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### Events



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# FACTORIZATION METHOD FOR SOLVING SYSTEMS OF SECOND-ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS

**I.N. Parasidis, E. Providas**

Communicated by R. Oinarov

**Key words:** systems of ordinary differential equations, nonlocal boundary value problems, multi-point boundary problems, integral boundary conditions, exact solution, correct problems, factorization.

**AMS Mathematics Subject Classification:** 34A30, 34B10, 47A68.

**Abstract.** We consider in a Banach space the following two abstract systems of first-order and second-order linear ordinary differential equations with general boundary conditions, respectively,

$$X'(t) - A_0(t)X(t) = F(t), \quad \Phi(X) = \sum_{j=1}^{n} M_j \Psi_j(X),$$

and

$$X''(t) - S(t)X'(t) - Q(t)X(t) = F(t),$$

$$\Phi(X) = \sum_{i=j}^{n} M_j \Psi_j(X), \quad \Phi(X') = C\Phi(X) + \sum_{j=1}^{r} N_j \Theta_j(X),$$

where $X(t) = \mathrm{col}\,(x_1(t), \dots, x_m(t))$ denotes a vector of unknown functions, $F(t)$ is a given vector and $A_0(t), S(t), Q(t)$ are given matrices, $\Phi, \Psi_1, \dots, \Psi_n$, $\Theta_1, \dots, \Theta_r$ are vectors of linear bounded functionals, and $M_1, \dots, M_n, C$, $N_1, \dots, N_r$ are constant matrices. We first provide solvability conditions and a solution formula for the first-order system. Then we construct in closed form the solution of a special system of $2m$ first-order linear ordinary differential equations with constant coefficients when the solution of the associated system of $m$ first-order linear ordinary differential equations is known. Finally, we construct in closed form the solution of the second-order system in the case in which it can be factorized into first-order systems.

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## 1 Introduction

Boundary value problems (BVPs) for ordinary differential equations (ODEs) appear in a wide range of sciences. Many of these are nonlocal problems with integral and multipoint boundary conditions, such as in the modeling of power networks, telecommunication lines, electric railway systems, kinetic reaction problems in chemistry, elasticity, and elsewhere [19, 18, 15].

Perhaps the first problem with nonlocal integral boundary conditions for a system of linear first order ODEs was Hilb's problem

$$LY = PY' + QY = f, \quad \int_0^1 K(\xi)Y(\xi)d\xi + \gamma Y(0) - \Gamma Y(1) = 0,$$

which was investigated in 1911 [12]. The multipoint boundary value problems for a system of Transferable Differential-Algebraic Equations was investigated in [15]. In [4] an approach is given to solving the overdetermined problem for a system of the first and second order ODEs. The unique exact solution to the BVP

$$Y'(t) - M(t)Y(t) = F(t), \quad \Phi(Y) = \vec{c},$$

was obtained in [10]. The solvability condition and exact solution to the BVP

$$Y'(t) - AY(t) = F(t), \quad \sum_{i=1}^{m} A_i Y(t_i) + \sum_{j=0}^{s} B_j \int_{z_j}^{z_{j+1}} C_j(t)Y(t)dt = \vec{0},$$

where $A, A_i, B_j$ are constant matrices, are given in [5]. Necessary and sufficient conditions are established in [7] for the existence of a unique holomorphic solution of the BVP

$$X'(t) = T(t)X(t) + F(t), \quad \sum_{i=1}^{m} A_i X(t_i) + \sum_{j=0}^{m} \int_{t_{i-1}}^{t_i} \Phi_i(t)X(t)dt = h$$

with holomorphic coefficients and general linear boundary conditions. The existence of positive solutions of nonlocal BVPs for ordinary second order differential systems is given in [8]. The existence of solutions of nonlocal BVPs for ordinary differential systems of higher order was investigated in [9]. A numerical method for solving systems of linear nonautonomous ODEs with nonseparated multipoint and integral conditions was considered in [1]. Numerical solutions of systems of loaded ordinary differential equations are given in [2]. Ordinary differential equations and systems of various types were studied by the parametrization method in [13], [3] (see also [16]). The factorization (decomposition) method is a powerful tool for finding solutions to systems of ODEs. The factorization method proposed here for systems of ODEs is essentially different from other factorization methods in the relevant literature, where usually approximate solutions to ordinary differential systems are found by using the Adomian decomposition method and its many modifications [17], [6]. Note that finding of the fundamental and particular solutions for the following system of linear second order ODEs

$$B_2X(t) = X''(t) - S(t)X'(t) - Q(t)X(t) = F,$$

with nonlocal boundary conditions, is usually a difficult problem. Our goal is to find special cases that allow factorization like $B_2X(t) = B^2X(t)$, where an operator $B$ corresponds to a system of linear first order ODEs with a simpler nonlocal boundary condition. The technique proposed in this article is simple to use and can be easily incorporated to any Computer Algebra System (CAS).

## 2 Preliminaries

Let $\mathcal{X}$ be a Banach space such as the space of continuous functions $C[0,1]$ or the space of Lebesgue integrable functions $L_p(0,1)$. Let $\mathcal{X}_m$ be the space of column vectors $X(t) = \mathrm{col}(x_1(t), \dots, x_m(t))$, $x_i(t) \in \mathcal{X}, i = 1, \dots, m$, i.e. $\mathcal{X}_m = C_m = C_m[0,1]$ or $\mathcal{X}_m = L_{p,m} = L_{p,m}(0,1)$ with the norm

$$||X(t)||_{\mathcal{X}_m} = \sum_{i=1}^{m} ||x_i(t)||_{\mathcal{X}}.$$

In addition, let $\mathcal{X}^k, k > 0$, be the space $C^k[0,1]$ with the norm

$$||x(t)||_{\mathcal{X}^k} = \sum_{\ell=0}^{k} ||x^{(\ell)}(t)||_C,$$

or the Sobolev space $\hat{W}_p^k(0,1)$ with the norm

$$||x(t)||_{\mathcal{X}^k} = \sum_{\ell=0}^{k} ||x^{(\ell)}(t)||_{L_p},$$

(in the case of the Sobolev spaces $x^{(l)}$ are weak derivatives), and $\mathcal{X}_m^k$ be the space $C_m^k[0,1]$ or $\hat{W}_{p,m}^k(0,1)$ with the norm

$$||X(t)||_{\mathcal{X}_m^k} = \sum_{\ell=0}^{k} ||X^{(\ell)}(t)||_{\mathcal{X}_m}.$$

