02} = \gamma - \frac{1}{4} - \frac{1}{\alpha} > 0$. According to the Sobolev embedding theorem $H^{2\beta}(0,1) \subset C[0,1]$ for $\beta = \frac{1}{4} + \varepsilon_0$, we have

$$\|e_n\|_{C[0,1]} \le c_3\|e_n\|_{H^{2\beta}(0,1)} \le c_3c_2\|\mathcal{L}^{\beta}e_n\|_{L^2(0,1)} \le c_4\lambda_n^{\beta}, \tag{2.26}$$

where $c_2, c_3, c_4 > 0$ depend only of β, λ_1.

For convenience, we split $\mathcal{L}u(x_i,t)$ in three parts, namely $\mathcal{L}u(x_i,t) := \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3$, where

$$\mathrm{I}_1 := \mathcal{L}Z_1(t)\varphi(x_i), \quad \mathrm{I}_2 := \mathcal{L}Z_2(t)\psi(x_i), \quad \mathrm{I}_3 := -\int_0^t Z_3(t-s)F(x_i,s)ds, \ i=1,2.$$

Note that

$$\lambda_n \ge c_5 n^2,$$

where $c_5 > 0$ depends only on ρ_0 and c_0 (see [12], p. 190). For I_1, by Lemma 1.1, and , we have

$$\begin{aligned}
|\mathrm{I}_1| &\le \sum_{n=1}^{\infty}\lambda_n|(\varphi,e_n)||E_{\alpha,1}(-\lambda_n t^{\alpha})||e_n(x_i)| \le c\sum_{n=1}^{\infty}\lambda_n^{\gamma_0+\frac{1}{\alpha}}|(\varphi,e_n)|\lambda_n^{-(\gamma_0+\frac{1}{\alpha}-\beta-1)} \\
&\le c\Big(\sum_{n=1}^{\infty}\lambda_n^{2\gamma_0+\frac{2}{\alpha}}|(\varphi,e_n)|^2\Big)^{1/2}\Big(\sum_{n=1}^{\infty}\lambda_n^{-2(\gamma_0+\frac{1}{\alpha}-\beta-1)}\Big)^{1/2} \\
&\le cc_5\|\varphi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0+\frac{1}{\alpha}})}\Big(\sum_{n=1}^{\infty}n^{-4(\gamma_0+\frac{1}{\alpha}-\beta-1)}\Big)^{1/2}.
\end{aligned} \tag{2.27}$$

By the choice of β, we have $4(\gamma_0 + \frac{1}{\alpha} - \beta - 1) = 1 + 8\varepsilon_{01} - 4\varepsilon_0 > 1$, which implies

$$\sum_{n=1}^{\infty} n^{-4(\gamma_0+\frac{1}{\alpha}-\beta-1)} < c(\alpha, \gamma_0, \gamma).$$

So, we get

$$|\mathrm{I}_1| \leq c(\alpha, \gamma_0, \gamma, \rho_0, c_0)\|\varphi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0+\frac{1}{\alpha}})}. \tag{2.28}$$

Further, by Lemma 1.1 and (1.4), we have the following estimate for I_2:

$$\begin{aligned}|\mathrm{I}_2| &\leq \sum_{n=1}^{\infty} \lambda_n |(\psi, e_n)| t |E_{\alpha,2}(-\lambda_n t^\alpha)||e_n(x_i)| \leq cc_4 \sum_{n=1}^{\infty} |(\psi, e_n)| \frac{\lambda_n t}{1+\lambda_n t^\alpha} \lambda_n^\beta \\ &\leq cc_4 \sum_{n=1}^{\infty} \lambda_n^{\gamma_0} |(\psi, e_n)| \frac{(\lambda_n t^\alpha)^{\frac{1}{\alpha}}}{1+\lambda_n t^\alpha} \lambda_n^{-(\gamma_0+\frac{1}{\alpha}-\beta-1)} \\ &\leq cc_4 \left(\sum_{n=1}^{\infty} \lambda_n^{2\gamma_0} |(\psi, e_n)|^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} \lambda_n^{-2(\gamma_0+\frac{1}{\alpha}-\beta-1)} \right)^{1/2} \leq c_6 \|\psi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0})},\end{aligned} \tag{2.29}$$

where $c_6 > 0$ depends only on $\alpha, \gamma, \gamma_0, \lambda_1, \rho_0, c_0$. Next, we estimate I_3. The estimate for I_3 is the same as in [18] for $\gamma - \beta - \frac{1}{\alpha} = 2\varepsilon_{02} - \varepsilon_0 > 0$, and we have

$$\begin{aligned}|\mathrm{I}_3|^2 &= \sum_{n=1}^{\infty} \left| \lambda_n \int_0^t (F(\cdot, s), e_n)(t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) ds \cdot e_n(x_i) \right|^2 \\ &\leq c^2 c_4^2 \left| \int_0^t \left(\sum_{n=1}^{\infty} \lambda_n^{2\gamma} |(F(\cdot, s), e_n)|^2 \right)^{1/2} \frac{(\lambda_n (t-s)^\alpha)^{\frac{\alpha-1}{\alpha}}}{1+\lambda_n (t-s)^\alpha} ds \right|^2 \lambda_n^{-2(\gamma-\beta-\frac{1}{\alpha})} \\ &\leq c^2 c_4^2 \lambda_1^{-2(\gamma-\beta-\frac{1}{\alpha})} \|F\|^2_{C([0,T];\mathcal{D}(\mathcal{L}))} t^2.\end{aligned} \tag{2.30}$$

So,

$$|\mathrm{I}_3| \leq c(\alpha, \gamma, \gamma_0, \lambda_1, \rho_0, c_0) t \|F\|_{C([0,T];\mathcal{D}(\mathcal{L}^\gamma))}, \qquad \forall t \in [0, T]. \tag{2.31}$$

According to (2.28)-(2.31), we obtain (2.24).

By differentiating (2.20) with respect to the variable t and taking into account Proposition 1, we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{L}u(x_i, t) &= -\sum_{n=1}^{\infty} \lambda_n^2 (\varphi, e_n) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) e_n(x_i) \\ &+ \sum_{n=1}^{\infty} \lambda_n (\psi, e_n) E_{\alpha,1}(-\lambda_n t^\alpha) e_n(x_i) \\ &+ \sum_{n=1}^{\infty} \lambda_n \left(\int_0^t (F(\cdot, s), e_n)(t-s)^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n (t-s)^\alpha) ds \right) e_n(x_i) \\ &:= \tilde{\mathrm{I}}_1 + \tilde{\mathrm{I}}_2 + \tilde{\mathrm{I}}_3.\end{aligned} \tag{2.32}$$

Let $\varepsilon_0 = \min\{\varepsilon_{10}, \varepsilon_{11}\}$ where $2\varepsilon_{10} = \gamma_0 - \frac{3}{2} > 0$ and $2\varepsilon_{11} = \gamma - \frac{5}{4} > 0$.

