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ON A METHOD OF FINDING APPROXIMATE SOLUTIONS OF Ill-CONDITIONED ALGEBRAIC SYSTEMS AND PARALLEL COMPUTATION

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Abstract. A new method of finding approximate solutions of linear algebraic systems with ill-conditioned or singular matrices is presented. This method can be effectively used for arranging parallel computations for matrices of large size.

Let A be a quadratic matrix of order $n \geq 1$ and f be an n-dimensional vector. If n is large, then in order to speed up the process of solving the equation $Ax = f$ several computers are usually used. In this case the problem of parallel computation arises.

In the well-known books [1]-[4] the problem of parallel computation is solved for band and for sparse matrices A. We suggest a method which allows to do this for an arbitrary matrix A. In particular, we cover the cases of matrices with small determinants or even non-invertible matrices. Instead of solving the equation $Ax = f$ we consider the problem of finding a vector $\mathring{x}$ such that

$$\inf_x \parallel Ax - f \parallel = \parallel A\mathring{x} - f \parallel \ . \tag{1}$$

Here the infimum is taken with respect to all n-dimensional vectors x. If the matrix A is invertible, then clearly $\mathring{x}$ is the unique solution of the equation $Ax = f$. If problem (1) has more that one solution, then we choose as $\mathring{x}$ the one of them which has the minimal norm.

Theorem 1. *Let E be the unit matrix, A^* be the adjoint of A, $KerA = \{x : Ax = 0\}$. Then*
a) the solution $\mathring{x}$ of problem (1) *exists, is unique and $\mathring{x} \in \mathbb{R}^n \ominus KerA$,*
b) if

$$0 < a < 2 \parallel A^*A \parallel^{-1},$$
$$\rho = \max\{|1 - a\tilde{\lambda}^2|, |1 - a\|A^*A\||\} < 1$$

where $\tilde{\lambda}^2$ is the minimal nonzero eigenvalue of A^*A, and

$$x_j = a \sum_{k=0}^{j-1} (E - aA^*A)^k A^* f, \tag{2}$$

then

$$x_j - \mathring{x} = -(E - aA^*A)^j \mathring{x}$$

and

$$\|x_j - \mathring{x}\| \leq \|\mathring{x}\| \rho^j . \tag{3}$$

Remark 1. Note that the existence of zero eigenvalues of A^*A does not affect the rate of convergence of x_j to $\mathring{x}$, but the existence of small nonzero eigenvalues of A^*A essentially diminishes the rate of convergence. For this reason in the case of an ill-conditioned matrix A we replace A by AP, where P is an orthogonal projector onto a subspace of dimension smaller than n. We manage to choose P in such a way that the eigenvalues of the matrix PA^*AP are not less than an a priori fixed number $\delta > 0$ and $\|\mathring{x} - \mathring{x}_P\| \leq \delta c_0$, where $\mathring{x}_P$ is the solution of problem (1) with A replaced by AP and $c_0 > 0$ is independent of δ.

A lot of attention will be paid to the statements of this remark in the detailed exposition of this work.

Remark 2. If A is ill-conditioned and its order n is not large, then the the statements of Theorem 1 and Remark 1 do not lead to essential improvements of the known methods of approximate solving the equation $Ax = f$. Our method is aimed at matrices of large size.

Because of the lack of space we cannot describe our method of arranging parallel computations in detail, but we can briefly describe the main idea of the method. It is based on the well-known fact that it is easy to arrange parallel computations for the procedure of multiplying a matrix by a vector. Therefore it is possible to arrange parallel computations of the vectors x_j $(j = 1, 2, ...)$ of Theorem 1 by using the known methods.

We would like to emphasize the following important fact. In our method of arranging parallel computations the number of multiplications and divisions does not depend on the number of computers (processors)!

Since the ideas applied in this communication are rather simple, they can also be effectively used for nonlinear finite-dimensional equations. We hope to discuss this in detail in one of further papers on this subject.

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