Let $\mathcal{X}^*$ be the adjoint space of $\mathcal{X}$, i.e. the set of all linear and bounded functionals $\Phi$ on $\mathcal{X}$. We denote by $\Phi(x)$ the value of $\Phi \in \mathcal{X}^*$ on $x \in \mathcal{X}$. Let $\Psi_j \in \mathcal{X}^*, j = 1, \ldots, n$, and the vector $\Psi = \mathrm{col}(\Psi_1, \ldots, \Psi_n) \in [\mathcal{X}_m]^*$. For $X \in \mathcal{X}_m$ we write

$$\Phi(X) = \begin{pmatrix} \Phi(x_1) \\ \vdots \\ \Phi(x_m) \end{pmatrix}, \quad \Psi_j(X) = \begin{pmatrix} \Psi_j(x_1) \\ \vdots \\ \Psi_j(x_m) \end{pmatrix}, \quad \Psi(X) = \begin{pmatrix} \Psi_1(X) \\ \vdots \\ \Psi_n(X) \end{pmatrix}.$$

**Remark 1.** Let $m = 2$, $k = 1$, $X(t) = \mathrm{col}(x_1(t), x_2(t)) \in \mathcal{X}_2$ and the functional vector $\Theta(X(t)) = \mathrm{col}(\Theta(x_1), \Theta(x_2))$. Then $\Theta \in [\mathcal{X}_2]^*$ if there exists a constant $c_1 > 0$, such that

$$\begin{aligned} |\Theta(X)| = \sqrt{[\Theta(x_1)]^2 + [\Theta(x_2)]^2} &\leq |\Theta(x_1)| + |\Theta(x_2)| \\ &\leq c_1||x_1||_{\mathcal{X}} + c_1||x_2||_{\mathcal{X}} = c_1||X(t)||_{\mathcal{X}_2}. \end{aligned}$$

Similarly $\Theta \in [\mathcal{X}_2^1]^*$ if there exists a constant $c_2 > 0$, such that

$$\begin{aligned} |\Theta(X)| = \sqrt{[\Theta(x_1)]^2 + [\Theta(x_2)]^2} &\leq |\Theta(x_1)| + |\Theta(x_2)| \\ &\leq c_2(||x_1||_{\mathcal{X}} + ||x_2||_{\mathcal{X}} + ||x_1'||_{\mathcal{X}} + ||x_2'||_{\mathcal{X}}) \\ &= c_2(||X(t)||_{\mathcal{X}_2} + ||X'(t)||_{\mathcal{X}_2}) = c_2||X(t)||_{\mathcal{X}_2^1}. \end{aligned}$$

Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces as above. Let the operator $A : \mathcal{X} \to \mathcal{Y}$ and let $D(A)$ and $R(A)$ denote its domain and the range, respectively. The operator $A$ is said to be *injective or uniquelly solvable* if for all $x_1, x_2 \in D(A)$ such that $Ax_1 = Ax_2$, it follows that $x_1 = x_2$. Recall that a linear operator $A$ is injective if and only if $\ker A = \{0\}$. The operator $A$ is called *surjective* or *everywhere solvable* if $R(A) = \mathcal{Y}$. The operator $A$ is called *bijective* if it is both injective and surjective. Finally, the operator $A$ is said to be *correct* if $A$ is bijective and its inverse $A^{-1}$ is bounded on $\mathcal{Y}$. Recall that *the problem* $Au = f$ *is said to be well-posed* if the operator $A$ is correct.

We denote by $0_m$ and $I_m$ the $m \times m$ zero and identity matrix, respectively, $0_{m,n}$ the $m \times n$ zero matrix, and $\vec{0}$ the zero column vector.

**Definition 1.** Two $n \times m$ matrices $P = P(t) = (P_1(t), \ldots, P_m(t))$ and $G = G(t) = (G_1(t), \ldots, G_m(t))$, where $P_i(t) = \mathrm{col}(p_{1i}(t), \ldots, p_{ni}(t))$ and $G_i(t) = \mathrm{col}(g_{1i}(t), \ldots, g_{ni}(t))$, $i = 1, \ldots, m$, respectively, are said to be linearly independent if the vectors $P_1(t), \ldots, P_m(t), G_1(t), \ldots, G_m(t)$ are linearly independent, that is, if $\vec{c}_1, \vec{c}_2$ are two $m$-dimensional constant column vectors and $P(t)\vec{c}_1 + G(t)\vec{c}_2 = \vec{0}$, then $\vec{c}_1 = \vec{c}_2 = \vec{0}$.

## 3 General systems of $m$ first-order ODEs

Let $A : \mathcal{X}_m \to \mathcal{X}_m$ be the differential operator defined by

$$AX(t) = X'(t) - A_0(t)X(t), \quad X(t) \in D(A) = \mathcal{X}_m^1, \tag{3.1}$$

where $A_0(t)$ is an $m \times m$ matrix with entries from $\mathcal{X}$. Let the $m \times m$ matrix $Z = Z(t) = (Z_1(t), \ldots, Z_m(t)) = (z_{ij}(t)), i, j = 1, \ldots, m$, be a fundamental matrix of the homogeneous system

$$AX(t) = \vec{0}, \tag{3.2}$$

such that

$$\Phi(Z) = \begin{pmatrix} \Phi(Z_1), \ldots, \Phi(Z_m) \end{pmatrix} = \begin{pmatrix} \Phi(z_{11}) & \ldots & \Phi(z_{1m}) \\ \vdots & \ddots & \vdots \\ \Phi(z_{m1}) & \ldots & \Phi(z_{mm}) \end{pmatrix} = I_m,$$

where $\Phi \in \mathcal{X}^*$.