By the asymptotic property of the eigenvalues $\lambda_n \geq c_5 n^2$, for $\tilde{\mathrm{I}}_1$, using Lemma 1.1 and (1.4), we have

$$
\begin{aligned}
|\tilde{\mathrm{I}}_1| &\leq \sum_{n=1}^{\infty} \lambda_n^2 |(\varphi, e_n)| t^{\alpha-1} |E_{\alpha,\alpha}(-\lambda_n t^\alpha)| |e_n(x_i)| \\
&\leq c c_4 \sum_{n=1}^{\infty} \lambda_n^{\gamma_0+\frac{1}{\alpha}} |(\varphi, e_n)| \frac{(\lambda_n t^\alpha)^{\frac{\alpha-1}{\alpha}}}{1+\lambda_n t^\alpha} \lambda_n^{-(\gamma_0-\beta-1)} \\
&\leq c c_4 \left(\sum_{n=1}^{\infty} \lambda_n^{2\gamma_0+\frac{2}{\alpha}} |(\varphi, e_n)|^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} \lambda_n^{-2(\gamma_0-\beta-1)} \right)^{1/2} \\
&\leq c c_4 c_5 \|\varphi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0+\frac{1}{\alpha}})} \left(\sum_{n=1}^{\infty} n^{-4(\gamma_0-\beta-1)} \right)^{1/2}.
\end{aligned}
$$

By choice of β, we have $4(\gamma_0 - \beta - 1) = 1 + 8\varepsilon_{10} - 4\varepsilon_0 > 1$, which implies

$$
\sum_{n=1}^{\infty} n^{-4(\gamma_0-\beta-1)} < c(\gamma, \gamma_0).
$$

Thus, we obtain

$$
|\tilde{\mathrm{I}}_1| \leq c(\alpha, \gamma_0, \rho_0, c_0) \|\varphi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0+\frac{1}{\alpha}})}. \tag{2.33}
$$

Similarly, we have the following estimate for $\tilde{\mathrm{I}}_2$:

$$
\begin{aligned}
|\tilde{\mathrm{I}}_2| &\leq \sum_{n=1}^{\infty} \lambda_n |(\psi, e_n)| |E_{\alpha,1}(-\lambda_n t^\alpha)| |e_n(x_i)| \\
&\leq c c_4 \sum_{n=1}^{\infty} \lambda_n^{\gamma_0} |(\psi, e_n)| \frac{\lambda_n^{-(\gamma_0-\beta-1)}}{1+\lambda_n t^\alpha} \leq c c_4 \left(\sum_{n=1}^{\infty} \lambda_n^{2\gamma_0} |(\psi, e_n)|^2 \right)^{1/2} \\
&\quad \times \left(\sum_{n=1}^{\infty} n^{-4(\gamma_0-\beta-1)} \right)^{1/2} \leq c(\alpha, \gamma, \gamma_0, \rho_0, c_0) \|\psi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0})}.
\end{aligned} \tag{2.34}
$$

Further, we estimate $\tilde{\mathrm{I}}_3$. By Lemma 1.1 and $\gamma - \beta - 1 = 2\varepsilon_{11} - \varepsilon_0 > 0$, we have

$$
\begin{aligned}
|\tilde{\mathrm{I}}_3|^2 &\leq \sum_{n=1}^{\infty} \left| \lambda_n \int_0^t (F(\cdot, s), e_n)(t-s)^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n (t-s)^\alpha) ds \cdot e_n(x_i) \right|^2 \\
&\leq c^2 c_4^2 \left| \int_0^t \left(\sum_{n=1}^{\infty} \lambda_n^{2\gamma} |(F(\cdot, s), e_n)|^2 \right)^{1/2} \frac{(t-s)^{\alpha-2}}{1+\lambda_1 (t-s)^\alpha} ds \right|^2 \cdot \lambda_n^{-2(\gamma-\beta-1)} \\
&\leq c^2 c_4^2 \max_{0 \leq s \leq t} \|F(\cdot, s)\|^2_{\mathcal{D}(\mathcal{L}^\gamma)} \left| \int_0^t s^{\alpha-2} ds \right|^2 \cdot \lambda_1^{-2(\gamma-\beta-1)}.
\end{aligned}
$$

So that

$$
|\tilde{\mathrm{I}}_3| \leq c(\alpha, \gamma, \gamma_0, \lambda_1, \rho_0, c_0) \|F\|_{C([0,T], \mathcal{D}(\mathcal{L}^\gamma))} t^{\alpha-1}, \quad \forall t \in [0, T]. \tag{2.35}
$$

Finally, by (2.33)-(2.35), we get (2.25), thereby completing the proof of this lemma. □

To study the main problem (1.1)-(1.3), (1.5), we consider the following auxiliary inverse initial and boundary value problem.

Lemma 2.6. *Let* (C1)-(C4) *be held. Then the problem of finding a solution to* (1.1)-(1.3), (1.5) *is equivalent to the problem of determining functions* $u(x,t) \in X_0^T$, $q(t) \in C^1[0,T]$ *and* $k(t) \in C[0,T]$ *satisfying*

$$\begin{cases} \partial_t^\alpha u(x,t) + \mathcal{L}u(x,t) = q(t)u_t(x,t) + (k * u)(t) + f(x,t), & (x,t) \in Q_0^T, \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x) & 0 < x < 1, \\ u(0,t) = u(1,t) = 0, & 0 < t < T, \end{cases} \tag{2.36}$$

and

$$q(t) = \frac{1}{p(t)}\Big(h_2(0)\mathcal{N}_1[u,l](t) - h_1(0)\mathcal{N}_2[u,l](t)\Big), \qquad 0 \le t \le T, \tag{2.37}$$

$$k(t) = D_t\left[\frac{1}{p(t)}\Big(h_1'(t)\mathcal{N}_2[u,l](t) - h_2'(t)\mathcal{N}_1[u,l](t)\Big)\right], \quad 0 \le t \le T, \tag{2.38}$$

where $D_t := (d/dt)$,

$$\mathcal{N}_i = \partial_t^\alpha h_i(t) + \mathcal{L}u(x_i,t) - (l * h_i')(t) - \tilde{f}_i(t),\ (i = 1,2) \tag{2.39}$$

and

$$l(t) = \int_0^t k(\tau)d\tau. \tag{2.40}$$

Remark 5. By Lemma 2.6, we know that problem (2.36)-(2.38) is an equivalent form of the original inverse problem (1.1)-(1.3), (1.5). Therefore, in the following sections, we will discuss problem (2.36)-(2.38), rather than the original one.