**Lemma 3.1.** *Let the operator $A$ be defined as in (3.1), $Z$ be a fundamental matrix of the homogeneous system (3.2), and $F = F(t) = \mathrm{col}(f_1(t), \ldots, f_m(t)) \in \mathcal{X}_m$. Then:*

*(i) the operator $\widehat{A} : \mathcal{X}_m \to \mathcal{X}_m$, corresponding to the problem*

$$\hat{A}X(t) = AX(t) = F(t), \;\; D(\hat{A}) = \{X(t) \in D(A) = \mathcal{X}_m^1 : \Phi(X) = \vec{0}\}, \tag{3.3}$$

*is correct and the unique solution $X(t)$ of equation (3.3) is given by*

$$\begin{aligned} X(t) &= \widehat{A}^{-1}F(t) \\ &= -Z(t)\Phi\left(Z(t)\int_0^t Z^{-1}(s)F(s)ds\right) + Z(t)\int_0^t Z^{-1}(s)F(s)ds, \end{aligned} \tag{3.4}$$

*(ii) if in (i), $\Phi(X) = X(0)$ then*

$$X(t) = \widehat{A}^{-1}F(t) = Z(t)\int_0^t Z^{-1}(s)F(s)ds. \tag{3.5}$$

*Proof.* (i) It is well known that every solution of the system $AX(t) = F(t)$, is given by

$$X(t) = Z(t)\vec{c} + Z(t)\int_0^t Z^{-1}(s)F(s)ds, \tag{3.6}$$

where $\vec{c}$ is an arbitrary $m$-dimensional constant column vector. Acting by functional vector $\Phi$ on both sides of (3.6) and taking into account the boundary condition in (3.3) and that $\Phi(Z) = I_m$, we obtain

$$\begin{aligned} \Phi(X) &= \vec{c} + \Phi\left(Z(t)\int_0^t Z^{-1}(s)F(s)ds\right) = \vec{0}, \\ \vec{c} &= -\Phi\left(Z(t)\int_0^t Z^{-1}(s)F(s)ds\right). \end{aligned}$$

Substituting $\vec{c}$ into (3.6), we get (3.4).

(ii) Equation (3.5) is derived directly from (3.4) using $\Phi(X) = X(0) = \vec{0}$. □

**Theorem 3.1.** *Let the operators $A$ and $\widehat{A}$, the vector $F$ and the matrix $Z$ be defined as in Lemma 3.1. In addition, let the $m \times (mn)$ constant matrix $M = (M_1, \ldots, M_n)$, where $M_j, j = 1, \ldots, n$, are $m \times m$ constant matrices, the functionals $\Phi, \Psi_j \in \mathcal{X}^*, j = 1, \ldots, n$, and the functional vector $\Psi = \mathrm{col}(\Psi_1, \ldots, \Psi_n)$ are given. Then:*

*(i) the operator $B : \mathcal{X}_m \to \mathcal{X}_m$, corresponding to the problem*

$$\begin{aligned} BX(t) &= AX(t) = F(t), \\ D(B) &= \{X(t) \in D(A) = \mathcal{X}_m^1 : \Phi(X) = \sum_{j=1}^{n} M_j \Psi_j(X)\} \end{aligned} \tag{3.7}$$

*is injective if and only if*

$$\det W = \det[I_{mn} - \Psi(Z)M] \neq 0, \tag{3.8}$$

*(ii) if the operator $B$ is injective, then it is also correct and the unique solution to problem (3.7) is given by*

$$X(t) = B^{-1}F(t) = \widehat{A}^{-1}F(t) + ZMW^{-1}\Psi(\widehat{A}^{-1}F), \tag{3.9}$$

*where $\widehat{A}^{-1}F(t)$ is the solution of system (3.3) given in (3.4).*

*Proof.* (i) Let $\det W \neq 0$ and $X(t) \in \ker B$. Then from problem (3.7) we get

$$AX(t) = \vec{0}, \quad \Phi(X) = M\Psi(X), \tag{3.10}$$

which, since $Z \in ker A$ and $\Phi(Z) = I_m$, can be written as

$$A\,(X(t) - ZM\Psi(X)) = \vec{0}, \quad \Phi(X(t) - ZM\Psi(X)) = \vec{0}. \tag{3.11}$$

From the second equation of (3.11) by taking into account (3.3) we get $X(t) - ZM\Psi(X) \in D(\widehat{A})$ and then from the first equation of (3.11), since $\ker \widehat{A} = \{0\}$ and $A$ is the extension of $\widehat{A}$, it follows that

$$X(t) = ZM\Psi(X). \tag{3.12}$$

Acting by the functional vector $\Psi$ on both sides we get

$$[I_{mn} - \Psi(Z)M]\Psi(X) = W\Psi(X) = \vec{0},$$

and since $\det W \neq 0$, it is implied that $\Psi(X) = \vec{0}$. Substitution into (3.10) yields $\widehat{A}X(t) = \vec{0}$. This means that $X(t) = \vec{0}$ and therefore the operator $B$ is injective.

Conversely, let $\det W = 0$. Then there exists a nonzero vector $\vec{c} = \mathrm{col}(c_1, \ldots, c_{mn})$, such that $W\vec{c} = \vec{0}$. Consider the element

$$X_0(t) = Z(t)M\vec{c}, \tag{3.13}$$

and note that $X_0(t) \neq \vec{0}$, since otherwise $W\vec{c} = [I_{mn} - \Psi(Z)M]\vec{c} = \vec{c} - \Psi(Z)M\vec{c} = \vec{c} = \vec{0}$. Then

$$\begin{aligned} BX_0(t) &= AX_0(t) = \vec{0}, \\ \Phi(X_0) - M\Psi(X_0) &= M\vec{c} - M\Psi(Z)M\vec{c} = M[I_{mn} - \Psi(Z)M]\vec{c} = MW\vec{c} = \vec{0}, \end{aligned}$$

and, hence, $X_0(t) \in \ker B$. Therefore, $B$ is not injective. Thus, we proved that if $B$ is injective, then $\det W \neq 0$.