Proof. The solution $(u(x,t), q(t), k(t)) \in Y_0^T$ of our inverse problem (1.1)-(1.3), (1.5) is also a solution to problem (2.36) in Y_0^T, because problem (2.36) is the same as (1.1)-(1.3). Therefore, we should show only (2.37) and (2.38). Let the three $\{u(x,t), q(t), k(t)\}$ functions be a solution to problem (1.1)-(1.3), (1.5). Taking into account the conditions $\partial_t^\alpha h_i(t) \in C[0,T]$ which imply that $h_i \in C^1[0,T]$, and fractional differentiating both sides of (1.5) with respect to t gives

$$\partial_t^\alpha u(x_i,t) = \partial_t^\alpha h_i(t), \quad u_t(x_i,t) = h_i'(t), \quad 0 \le t \le T. \tag{2.41}$$

Set $x = x_i$ in equation (1.1), the procedure yields

$$\partial_t^\alpha u(x_i,t) + \mathcal{L}u(x_i,t) = q(t)u_t(x_i,t) + \int_0^t k(t-\tau)u(x_i,\tau)d\tau + f(x_i,t),\ \ i = 1,2. \tag{2.42}$$

We note that $l(t) = \int_0^t k(\tau)d\tau$. Then by integration by parts, we get the following equality:

$$\int_0^t k(\tau)h_i(t-\tau)d\tau = h_i(0)l(t) + \int_0^t l(t-\tau)h_i'(\tau)d\tau. \tag{2.43}$$

With the help of (2.41) and (2.43), we can rewrite (2.42) as

$$h_i'(t)q(t) + h_i(0)l(t) = \partial_t^\alpha h_i(t) + \mathcal{L}u(x_i,t) - (l * h_i')(t) - \tilde{f}_i(t), \quad i = 1,2.$$

Due to (C4), we can solve this system to get (2.37) and

$$l(t) = \frac{1}{p(t)}\Big(h_1'(t)\mathcal{N}_2[u,l](t) - h_2'(t)\mathcal{N}_1[u,l](t)\Big). \tag{2.44}$$

Furthermore, by differentiating (2.44) with respect to t, we get (2.38).

Now, assume that (u,q,k) satisfies (2.36)-(2.38). To prove that $\{u,q,k\}$ is a solution to the inverse problem (1.1)-(1.3), (1.5), it suffices to show that $\{u,q,k\}$ satisfies (1.5).

Setting $x=x_i$ in equation (2.36), we have

$$\partial_t^\alpha u(x_i,t)+\mathcal{L}u(x_i,t)=q(t)u_t(x_i,t)+(k*u)(t)+\tilde{f}_i(t). \tag{2.45}$$

On the other hand, from (C2), we easily see that

$$\frac{1}{p(0)}\Big(h_1'(0)\mathcal{N}_2[u,l](0)-h_2'(0)\mathcal{N}_1[u,l](0)\Big)=0.$$

We get (2.44) by integrating (2.38) over $[0,t]$. From (2.37) and (2.44), we conclude that

$$\begin{aligned}h_i'(t)q(t)&=-h_i(0)l(t)+\partial_t^\alpha h_i(t)+\mathcal{L}u(x_i,t)-(l*h_i')(t)-\tilde{f}_i(t)\\&=\partial_t^\alpha h_i(t)+\mathcal{L}u(x_i,t)-(k*h_i)(t)-\tilde{f}_i(t)\end{aligned}$$

or

$$\tilde{f}_i(t)=-h_i'(t)q(t)+\partial_t^\alpha h_i(t)+\mathcal{L}u(x_i,t)-(k*h_i)(t). \tag{2.46}$$

Then substituting (2.46) into (2.45), and using (C3), we have that $P_i(t):=u(x_i,t)-h_i(t)\,(i=1,2)$ satisfies

$$\begin{cases}\partial_t^\alpha P_i(t)=q(t)P_i'(t)+(k*P_i)(t), & t>0,\\ P_i(0)=P_i'(0)=0.\end{cases} \tag{2.47}$$

Then, the fractional initial value problem (2.47) is equivalent to the integral equation (see, [9], p. 199)

$$\begin{aligned}P_i(t)=\frac{1}{\Gamma(\alpha)}\int_0^t\left(\int_s^t(t-\tau)^{\alpha-1}k(\tau-s)d\tau\right)P_i(s)ds\\-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}q'(s)P_i(s)ds\\+\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}q(s)P_i(s)ds,\quad i=1,2.\end{aligned} \tag{2.48}$$

This is a weakly singular homogeneous integral equation, and it has only a trivial solution for $q(t)\in C^1[0,T]$ and $k(t)\in C[0,T]$ (see [8]). Therefore, $u(x_i,t)-h_i(t)=0$, for $0\le t\le T$, i.e., condition (1.5) is satisfied. □

At the end of this section, we present a lemma that will be used to estimate q and k.

Lemma 2.7. *Let* (C1) *hold. Then for all $(u,q,k)\in Y_0^T$ and $l\in C^1[0,T]$, there exists a constant $c>0$ depending only on $\alpha,\gamma,\gamma_0,\lambda_1,\rho_0,c_0$, in particular, independent of T, φ,ψ, such that*

$$\begin{aligned}\|\mathcal{N}_i[u,l]\|_{C^1[0,T]}\le\|\partial_t^\alpha h_i\|_{C^1[0,T]}+c\left(\|\varphi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0+\frac{1}{\alpha}})}+\|\psi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0})}\right)+\|\tilde{f}_i\|_{C^1[0,T]}\\+c(T+T^{\alpha-1})\|q\|_{C[0,T]}\|u_t\|_{C([0,T];\mathcal{D}(\mathcal{L}^\gamma))}+c(T^2+T^\alpha)\|k\|_{C[0,T]}\|u\|_{C([0,T];\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}}))}\\+c(T+T^{\alpha-1})\|f\|_{C([0,T];\mathcal{D}(\mathcal{L}^\gamma))}+T^{\frac{1}{2}}\|l\|_{C^1[0,T]}\Big],\end{aligned} \tag{2.49}$$

where $\mathcal{N}_i\,(i=1,2)$ are the same as those in (2.39) *and $l(t)$ as in* (2.40).

Proof. By Lemma 2.5 and condition (C1), we see that

$$\|\mathcal{N}_i[u,l]\|_{C[0,T]} \le \|\partial_t^\alpha h_i\|_{C[0,T]} + \|\mathcal{L}u(x_i,t)\|_{C[0,T]} + \|l * h_i'\|_{C[0,T]}$$
$$+\|f_i\|_{C[0,T]} \le \|\partial_t^\alpha h_i\|_{C[0,T]} + c\Big(\|\varphi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0+\frac{1}{\alpha}})} + \|\psi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0})}$$
$$+T^{\frac{\alpha}{2}}\|F\|_{C([0,T];\mathcal{D}(\mathcal{L}^{\gamma}))}\Big) + T^{\frac{1}{2}}\|l\|_{C[0,T]}\|h_i'\|_{L^2(0,T)} + \|\tilde{f}_i\|_{C[0,T]}.$$

By the definition of F, the last inequality gives

$$\|\mathcal{N}_i[u,l]\|_{C[0,T]} \le \|\partial_t^\alpha h_i\|_{C[0,T]} + c\Big[\|\varphi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0+\frac{1}{\alpha}})} + \|\psi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0})}$$
$$+ T\Big(\|q\|_{C[0,T]}\|u_t\|_{C([0,T];\mathcal{D}(\mathcal{L}^{\gamma}))} + \lambda_1^{-\frac{1}{\alpha}}T\|k\|_{C[0,T]}\|u\|_{C([0,T];\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}}))}$$
$$+ \|f\|_{C([0,T];\mathcal{D}(\mathcal{L}^{\gamma}))}\Big)\Big] + T^{\frac{1}{2}}\|l\|_{C[0,T]}\|h_i'\|_{L^2(0,T)} + \|\tilde{f}_i\|_{C[0,T]}, \quad (2.50)$$

where we have used that

$$\|v\|^2_{D(\mathcal{L}^{\gamma})} = \sum_{n=1}^{\infty} \lambda_n^{2\gamma+\frac{2}{\alpha}}(v,e_n)^2\lambda_n^{-\frac{2}{\alpha}} \le \lambda_1^{-\frac{2}{\alpha}}\|v\|^2_{\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}})}.$$

On the other hand, direct calculations yields

$$D_t\mathcal{N}_i[u,l](t) = (\partial_t^\alpha h_i)' + \mathcal{L}u_t(x_i,t) - (l' * h_i')(t) - \tilde{f}_i'(t). \quad (2.51)$$

Here we have taken into account that $l(0) = 0$. By Lemma 2.5, we have

$$\|D_t\mathcal{N}_i[u,l]\|_{C[0,T]} \le \|(\partial_t^\alpha h_i)'\|_{C[0,T]} + c\Big[\|\varphi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0+\frac{1}{\alpha}})} + \|\psi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0})}$$
$$+T^{\alpha-1}\Big(\|q\|_{C[0,T]}\|u_t\|_{C([0,T];\mathcal{D}(\mathcal{L}^{\gamma}))} + \lambda_1^{-\frac{1}{\alpha}}T\|k\|_{C[0,T]}\|u\|_{C([0,T];\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}}))}$$
$$+ \|f\|_{C([0,T];\mathcal{D}(\mathcal{L}^{\gamma}))}\Big)\Big] + T^{\frac{1}{2}}\|l'\|_{C[0,T]}\|h_i'\|_{L^2(0,T)} + \|\tilde{f}_i'\|_{C[0,T]}. \quad (2.52)$$

Using (2.50) and (2.52), we obtain the desired estimate given in (2.49). □

3 Well-posedness of the inverse problem

We can now prove the existence of a solution to our inverse problem, i.e. Theorem 1.1, which proceeds by a fixed point argument. First, we define the function set

$$B_{\rho,T} = \Big\{(\bar{u},\bar{q},\bar{k}) \in Y_0^T: \ \bar{u}(x,0) = \varphi(x), \bar{u}_t(x,0) = \psi(x), \ \bar{u}(0,t) = \bar{u}(1,t) = 0,$$
$$\|\bar{u}\|_{X_0^T} + \|\bar{q}\|_{C^1[0,T]} + \|\bar{k}\|_{C[0,T]} \le \rho\Big\}.$$

Here ρ is a large constant that depends on the initial and source data φ, ψ, f, as well on the measurement data h_i. For a given $(\bar{u},\bar{q},\bar{k}) \in B_{\rho,T}$, we consider

$$\begin{cases} \partial_t^\alpha u(x,t) + \mathcal{L}u(x,t) = F(x,t), & (x,t) \in Q_0^T, \\ u(x,0) = \varphi(x), \ u_t(x,0) = \psi(x), & 0 < x < 1, \\ u(0,t) = u(1,t) = 0, & 0 < t < T, \end{cases} \quad (3.1)$$

where

$$F(x,t) = \bar{q}(t)\bar{u}_t(x,t) + (\bar{k} * \bar{u})(t) + f(x,t),$$

and

$$q(t) = \frac{1}{p(t)}\Big(h_2(0)\mathcal{N}_1[u,\bar{l}](t) - h_1(0)\mathcal{N}_2[u,\bar{l}](t)\Big), \tag{3.2}$$

$$k(t) = \frac{d}{dt}\left(\frac{h_1'(t)\mathcal{N}_2[u,\bar{l}](t) - h_2'(t)\mathcal{N}_1[u,\bar{l}](t)}{p(t)}\right) \tag{3.3}$$

to generate (u,q,k), where $\bar{l}(t) = \int_0^t \bar{k}(\tau)d\tau$, $\mathcal{N}_i\,(i=1,2)$ are the same as those in (2.37).

By Hölder's inequality, we have

$$\|(\bar{k} * \bar{u})(t)\|^2_{\mathcal{D}(\mathcal{L}^\gamma)} \le \int_0^t |\bar{k}(t-\tau)|^2 d\tau \int_0^t \|u(\cdot,\tau)\|^2_{\mathcal{D}(\mathcal{L}^\gamma)} d\tau \le \lambda_1^{-\frac{2}{\alpha}} t^2 \|\bar{k}\|^2_{C[0,t]} \|\bar{u}\|^2_{\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}})} \tag{3.4}$$

which implies

$$\|(\bar{k} * \bar{u})(t)\|_{C([0,T];\mathcal{D}(\mathcal{L}^\gamma))} \le \lambda_1^{-\frac{1}{\alpha}} \rho^2 T.$$

Furthermore

$$\|\bar{q}\bar{u}_t\|^2_{C([0,T];\mathcal{D}(\mathcal{L}^\gamma))} = \max_{0\le t\le T}\left|\sum_{n=1}^\infty \lambda_n^{2\gamma}(\bar{q}(t)\bar{u}_t(\cdot,t),e_n)^2\right| \le \|\bar{q}\|^2_{C[0,T]}\|\bar{u}_t\|^2_{C([0,T];\mathcal{D}(\mathcal{L}^\gamma))} \le \rho^4. \tag{3.5}$$

Using these results, along with $f \in C^1([0,T];\mathcal{D}(\mathcal{L}^\gamma))$, we have

$$\bar{q}(t)\bar{u}_t(x,t) + (\bar{k} * \bar{u})(t) + f(x,t) \in C([0,T];\mathcal{D}(\mathcal{L}^\gamma)).$$

By Lemma 2.4, the unique solution $u \in X_0^T$ to problem (3.1), given by (2.16) satisfies

$$\|u\|_{X_0^T} \le c\left(\|\varphi\|_{\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}})} + \|\psi\|_{\mathcal{D}(\mathcal{L}^\gamma)} + T^{\alpha-1}\|F\|_{C([0,T];\mathcal{D}(\mathcal{L}^\gamma))}\right), \tag{3.6}$$

where $c>0$ depends only on α. Further, (3.2)-(3.3) define the functions $q(t)$ and $k(t)$ in terms of u. Furthermore, by Lemma 2.7, we have

$$\begin{aligned}\|q\|_{C^1[0,T]} + \|k\|_{C[0,T]} \le{}& c\|1/p\|_{C^1[0,T]}\Big(|h_1(0)| + |h_2(0)| + \|h_1'\|_{C^1[0,T]} + \|h_2'\|_{C^1[0,T]}\Big)\\ &\times\Big(1 + (T+T^{\alpha-1})(1 + \|\bar{q}\|_{C[0,T]}\|u_t\|_{C([0,T];\mathcal{D}(\mathcal{L}^\gamma))})\\ &\quad + (T^2+T^\alpha)\|\bar{k}\|_{C[0,T]}\|u\|_{C([0,T];\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}}))} + T^{\frac{1}{2}}\|\bar{l}\|_{C^1[0,T]}\Big).\end{aligned} \tag{3.7}$$