(ii) Let $\det W \neq 0$, then the operator $B$ is injective. Problem (3.7) can be written as

$$A\,(X(t) - Z(t)M\Psi(X)) = F(t), \quad \Phi\,(X(t) - Z(t)M\Psi(X)) = \vec{0}. \tag{3.14}$$

Then, since (3.3) we get $X(t) - Z(t)M\Psi(X) \in D(\widehat{A})$, and from (3.14) it follows that

$$X(t) = Z(t)M\Psi(X) + \widehat{A}^{-1}F. \tag{3.15}$$

Acting by the functional vector $\Psi$ on both sides of the above equation we get

$$[I_{mn} - \Psi(Z)M]\Psi(X) = \Psi(\widehat{A}^{-1}F),$$
$$\Psi(X) = [I_{mn} - \Psi(Z)M]^{-1}\Psi(\widehat{A}^{-1}F) = W^{-1}\Psi(\widehat{A}^{-1}F).$$

Substituting into (3.12), we get solution (3.9). Since the functionals $\Psi_1, \ldots, \Psi_n$ and the operator $\widehat{A}^{-1}$ in (3.9) are bounded, then the operator $B^{-1}$ is also bounded. Note that solution (3.9) is obtained for any arbitrary vector $F(t) \in \mathcal{X}_m$. This means that $R(B) = \mathcal{X}_m$, i.e. the operator $B$ is everywhere solvable. So, the operator $B$ is correct. □

**Lemma 3.2.** *Let the operators $A, \widehat{A}$ and the $m \times m$ fundamental matrix $Z$ be defined as in Lemma 3.1. Then:*

*(i) the set $Z \cup \widehat{A}^{-1}Z$, with $\widehat{A}^{-1}Z = (\widehat{A}^{-1}Z_1(t), \ldots, \widehat{A}^{-1}Z_m(t))$, is linearly independent,*

*(ii) the set $Z \cup tZ, 0 < t < 1$, is also linearly independent.*

*Proof.* (i) The vectors $Z_1(t), \ldots, Z_m(t)$ are linearly independent since they are the columns of the fundamental matrix. Furthermore, the vectors $\widehat{A}^{-1}Z_1(t), \ldots, \widehat{A}^{-1}Z_m(t)$ are also linearly independent since $\ker \widehat{A} = \{0\}$. Let $Z\vec{c}_1 + \widehat{A}^{-1}Z\vec{c}_2 = \vec{0}$, where $\vec{c}_1, \vec{c}_2$ are two $m$-dimensional constant column vectors. Then, since $\ker A \cap D(\widehat{A}) = \{0\}$ [11], we have $Z\vec{c}_1 = \vec{0}$ and $\widehat{A}^{-1}Z\vec{c}_2 = \vec{0}$, and hence $\vec{c}_1 = \vec{c}_2 = \vec{0}$ since which since $Z_1, \ldots, Z_m$ and $\widehat{A}^{-1}Z_1(t), \ldots, \widehat{A}^{-1}Z_m(t)$ are linealy independent. Thus, the set $Z \cup \widehat{A}^{-1}Z$ is linearly independent.

(ii) Let $\Phi(X) = X(0)$. Then from (3.5) and (i) it follows that $\widehat{A}^{-1}Z(t) = tZ(t)$ and so the set $Z \cup tZ, 0 < t < 1$, is linearly independent. □

## 4 Special type systems of $2m$ first-order ODEs with constant coefficients

In this section, we consider the solvability and the construction of the exact solution of a special type of systems of $2m$ first-order ODEs with constant coefficients.

**Lemma 4.1.** *Let the operator $A$, where $A_0$ is an $m \times m$ nonsingular constant matrix, and the associated fundamental matrix $Z = Z(t)$ be defined as in Lemma 3.1. Let the operator $\mathbf{A} : \mathcal{X}_{2m} \to \mathcal{X}_{2m}$ be defined by*

$$\mathbf{A}U(t) = U'(t) - D_0U(t) = \vec{0}, \quad D(\mathbf{A}) = \mathcal{X}^1_{2m}, \tag{4.1}$$

*where the vector $U = U(t) = \mathrm{col}\,(u_1(t), \ldots, u_{2m}(t)) \in \mathcal{X}^1_{2m}$ and the $2m \times 2m$ constant matrix $D_0$ has the special form*

$$D_0 = \begin{pmatrix} 2A_0 & -A_0^2 \\ I_m & 0_m \end{pmatrix}.$$

*Then the $2m \times 2m$ matrix*

$$\mathbf{Z}(t) = \begin{pmatrix} Z(t) & tZ(t) \\ \int_0^t Z(s)ds + A_0^{-1} & \int_0^t sZ(s)ds - [A_0^{-1}]^2 \end{pmatrix} \tag{4.2}$$

*is a fundamental matrix of the homogeneous system (4.1).*

*Proof.* The two $2m \times m$ matrices

$$\begin{pmatrix} Z \\ \int_0^t Z(s)ds + A_0^{-1} \end{pmatrix}, \quad \begin{pmatrix} tZ \\ \int_0^t sZ(s)ds - [A_0^{-1}]^2 \end{pmatrix} \tag{4.3}$$

satisfy equation (4.1). Indeed, since $Z' = A_0 Z$ and $Z(0) = I_m$, we have

$$\begin{aligned} &\begin{pmatrix} Z \\ \int_0^t Z(s)ds + A_0^{-1} \end{pmatrix}' - \begin{pmatrix} 2A_0 & -A_0^2 \\ I_m & 0_m \end{pmatrix} \begin{pmatrix} Z \\ \int_0^t Z(s)ds + A_0^{-1} \end{pmatrix} \\ &= \begin{pmatrix} -A_0 Z + A_0 \int_0^t A_0 Z(s)ds + A_0 \\ Z - Z \end{pmatrix} \\ &= \begin{pmatrix} -A_0 Z + A_0 \int_0^t Z'(s)ds + A_0 \\ 0_m \end{pmatrix} = \begin{pmatrix} 0_m \\ 0_m \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} &\begin{pmatrix} tZ \\ \int_0^t sZ(s)ds - [A_0^{-1}]^2 \end{pmatrix}' - \begin{pmatrix} 2A_0 & -A_0^2 \\ I_m & 0_m \end{pmatrix} \begin{pmatrix} tZ \\ \int_0^t sZ(s)ds - [A_0^{-1}]^2 \end{pmatrix} \\ &= \begin{pmatrix} Z + tZ' \\ tZ \end{pmatrix} - \begin{pmatrix} 2A_0 tZ - A_0 \int_0^t sA_0 Z(s)ds + I_m \\ tZ \end{pmatrix} \\ &= \begin{pmatrix} Z + tZ' - 2A_0 tZ + A_0 \int_0^t sZ'(s)ds - I_m \\ 0_m \end{pmatrix} \\ &= \begin{pmatrix} Z + t(Z' - A_0 Z) - A_0 tZ + A_0 [sZ(s)]_0^t - \int_0^t A_0 Z(s)ds - I_m \\ 0_m \end{pmatrix} \\ &= \begin{pmatrix} Z - \int_0^t Z'(s)ds - I_m \\ 0_m \end{pmatrix} \\ &= \begin{pmatrix} Z(t) - [Z(t) - Z(0)] - I_m \\ 0_m \end{pmatrix} = \begin{pmatrix} 0_m \\ 0_m \end{pmatrix}. \end{aligned}$$

Furthermore, as shown below, the two matrices in (4.3) are linearly independent. Let

$$\begin{pmatrix} Z \\ \int_0^t Z(s)ds + A_0^{-1} \end{pmatrix} \vec{c}_1 + \begin{pmatrix} tZ \\ \int_0^t sZ(s)ds - [A_0^{-1}]^2 \end{pmatrix} \vec{c}_2 = \vec{0},$$

where $\vec{c}_1, \vec{c}_2$ are $m$-dimensional constant column vectors. Then we have $Z\vec{c}_1 + tZ\vec{c}_2 = \vec{0}$ and since $Z, tZ$, are linearly independent by Lemma 3.2, it follows that $\vec{c}_1 = \vec{c}_2 = \vec{0}$. Thus, the two matrices in (4.3) are linearly independent and $\mathbf{Z}$ is a fundamental matrix for the system (4.1). □

**Theorem 4.1.** *Let the operator* $\mathbf{A}$ *and the* $2m \times 2m$ *fundamental matrix* $\mathbf{Z}$ *be defined as in Lemma 4.1. Let the vector* $\mathbf{F} = \mathbf{F}(t) = \operatorname{col}(f_1(t), \ldots, f_{2m}(t)) \in \mathcal{X}_{2m}$*, the vector of functionals* $\Psi = (\Psi_1, \ldots, \Psi_n)$*,* $\Psi_j \in \mathcal{X}^*, j = 1, \ldots, n$*, and* $\mathbf{M}$ *be a* $2m \times 2mn$ *constant matrix. Then:*

*(i) the operator* $\mathbf{B} : \mathcal{X}_{2m} \to \mathcal{X}_{2m}$ *defined by the problem*

$$\mathbf{B}U(t) = \mathbf{A}U(t) = \mathbf{F}, \quad D(\mathbf{B}) = \{U(t) \in D(\mathbf{A}) : U(0) = \mathbf{M}\Psi(U)\} \tag{4.4}$$

*is injective if and only if*

$$det\mathbf{W} = det[\mathbf{Z}(0) - \mathbf{M}\Psi(\mathbf{Z})] \neq 0, \tag{4.5}$$

*where*

$$\mathbf{Z}(0) = \begin{pmatrix} I_m & 0_m \\ A_0^{-1} & -[A_0^{-1}]^2 \end{pmatrix}, \tag{4.6}$$

*(ii) the unique solution of problem (4.4) for every* $\mathbf{F} \in \mathcal{X}_{2m}$ *is given by*

$$U(t) = \mathbf{Z}\mathbf{W}^{-1}\mathbf{M}\Psi\left(\mathbf{Z}(t)\int_0^t \mathbf{Z}^{-1}(s)\mathbf{F}(s)ds\right) + \mathbf{Z}(t)\int_0^t \mathbf{Z}^{-1}(s)\mathbf{F}(s)ds. \tag{4.7}$$

*Proof.* The proof follows the same procedure as for the proof of Theorem 3.1. □

## 5 Factorization of systems of second-order ODEs

In this section, we present the main results regarding the factorization method for solving nonlocal systems of second-order linear differential equations.

**Lemma 5.1.** *Let the operators* $A, \widehat{A}$*, where the elements of* $A_0(t)$ *belong to* $\mathcal{X}^1$ *and the functional* $\Phi \in [\mathcal{X}^1]^*$*, the vectors* $X, F$ *and the fundamental matrix* $Z$ *be defined as in Lemma 3.1. Then:*

*(i) for the operator* $A^2 : \mathcal{X}_m \to \mathcal{X}_m$ *defined as*

$$A^2X(t) = X''(t) - 2A_0(t)X'(t) + [A_0^2(t) - A_0'(t)]X(t), \quad D(A^2) = \mathcal{X}_m^2, \tag{5.1}$$

*(ii) the operator* $\widehat{A}^2$ *defined by*

$$\widehat{A}^2X = A^2X = F, \quad D(\widehat{A}^2) = \{X(t) \in D(A^2) : \Phi(X) = \vec{0},\ \Phi(AX) = \vec{0}\} \tag{5.2}$$

*is correct and the unique solution of system (5.2) is given by*

$$\begin{aligned} X(t) &= \widehat{A}^{-2}F(t) = \widehat{A}^{-1}Y(t) \\ &= -Z(t)\Phi\left(Z(t)\int_0^t Z^{-1}(s)Y(s)ds\right) + Z(t)\int_0^t Z^{-1}(s)Y(s)ds, \end{aligned} \tag{5.3}$$

*where*

$$\begin{aligned} Y(t) &= \widehat{A}^{-1}F(t) \\ &= -Z(t)\Phi\left(Z(t)\int_0^t Z^{-1}(s)F(s)ds\right) + Z(t)\int_0^t Z^{-1}(s)F(s)ds, \end{aligned} \tag{5.4}$$