Note $\bar{l}(t) = \int_0^t \bar{k}(\tau)d\tau$. Then, we get

$$\|\bar{l}\|_{C^1[0,T]} = \left\|\int_0^t \bar{k}(\tau)d\tau\right\|_{C[0,T]} + \|\bar{k}\|_{C[0,T]} \le (1+T)\|\bar{k}\|_{C[0,T]}. \tag{3.8}$$

Substituting (3.8) into (3.7) yields

$$\begin{aligned}\|q\|_{C^1[0,T]} + \|k\|_{C[0,T]} \le c(T)\Big[&1 + \|\bar{q}\|_{C[0,T]}\|u_t\|_{C([0,T];\mathcal{D}(\mathcal{L}^\gamma))}\\ &+ \|\bar{k}\|_{C[0,T]}\|u\|_{C([0,T];\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}})} + \|\bar{k}\|_{C[0,T]}\Big].\end{aligned} \tag{3.9}$$

This implies that $q(t) \in C^1[0,T]$ and $k(t) \in C[0,T]$.

Thus the mapping

$$Z : B_{\rho,T} \to Y_0^T, \qquad (\bar{u},\bar{q},\bar{k}) \mapsto (u,q,k) \tag{3.10}$$

given by (3.1)-(3.3) is well defined.

The next lemma shows that Z is a contraction map on $B_{\rho,T}$ for sufficiently small $T>0$. More precisely, we have the following result.

Lemma 3.1. *Let* (C1)-(C4) *be hold. For* $(\bar{u},\bar{q},\bar{k})$, $(\bar{U},\bar{Q},\bar{K}) \in B_{\rho,T}$, *define*

$$(u,q,k) = Z(\bar{u},\bar{q},\bar{k}), \quad (U,Q,K) = Z(\bar{U},\bar{Q},\bar{K}).$$

Then for any sufficiently large ρ *and suitably small* $\tau(\rho) > 0$, *we have*

$$\|(u,q,k)\|_{Y_0^T} \le \rho$$

and

$$\|(u-U,q-Q,k-K)\|_{Y_0^T} \le \frac{1}{2}\|(\bar{u}-\bar{U},\bar{q}-\bar{Q},\bar{k}-\bar{K})\|_{Y_0^T} \tag{3.11}$$

for all $T \in (0,\tau(\rho)]$.

In the following proof, we use $c_j\ (j=7,\ldots)$ to denote a constant that depends on $\alpha,\gamma,\gamma_0,\lambda_1,\rho_0,c_0$ and the known functions φ, ψ, f and measurement data $h_i, i=1,2$, but is independent of ρ.

Proof. First, we prove that $Z(B_{\rho,T}) \subset B_{\rho,T}$ for sufficiently large ρ and suitably small T. Without loss of generality, we assume that $\rho \in [1,\infty)$ and $T \in (0,1]$.

By Lemma 2.4 and inequalities (3.4)-(3.6), we have

$$\begin{aligned}\|u\|_{X_0^T} \le c\lambda_1^{-(\gamma_0-\gamma)}(\|\varphi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0+\frac{1}{\alpha}})} + \|\psi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0})}) \\ + cT^{\alpha-1}\Big[\|\bar{q}(t)\bar{u}_t\|_{C([0,T],\mathcal{D}(\mathcal{L}^{\gamma}))} + \|(\bar{k}*\bar{u})\|_{C([0,T],\mathcal{D}(\mathcal{L}^{\gamma}))} \\ + \|f\|_{C([0,T],\mathcal{D}(\mathcal{L}^{\gamma}))})\Big] \le c_7\left[1+\rho^2T^{\alpha-1}\right].\end{aligned} \tag{3.12}$$

Here we have used the assumptions $\rho \in [1,\infty)$ and $T \in (0,1]$ (and we shall use them further on). On the other hand, by (3.2)-(3.3), together with Lemma 2.7 and (3.8), we have

$$\begin{aligned}\|q\|_{C^1[0,T]} + \|k\|_{C[0,T]} &\le c_8\Big(\|\mathcal{N}_1[u,\bar{l}]\|_{C^1[0,T]} + \|\mathcal{N}_2[u,\bar{l}]\|_{C^1[0,T]}\Big) \\ &\le c_9\Big[1+T+T^{\alpha-1}+\rho(T+T^{\alpha-1})\|u_t\|_{C([0,T];\mathcal{D}(\mathcal{L}^{\gamma}))} \\ &\quad + \rho(T^2+T^{\alpha})\|u\|_{C([0,T];\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}}))} + \rho T^{\frac{1}{2}}(1+T)\Big] \\ &\le c_{10}\Big[1+T+T^{\alpha-1}+\rho(T^2+T^{\alpha-1})\|u\|_{X_0^T} + \rho T^{\frac{1}{2}}(1+T)\Big] \\ &\le c_{11}[1+c_7\rho T^{\alpha-1}(1+\rho^2T^{\alpha-1}) + \rho T^{1/2}] \\ &\le c_{12}[1+\rho^3(T^{\alpha-1}+T^{1/2})].\end{aligned} \tag{3.13}$$

Adding inequalities (3.12) and (3.13) gives us

$$\begin{aligned}\|(u,q,k)\|_{Y_0^T} \le c_7\left[1+\rho^2T^{\alpha-1}\right] \\ + c_{12}\left[1+\rho^3(T^{\alpha-1}+T^{1/2})\right] \le c_{13}\Big[1+\rho^3\left(T^{\alpha-1}+T^{1/2}\right)\Big].\end{aligned} \tag{3.14}$$

For $\rho > c_{13}$, we choose a sufficiently small $\tau_1(\rho)$ such that, for $\rho > c_{13}$ and $0 < T < \tau_1(\rho)$

$$c_{13}\Big[1+\rho^3\left(T^{\alpha-1}+T^{1/2}\right)\Big] \le \rho. \tag{3.15}$$

Therefore, for all $T < \min\{1,\tau_1(\rho)\}$, we have

$$\|(u,q,k)\|_{Y_0^T} \le \rho. \tag{3.16}$$

That is, Z maps $B_{\rho,T}$ into itself for each fixed $\rho > c_{13}$ and $T \in (0, \min\{1, \tau_1(\rho)\}]$.