*(iii) in the case that* $\Phi(X) = X(0)$*,* $Z, tZ \in ker A^2$ *and* $(Z,\, tZ), 0 < t < 1$*, is a fundamental matrix of the homogeneous system*

$$A^2X(t) = \vec{0}, \tag{5.5}$$

*and*

$$\widehat{A}^{-2}F(t) = Z(t)\int_0^t (t-s)Z^{-1}(s)F(s)ds. \tag{5.6}$$

*Proof.* (i) Let $Y(t) = AX(t) = X'(t) - A_0(t)X(t)$. Then

$$\begin{aligned} A^2X(t) &= AY(t) = Y'(t) - A_0(t)Y(t) \\ &= [X'(t) - A_0(t)X(t)]' - A_0(t)[X'(t) - A_0(t)X(t)] \\ &= X''(t) - A_0'(t)X(t) - A_0(t)X'(t) - A_0(t)X'(t) + A_0^2(t)X(t) \\ &= X''(t) - 2A_0(t)X'(t) + [A_0^2(t) - A_0'(t)]X(t). \end{aligned} \tag{5.7}$$

It easily follows that if $D(A) = \mathcal{X}_m^1$, then $D(A^2) = \mathcal{X}_m^2$.

(ii) By using (5.7) system (5.2) can be factorized into the following two systems of first order differential equations

$$\widehat{A}Y(t) = AY(t) = Y'(t) - A_0(t)Y(t) = F(t), \quad \Phi(Y) = \vec{0},$$

$$\widehat{A}X(t) = AX(t) = X'(t) - A_0(t)X(t) = Y(t), \quad \Phi(X) = \vec{0},$$

which, by Lemma 3.1, are well-posed and their solutions are given by $Y(t) = \widehat{A}^{-1}F(t)$ and $X(t) = \widehat{A}^{-1}Y(t)$, respectively, from where (5.3) and (5.4) are derived. The operator $\widehat{A}^2$ is correct because it is a superposition of two correct operators [14].

(iii) Let $A^2X = \vec{0}$. Setting $Y = AX$ we get $AY = \vec{0}$. Then $Y = Z\vec{c}_1$ or $AX = Z\vec{c}_1$, which gives $X = Z\vec{c}_2 + \widehat{A}^{-1}Y = Z\vec{c}_2 + \widehat{A}^{-1}Z\vec{c}_1$, where $\vec{c}_1, \vec{c}_2$ are $m$-dimensional constant column vectors. From here, taking into account (5.3) and $\Phi(X) = X(0)$, for $F = Z\vec{c}_1$ we obtain $\widehat{A}^{-1}Z = tZ$ and $X(t) = Z(t)\vec{c}_2 + tZ(t)\vec{c}_1 \in ker\, A^2$. By Lemma 3.2, the system $Z \cup tZ$ is linearly independent. Hence $Z, tZ \in ker\, A^2$ and the system $(Z, tZ)$ constitutes a fundamental solution to (5.5). From (5.3), (5.4), because of $\Phi(0) = X(0) = \vec{0}$, by Fubini's theorem, equality (5.6) easily follows. □

**Theorem 5.1.** *Let the operator $\mathcal{A} : \mathcal{X}_m \to \mathcal{X}_m$ be defined by*

$$\mathcal{A}X(t) = X''(t) - S(t)X'(t) - Q(t)X(t), \quad D(\mathcal{A}) = \mathcal{X}_m^2, \tag{5.8}$$

*where $Q(t)$ and $S(t)$ are $m \times m$ matrices with entries from $\mathcal{X}$ and $\mathcal{X}^1$, respectively, and the operator $B_2 : \mathcal{X}_m \to \mathcal{X}_m$ be defined as*

$$\begin{aligned} B_2X(t) &= \mathcal{A}X(t) = F(t), \\ D(B_2) &= \{X(t) \in \mathcal{X}_m^2 : \Phi(X) = \sum_{i=1}^{n} M_i\Psi_i(X), \\ &\qquad \Phi(X') = \Phi(TX) + \sum_{j=1}^{r} N_j\Theta_j(X)\}, \end{aligned} \tag{5.9}$$

*where $F \in \mathcal{X}_m$, $T(t)$ is an $m \times m$ matrix with entries from $\mathcal{X}$, $M_j, j = 1, \ldots, n$, and $N_j, j = 1, \ldots, r$, are $m \times m$ constant matrices, $\Phi \in [\mathcal{X}^1]^*$, $\Psi_j \in \mathcal{X}^*, j = 1, \ldots, n$, and $\Theta_j \in \mathcal{X}^*, j = 1, \ldots, r$. Then:*

*(i) if*

$$Q(t) = \frac{1}{2}S'(t) - \frac{1}{4}S^2(t), \tag{5.10}$$

*the operator $\mathcal{A}$ can be factorized as follows*

$$\mathcal{A}X(t) = A^2X(t), \quad X(t) \in D(\mathcal{A}), \tag{5.11}$$

*where*

$$AX(t) = X'(t) - \frac{1}{2}S(t)X(t), \quad D(A) = \mathcal{X}_m^1, \tag{5.12}$$

*(ii) if, in addition to (i), we have $T(t) = \frac{1}{2}S(t)$, the operator $B_2$ is injective if and only if*

$$\det W_2 = \det \begin{pmatrix} I_{mn} - \Psi(Z)M & -\Psi(\widehat{A}^{-1}Z)N \\ -\Theta(Z)M & I_{mk} - \Theta(\widehat{A}^{-1}Z)N \end{pmatrix} \neq 0, \tag{5.13}$$

*where $W_2$ is an $m(n+r) \times m(n+r)$ matrix, $Z$ is a fundamental matrix of the system $AX = \vec{0}$,*

$$\widehat{A}X(t) = AX(t) = F(t), \quad D(\widehat{A}) = \{X(t) \in D(A) : \Phi(X) = \vec{0}\}, \tag{5.14}$$