Next, we check the second condition of contractive mapping Z. Let $(u, q, k) = Z(\bar{u}, \bar{q}, \bar{k})$ and $(U, Q, K) = Z(\bar{U}, \bar{Q}, \bar{K})$. Then we obtain that $(u - U, q - Q, k - K)$ satisfies the equalities

$$u(x,t) - U(x,t) = \int_0^t A^{-1} Y(t-s) \bar{F}(x,s) ds, \quad (x,t) \in Q_0^T, \tag{3.17}$$

and

$$q(t) - Q(t) = \frac{1}{p(t)} \Big(h_2(0)(\mathcal{N}_1[u, \bar{l}](t) - \mathcal{N}_1[U, \bar{L}](t)) - h_1(0)(\mathcal{N}_2[u, \bar{l}](t) - \mathcal{N}_2[U, \bar{L}](t)) \Big), \tag{3.18}$$

$$k(t) - K(t) = \frac{d}{dt} \Big(\frac{h_1'(t)(\mathcal{N}_2[u, \bar{l}](t) - \mathcal{N}_2[U, \bar{L}](t))}{p(t)} - \frac{h_2'(t)(\mathcal{N}_1[u, \bar{l}](t) - \mathcal{N}_1[U, \bar{L}](t))}{p(t)} \Big) \tag{3.19}$$

where $\bar{L}(t) = \int_0^t \bar{K}(\tau) d\tau$ and

$$\bar{F} := q(u_t - U_t) + (q - Q) U_t + k * (u - U) + (k - K) * U.$$

Using Lemma 2.4, (3.5) and (3.6), we get

$$\begin{aligned}
\|u - U\|_{X_0^T} &\le c_{14} T^{\alpha-1} \Big[\|(\bar{q} - \bar{Q}) \bar{u}_t\|_{C([0,T],\mathcal{D}(\mathcal{L}^\gamma))} + \|(\bar{u}_t - \bar{U}_t) \bar{q}\|_{C([0,T],\mathcal{D}(\mathcal{L}^\gamma))} \\
&\quad + \|(\bar{k} - \bar{K}) * \bar{u})\|_{C([0,T],\mathcal{D}(\mathcal{L}^\gamma))} + \|\bar{k} * (\bar{u} - \bar{U})\|_{C([0,T],\mathcal{D}(\mathcal{L}^\gamma))} \Big] \\
&\le c_{14} T^{\alpha-1} \Big[\|\bar{q} - \bar{Q}\|_{C[0,T]} \|\bar{u}_t\|_{C([0,T];\mathcal{D}(\mathcal{L}^\gamma))} \\
&\quad + \|\bar{u}_t - \bar{U}_t\|_{C([0,T],\mathcal{D}(\mathcal{L}^\gamma))} \|\bar{q}\|_{C[0,T]} + T^2 \lambda_1^{-\frac{1}{\alpha}} \|\bar{k} - \bar{K}\|_{C[0,T]} \|\bar{u}\|_{C([0,T],\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}}))} \\
&\quad + T^2 \lambda_1^{-\frac{1}{\alpha}} \|\bar{u} - \bar{U}\|_{C([0,T],\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}}))} \|\bar{k}\|_{C[0,T]} \Big] \\
&\le c_{14} \rho T^{\alpha-1} \max\{1, T^2 \lambda_1^{-\frac{1}{\alpha}}\} \Big[\|\bar{q} - \bar{Q}\|_{C[0,T]} + \|\bar{u} - \bar{U}\|_{X_0^T} + \|\bar{k} - \bar{K}\|_{C[0,T]} \Big].
\end{aligned} \tag{3.20}$$

Similarly, by (3.18)-(3.19) and Lemma 2.7, we have

$$\begin{aligned}
\|q - Q\|_{C^1[0,T]} + \|k - K\|_{C[0,T]} &\le c_{15} \rho (T^2 + T^{\alpha-1}) \max\{1, T^2 \lambda_1^{-\frac{1}{\alpha}}, T^{\frac{3}{2}}\} \\
&\quad \times \Big[\|\bar{q} - \bar{Q}\|_{C[0,T]} + \|\bar{u} - \bar{U}\|_{X_0^T} + \|\bar{k} - \bar{K}\|_{C[0,T]} \Big].
\end{aligned} \tag{3.21}$$

Therefore, by (3.20) and (3.21), we have

$$\begin{aligned}
\|(u - U, q - Q, k - K)\|_{Y_0^T} &\le c_{16} \rho \Big[T^{\alpha-1} \max\{1, T^2 \lambda_1^{-\frac{1}{\alpha}}\} \\
&\quad + (T^2 + T^{\alpha-1}) \max\{1, T^2 \lambda_1^{-\frac{1}{\alpha}}, T^{\frac{3}{2}}\} \Big] \|(\bar{u} - \bar{U}, \bar{q} - \bar{Q}, \bar{k} - \bar{K})\|_{Y_0^T}.
\end{aligned} \tag{3.22}$$

Hence, we can choose a sufficiently small τ_2 such that

$$c_{16} \rho \Big[T^{\alpha-1} \max\{1, T^2 \lambda_1^{-\frac{1}{\alpha}}\} + (T^2 + T^{\alpha-1}) \max\{1, T^2 \lambda_1^{-\frac{1}{\alpha}}, T^{\frac{3}{2}}\} \Big] \le 1/2 \tag{3.23}$$

for all $T \in (0, \tau_2]$ to obtain

$$\|(u - U, q - Q, k - K)\|_{Y_0^T} \le \frac{1}{2} \|(\bar{u} - \bar{U}, \bar{q} - \bar{Q}, \bar{k} - \bar{K})\|_{Y_0^T}. \tag{3.24}$$

Estimates (3.16) and (3.24) show that Z is a contraction map on $B_{\rho,T}$ for all $T \in (0, \tau]$, if we choose $\tau \le \min\{1, \tau_1, \tau_2\}$. $\square$

Let us now prove Theorem 1.1.

Proof. Lemma 3.1 shows that there exists a sufficiently large $\rho > 0$ and a corresponding sufficiently small $\tau(\rho) > 0$, such that, for any $0 < T < \tau(\rho)$, the mapping Z is a contraction on $B_{\rho,T}$. Hence, the Banach fixed point theorem guarantees the existence of a unique solution $(u, q, k) \in B_{\rho,T} \subset Y_0^\tau$ to the system (2.36)-(2.38), for sufficiently small τ. As a consequence, the problem constituted by (1.1)-(1.3) and (1.5) also admits a unique solution $(u, q, k) \in B_{\rho,T} \subset Y_0^\tau$ by Lemma 2.6. □

Now, we present a global uniqueness result in time.

Lemma 3.2. *Under conditions* (C1)-(C4)*, for given measurement data $h_i(t)$ for $i = 1, 2$ in* (1.5)*, if the inverse problem* (1.1)-(1.3)*,* (1.5) *has two solutions $(u_j, q_j, k_j) \in Y_0^T$ $(j = 1, 2)$ for any time, then $(u_1, q_1, k_1) = (u_2, q_2, k_2)$ in $[0, T]$.*

According to Remark 5, we know that (2.36)-(2.38) is equivalent to (1.1)-(1.3), (1.5). In Lemma 3.2, we discuss the global uniqueness of inverse problem (2.36)-(2.38).

Proof. Given any time T, let (u_i, q_i, k_i) $(i = 1, 2)$ be two solutions to inverse problem (2.36)-(2.38) in $[0, T]$ such that $(u_i, q_i, k_i) \in Y_0^T$. This implies

$$\|(u_i, q_i, k_i)\|_{Y_0^T} \le C^*, \quad i = 1, 2, \tag{3.25}$$

where $C^* > 0$ depends only on α, T, initial data φ and ψ, the known function f and measurement data h_i.