*and $\Psi = \mathrm{col}(\Psi_1, \ldots, \Psi_n)$ and $\Theta = \mathrm{col}(\Theta_1, \ldots, \Theta_r)$,* *(iii) under (ii), the operator $B_2$ is correct and the unique solution of system (5.9) is given by*

$$X(t) = B_2^{-1}F(t) = \widehat{A}^{-2}F(t) + \left(Z(t)M, \widehat{A}^{-1}Z(t)N\right) W_2^{-1} \begin{pmatrix} \Psi(\widehat{A}^{-2}F) \\ \Theta(\widehat{A}^{-2}F) \end{pmatrix}, \tag{5.15}$$

*where $\widehat{A}^{-2}F(t)$, $\widehat{A}^{-1}F(t)$ are given by (5.3), (5.4), respectively.*

*Proof.* (*i*) Denote $Y(t) = X'(t) - \frac{1}{2}S(t)X(t)$. Then since (5.10) and (5.12), we get

$$\begin{aligned} \mathcal{A}X(t) &= X''(t) - S(t)X'(t) - Q(t)X(t) \\ &= X''(t) - S(t)X'(t) - \left[\frac{1}{2}S'(t) - \frac{1}{4}S^2(t)\right] X(t) \\ &= X''(t) - \frac{1}{2}\left(S(t)X(t)\right)' - \frac{1}{2}S\left(X' - \frac{1}{2}SX\right) \\ &= \left(X' - \frac{1}{2}SX\right)' - \frac{1}{2}S\left(X' - \frac{1}{2}SX\right) = Y' - \frac{1}{2}SY = AY = A^2X. \end{aligned}$$

From $D(A) = \mathcal{X}_m$ it easily follows that $D(A^2) = \mathcal{X}_m^2$. Thus, we proved that $B_2X(t) = \mathcal{A}X(t) = A^2X(t)$.

(*ii*) If $T(t) = \frac{1}{2}S(t)$, then $\Phi(X') - \Phi(TX) = \Phi\left(X' - \frac{1}{2}SX\right) = \Phi(AX)$ and problem (5.9) is reduced to

$$B_2X(t) = A^2X(t) = F(t), \quad \Phi(X) = M\Psi(X), \quad \Phi(AX) = N\Theta(X). \tag{5.16}$$

Let $\det W_2 \neq 0$ and $X(t) \in \ker B_2$. Then from problem (5.16) we get

$$B_2X(t) = A^2X(t) = \vec{0}, \quad \Phi(X) = M\Psi(X), \quad \Phi(AX) = N\Theta(X), \tag{5.17}$$

which, since $\Phi(Z) = I_m$ and $AZ = 0_m$, can be represented as

$$A\left(AX(t) - ZN\Theta(X)\right) = \vec{0}, \tag{5.18}$$
$$\Phi(X(t) - ZM\Psi(X)) = \vec{0}, \tag{5.19}$$
$$\Phi(AX(t) - ZN\Theta(X)) = \vec{0}. \tag{5.20}$$

Further taking into account (3.3), we get $X(t) - ZM\Psi(X),\ AX(t) - ZN\Theta(X) \in D(\widehat{A})$ and from (5.18), because of $A$ is an extension of $\widehat{A}$ and $\ker \widehat{A} = \{0\}$, it follows that

$$\begin{gathered} AX(t) = ZN\Theta(X), \\ A(X(t) - ZM\Psi(X)) = ZN\Theta(X), \\ \widehat{A}(X(t) - ZM\Psi(X)) = ZN\Theta(X), \\ X(t) = ZM\Psi(X) + \widehat{A}^{-1}ZN\Theta(X). \end{gathered}$$

Then acting by functional vectors $\Psi, \Theta$ on both sides of the above equation we get

$$[I_{mn} - \Psi(Z)M]\Psi(X) - \Psi(\widehat{A}^{-1}Z)N\Theta(X) = \vec{0}, \tag{5.21}$$
$$-\Theta(Z)M\Psi(X) + [I_{mk} - \Theta(\widehat{A}^{-1}Z)N]\Theta(X) = \vec{0}. \tag{5.22}$$

From the last system, since $\det W_2 \neq 0$, it follows that $\Psi(X) = \vec{0}$, $\Theta(X) = \vec{0}$. Substituting these values into (5.17), we obtain that $\widehat{A}^2X(t) = \vec{0}$, and so, because $\widehat{A}^2$ is correct, we have $X(t) = \vec{0}$. Then $\ker B_2 = \{\vec{0}\}$ and $B_2$ is injective.

Conversely, let $\det W_2 = 0$. Then there exists a nonzero constant vector $\vec{c} = \mathrm{col}(\mathbf{c}_1, \mathbf{c}_2)$, where $\mathbf{c}_1 = \mathrm{col}(c_{11}, \dots, c_{1,mn})$, $\mathbf{c}_2 = \mathrm{col}(c_{21}, \dots, c_{2,mk})$, such that

$$W_2\vec{c} = \begin{pmatrix} I_{mn} - \Psi(Z)M & -\Psi(\widehat{A}^{-1}Z)N \\ -\Theta(Z)M & I_{mk} - \Theta(\widehat{A}^{-1}Z)N \end{pmatrix} \begin{pmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{pmatrix} = \begin{pmatrix} \vec{0} \\ \vec{0} \end{pmatrix}. \tag{5.23}$$