Let

$$\tilde{u} = u_1 - u_2, \quad \tilde{q} = q_1 - q_2, \quad \tilde{k} = k_1 - k_2.$$

Then $(\tilde{u}, \tilde{q}, \tilde{k})$ satisfies

$$\begin{cases} \partial_t^\alpha \tilde{u} + \mathcal{L}\tilde{u} = q_1 \tilde{u}_t + \tilde{q} u_{2t} + k_1 * \tilde{u} + \tilde{k} * u_2, & (x, t) \in Q_0^T, \\ \tilde{u}(x, 0) = \tilde{u}_t(x, 0) = 0, & 0 < x < 1, \\ \tilde{u}(0, t) = \tilde{u}(1, t) = 0, & 0 < t < T, \end{cases} \tag{3.26}$$

and

$$\tilde{q}(t) = \frac{1}{p(t)} \Big(h_2(0)\mathcal{L}\tilde{u}(x_1, t) - h_1(0)\mathcal{L}\tilde{u}(x_2, t) - \tilde{l} * p \Big), \tag{3.27}$$

$$\tilde{k}(t) = \frac{d}{dt}\left[\frac{h_1'(t)\left(\mathcal{L}\tilde{u}(x_2, t) - \tilde{l} * h_2'\right) - h_2'(t)\left(\mathcal{L}\tilde{u}(x_1, t) - \tilde{l} * h_1'\right)}{p(t)} \right], \tag{3.28}$$

where $\tilde{l}(t) = l_1 - l_2$ and $l_i(t) = \int_0^t k_i(s)ds$. We have to show

$$\|(\tilde{u}, \tilde{q}, \tilde{k})\|_{Y_0^T} = 0. \tag{3.29}$$

Define

$$\sigma = \inf \Big\{ t \in (0, T] : \|(\tilde{u}, \tilde{q}, \tilde{k})\|_{Y_0^t} > 0 \Big\}. \tag{3.30}$$

If (3.29) does not hold, then it is clear that σ is well-defined and satisfies $0 \le \sigma \le T$. Moreover, by Theorem 1.1, we have $\sigma > 0$, and $\sigma < T$ follows from the fact that $\|(\tilde{u}, \tilde{q}, \tilde{k})\|_{Y_0^T} > 0$ and the continuity of the norm with respect to time t.

Let $0 < \epsilon < T - \sigma$. Further, by (2.16), we can write the solution $\tilde{u}$ as

$$\tilde{u}(x, t) = -\int_0^t \mathcal{L}^{-1} Z_3(t - s)\tilde{F}(x, s)ds, \quad (x, t) \in Q_\sigma^{\sigma+\epsilon}, \tag{3.31}$$

where

$$\tilde{F}(x,t) = q_1\tilde{u}_t + \tilde{q}u_{2t} + k_1 * \tilde{u} + \tilde{k} * u_2.$$

Then, similarly to the proofs of Lemma 2.4 and 2.5, we have

$$\|\tilde{u}\|_{X_\sigma^{\sigma+\epsilon}} \le c\epsilon^{\alpha-1}\|\tilde{F}\|_{C([\sigma,\sigma+\epsilon];\mathcal{D}(\mathcal{L}^\gamma))}, \tag{3.32}$$

and

$$\begin{cases} \|\mathcal{L}\tilde{u}(x_i,\cdot)\|_{C[\sigma,\sigma+\epsilon]} \le c_{17}\epsilon\|\tilde{F}\|_{C([\sigma,\sigma+\epsilon];\mathcal{D}(\mathcal{L}^\gamma))}, \\ \|\mathcal{L}\tilde{u}_t(x_i,\cdot)\|_{C[\sigma,\sigma+\epsilon]} \le c_{18}\epsilon^{\alpha-1}\|\tilde{F}\|_{C([\sigma,\sigma+\epsilon];\mathcal{D}(\mathcal{L}^\gamma))}. \end{cases} \tag{3.33}$$

From the definition of σ, we see that

$$\tilde{u} = \tilde{q} = \tilde{k} = 0 \quad \text{in} \quad [0,\sigma]. \tag{3.34}$$

By the definition of $\tilde{F}$, and using (3.4), (3.5) and (3.25), we have

$$\begin{aligned} \|\tilde{u}\|_{X_\sigma^{\sigma+\epsilon}} \le c_{19}\epsilon^{\alpha-1}\Big(&\|q_1\tilde{u}_t\|_{C([\sigma,\sigma+\epsilon];\mathcal{D}(\mathcal{L}^\gamma))} + \|\tilde{q}u_{2t}\|_{C([\sigma,\sigma+\epsilon];\mathcal{D}(\mathcal{L}^\gamma))} \\ &+ \|k_1 * \tilde{u}\|_{C([\sigma,\sigma+\epsilon];\mathcal{D}(\mathcal{L}^\gamma))} + \|\tilde{k} * u_2\|_{C([\sigma,\sigma+\epsilon];\mathcal{D}(\mathcal{L}^\gamma))}\Big) \\ \le c_{20}C^*\epsilon^{\alpha-1}\Big(&\|\tilde{u}_t\|_{C([\sigma,\sigma+\epsilon];\mathcal{D}(\mathcal{L}^\gamma))} + \|\tilde{q}\|_{C[\sigma,\sigma+\epsilon]} \\ &+ \lambda_1^{-\frac{1}{\alpha}}\epsilon\|\tilde{u}\|_{C([\sigma,\sigma+\epsilon];\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}}))} + \lambda_1^{-\frac{1}{\alpha}}\epsilon\|\tilde{k}\|_{C[\sigma,\sigma+\epsilon]}\Big). \end{aligned} \tag{3.35}$$

Due to $\tilde{q}(\sigma) = 0$, then implies

$$\|\tilde{q}\|_{C[\sigma,\sigma+\epsilon]} = \max_{\sigma\le t\le\sigma+\epsilon}\Big|\int_\sigma^t \tilde{q}'(s)ds\Big| \le \epsilon\|\tilde{q}\|_{C^1[\sigma,\sigma+\epsilon]}. \tag{3.36}$$

Substituting (3.36) into (3.35), we have

$$\|\tilde{u}\|_{X_\sigma^{\sigma+\epsilon}} \le c_{20}C^*\epsilon^{\alpha-1}\max\{1,\epsilon,\lambda_1^{-\frac{1}{\alpha}}\epsilon\}\|(\tilde{u},\tilde{q},\tilde{k})\|_{Y_\sigma^{\sigma+\epsilon}}. \tag{3.37}$$

Note $\|\tilde{q}\|_{C^1[0,\sigma]} = \|\tilde{k}\|_{C[0,\sigma]} = 0$. On the other hand, by (3.27), and using (3.33), we have the following estimate for $\tilde{q}$

$$\begin{aligned} &\|\tilde{q}\|_{C^1[\sigma,\sigma+\epsilon]} \\ &\quad\le c_{21}(\epsilon+\epsilon^{\alpha-1})\Big(\|h_2(0)/p(t)\|_{C^1[\sigma,\sigma+\epsilon]} + \|h_2(0)/p(t)\|_{C^1[\sigma,\sigma+\epsilon]}\Big)\|\tilde{F}\|_{C([\sigma,\sigma+\epsilon];\mathcal{D}(\mathcal{L}^\gamma))} \\ &\quad+ \epsilon^{1/2}\|p\|_{C[\sigma,\sigma+\epsilon]}\|\tilde{l}\|_{C[\sigma,\sigma+\epsilon]} \\ &\quad\le c_{21}C(\|h_1\|_{C^1[0,T]},\|h_2\|_{C^1[0,T]})(\epsilon+\epsilon^{\alpha-1})\Big(\|\tilde{u}_t\|_{C([\sigma,\sigma+\epsilon];\mathcal{D}(\mathcal{L}^\gamma))} + \epsilon\|\tilde{q}\|_{C^1[\sigma,\sigma+\epsilon]} \\ &\qquad+ \lambda_1^{-\frac{1}{\alpha}}\epsilon\|\tilde{u}\|_{C([\sigma,\sigma+\epsilon];\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}}))}\Big) + C(\|h_1\|_{C[0,T]},\|h_2\|_{C[0,T]})\epsilon^{3/2}\|\tilde{k}\|_{C[\sigma,\sigma+\epsilon]}, \end{aligned} \tag{3.38}$$

where we have used that

$$\|\tilde{l}\|_{C[\sigma,\sigma+\epsilon]} = \max_{\sigma\le t\le\sigma+\epsilon}\Big|\int_\sigma^t \tilde{k}(s)ds\Big| \le \epsilon\|\tilde{k}\|_{C[\sigma,\sigma+\epsilon]}.$$

Similarly to (3.38), by (3.28) we can easily estimate $\tilde{k}$

$$\begin{aligned} \|\tilde{k}\|_{C[\sigma,\sigma+\epsilon]} \le C(\|h_1\|_{C^2[0,T]},\|h_2\|_{C^2[0,T]})\Big[&c_{21}(\epsilon+\epsilon^{\alpha-1})\big(\|\tilde{u}_t\|_{C([\sigma,\sigma+\epsilon];\mathcal{D}(\mathcal{L}^\gamma))} \\ &+ \epsilon\|\tilde{q}\|_{C^1[\sigma,\sigma+\epsilon]} + \lambda_1^{-\frac{1}{\alpha}}\epsilon\|\tilde{u}\|_{C([\sigma,\sigma+\epsilon];\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}}))}\big) + \epsilon^{3/2}\|\tilde{k}\|_{C[\sigma,\sigma+\epsilon]}\Big]. \end{aligned} \tag{3.39}$$

From (3.37)-(3.39), we obtain

$$\|(\tilde{u},\tilde{q},\tilde{k})\|_{Y_\sigma^{\sigma+\epsilon}} \le C(\|h_i\|_{C^2[0,T]}, C^*)\eta(\epsilon)\|(\tilde{u},\tilde{q},\tilde{k})\|_{Y_\sigma^{\sigma+\epsilon}} \tag{3.40}$$

with

$$\lim_{\epsilon\to+0}\eta(\epsilon) = \lim_{\epsilon\to+0}\left(\epsilon + 2\epsilon^{\alpha-1} + \epsilon^{3/2}\right)\max\{1,\epsilon,\lambda_1^{-\frac{1}{\alpha}}\epsilon\} = 0,$$

and implying

$$\|(\tilde{u},\tilde{q},\tilde{k})\|_{Y_\sigma^{\sigma+\epsilon}} = 0$$

for some sufficiently small positive constant ϵ. This means that $(u_1-u_2, q_1-q_2, k_1-k_2)$ vanishes in $[0,\sigma+\epsilon]$, which contradicts with the definition of σ. Therefore (3.29) is proved. From here, we can conclude that

$$(u_1,q_1,k_1) = (u_2,q_2,k_2) \quad \text{in} \quad [0,T]$$

for any time T. □

4 Examples

In this section, as an illustration, we provide two examples of inverse problem (1.1)-(1.3), (1.5).
Example 1. In this example, we consider inverse problem (1.1)-(1.3), (1.5) with the following input data:

$$\begin{cases} \varphi(x) = \sin 2\pi x, \quad \psi(x) = \left((8-16\sqrt{2})x^2 + (8\sqrt{2}-2)x\right)\sin\pi x, \quad x_1 := \frac{1}{4},\ x_2 := \frac{1}{2}, \\ f(x,t) = -\varphi''(x) - \left[\left(\psi''(x) + (64-74\sqrt{2}+7\pi^2)\psi(x)\right)t + (48-42\sqrt{2}+8\pi^2)t^2\psi(x)\right] \\ \qquad\qquad - \left[(48-42\sqrt{2}+8\pi^2)t + 4\pi^2t^2\right]\varphi(x), \\ h_1(t) := 1+t, \quad h_2(t) := t. \end{cases}$$

It is easy to see that all functions given in Example 1 satisfy conditions (C1)-(C4).

Then, the exact solution of the inverse problem is

$$u(x,t) = \varphi(x) + \psi(x)t, \quad k(t) = 48 - 42\sqrt{2} + 8\pi^2 + 8\pi^2 t,$$
$$q(t) = (64-74\sqrt{2}+7\pi^2)t + (24-21\sqrt{2}+4\pi^2)t^2 - \frac{4\pi^2}{3}t^3.$$

Example 2. In this example, we consider inverse problem (1.1)-(1.3), (1.5) with the following input data:

$$\begin{cases} \varphi(x) = \sin 2\pi x, \quad \psi(x) = \left((8-16\sqrt{2})x^2 + (8\sqrt{2}-2)x\right)\sin\pi x, \quad x_1 := \frac{1}{4},\ x_2 := \frac{1}{2}, \\ f(x,t) = -\varphi''(x) + \varphi(x)\varphi''(x_1) \\ \qquad\qquad -\Big[\psi''(x) + (\varphi(x)-\psi(x))\psi''(x_2) + (\varphi''(x_1)-\psi''(x_1))\varphi(x)\Big]\cos t, \\ h_1(t) := 1+t, \quad h_2(t) := t. \end{cases}$$

Then the exact solution of the inverse problem is

$$u(x,t) = \varphi(x) + \psi(x)t, \quad k(t) = 48 - 42\sqrt{2} + 4\pi^2\sin t,$$
$$q(t) = (-64+74\sqrt{2}-3\pi^2)t + (24-21\sqrt{2})t^2 + 4\pi^2\sin t.$$

Since the inverse problem considered in equations (1.1)-(1.3), (1.5) is nonlinear, hence, an analytical solution cannot be found, however, to obtain the exact solutions provided in Examples 1 and 2, we employed a reverse approach: first, we define the functions φ,ψ,h_1,h_2 that satisfy conditions (C1)-(C4) and the function $u(t,x)$ that satisfies conditions (1.2), (1.3) and (1.5), then we determine the remaining functions q,k,f from the system of equations (2.36)-(2.38).

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