Consider the vector

$$X_0(t) = Z(t)M\mathbf{c}_1 + \widehat{A}^{-1}Z(t)N\mathbf{c}_2. \tag{5.24}$$

Note that $X_0(t) = \vec{0}$, if and only if $M\mathbf{c}_1 = \vec{0}$, $N\mathbf{c}_2 = \vec{0}$, since $Z(t)$ is the fundamental matrix and the set $Z(t) \cup \widehat{A}^{-1}Z(t)$, by Lemma 3.2, is linearly independent. But if $M\mathbf{c}_1 = \vec{0}$, $N\mathbf{c}_2 = \vec{0}$, then from (5.23) follows that $\mathbf{c}_1 = \vec{0}$, $\mathbf{c}_2 = \vec{0}$. Thus, we obtain $\vec{c} = \vec{0}$. But by hypothesis $\vec{c} \neq \vec{0}$. So $X_0(t) \neq \vec{0}$. Further using (5.24) and taking into account (5.23), we find

$$\begin{gathered} B_2X_0(t) = A^2X_0(t) = \vec{0}, \\ \Phi(X_0) - M\Psi(X_0) = M[I_{mn} - \Psi(Z)M]\mathbf{c}_1 - M\Psi(\widehat{A}^{-1}Z)N\mathbf{c}_2 = \vec{0}, \\ AX_0(t) = Z(t)N\mathbf{c}_2, \\ \Phi(AX_0) - N\Theta(X_0) = -N\Theta(Z)M\mathbf{c}_1 + N[I_{mk} - \Theta(\widehat{A}^{-1}Z)N]\mathbf{c}_2 = \vec{0}. \end{gathered}$$

From here it follows that $X_0(t) \in \ker B_2$ and $B_2$ is not injective. Thus, by way of contradiction we proved that if $B_2$ is injective, then $\det W \neq 0$.

$(iii)$ Let $\det W_2 \neq 0$, then the operator $B_2$ is injective. From (5.16) we obtain

$$A\left(AX(t) - Z(t)N\Theta(X)\right) = F(t), \quad \Phi\left(AX(t) - Z(t)N\Theta(X)\right) = \vec{0}. \tag{5.25}$$

Then since $\widehat{A}$ is a restriction of $A$ and (3.3), we get $AX(t) - Z(t)N\Theta(X) \in D(\widehat{A})$. From (5.25) and first boundary condition (5.16) it follows that

$$AX(t) = Z(t)N\Theta(X) + \widehat{A}^{-1}F(t), \quad \Phi(X(t) - Z(t)M\Psi(X)) = \vec{0}. \tag{5.26}$$

By means (3.3) we get $X(t) - Z(t)M\Psi(X) \in D(\widehat{A})$. Then from (5.26), taking into account that $\widehat{A}$ is a restriction of $A$, we get

$$\begin{gathered} \widehat{A}[X(t) - Z(t)M\Psi(X)] - Z(t)N\Theta(X) = \widehat{A}^{-1}F(t), \\ X(t) - Z(t)M\Psi(X) - \widehat{A}^{-1}Z(t)N\Theta(X) = \widehat{A}^{-2}F(t). \end{gathered} \tag{5.27}$$

Acting by functional vectors $\Psi, \Theta$ on both sides of the above equation, obtain

$$[I_{mn} - \Psi(Z)M]\Psi(X) - \Psi(\widehat{A}^{-1}Z)N\Theta(X) = \Psi(\widehat{A}^{-2}F), \tag{5.28}$$

$$-\Theta(Z)M\Psi(X) + [I_{mk} - \Theta(\widehat{A}^{-1}Z)N]\Theta(X) = \Theta(\widehat{A}^{-2}F), \tag{5.29}$$

or

$$W_2 \begin{pmatrix} \Psi(X) \\ \Theta(X) \end{pmatrix} = \begin{pmatrix} \Psi(\widehat{A}^{-2}F) \\ \Theta(\widehat{A}^{-2}F) \end{pmatrix}.$$

The last equation yelds

$$\begin{pmatrix} \Psi(X) \\ \Theta(X) \end{pmatrix} = W_2^{-1} \begin{pmatrix} \Psi(\widehat{A}^{-2}F) \\ \Theta(\widehat{A}^{-2}F) \end{pmatrix}.$$

Substituting this value into (5.27), we get solution (5.15). Since the functionals $\Psi_1, \dots, \Psi_n, \Theta_1, \dots, \Theta_k$ and the operators $\widehat{A}^{-1}, \widehat{A}^{-2}$ in (5.15) are bounded, then the operator $B_2^{-1}$ is also bounded. Note that formula (5.15) was proved for any arbitrary vector $F(t) \in \mathcal{X}_m$. This means that $R(B_2) = \mathcal{X}_m$, i.e. the operator $B_2$ is everywhere solvable. Before we proved that $B_2$ is injective and $B_2^{-1}$ is bounded. Hence, $B_2$ is correct. □

**Corollary 5.1.** *In Theorem 5.1 let* $\Phi(X) = X(t_0)$, $t_0 \in [0,1]$, $r = n$, $\Psi_j, \Theta_j \in [\mathcal{X}^1]^*$, $j = 1, \ldots, n$, $Q(t)$ *satisfies* (5.10) *and* $T$ *be a constant matrix. Then*

$$\begin{aligned} B_2 X(t) &= \mathcal{A}X(t) = F(t), \\ D(B_2) &= \{X(t) \in \mathcal{X}_m^2 : X(t_0) = \sum_{j=1}^{n} M_j \Psi_j(X), \\ & \qquad X'(t_0) = TX(t_0) + \sum_{j=1}^{n} N_j \Theta_j(X)\}. \end{aligned} \tag{5.30}$$

*(i) If*

$$T = \frac{1}{2}S(t_0), \quad N_j = M_j, \quad \Theta_j(X) = \Psi_j(AX),\ i = 1, \ldots, n, \tag{5.31}$$

*then there exists an operator* $B : \mathcal{X}_m \to \mathcal{X}_m$ *defined by*

$$BX(t) = AX(t), \quad D(B) = \{X(t) \in D(A) : X(t_0) = \sum_{j=1}^{n} M_j \Psi_j(X)\}, \tag{5.32}$$

*such that* $B_2$ *can be factorized into* $B_2 = B^2$,

*(ii) in addition, problem* (5.30) *is uniquely solvable if and only if*

$$\det W_3 = \det[I_{mn} - \Psi(Z)M] \neq 0, \tag{5.33}$$

*and its unique solution for all* $F \in \mathcal{X}_m$ *is given by*

$$X(t) = B_2^{-1}F(t) = \widehat{A}^{-1}Y(t) + ZMW_3^{-1}\Psi(\widehat{A}^{-1}Y), \tag{5.34}$$

*where*

$$Y(t) = \widehat{A}^{-1}F(t) + ZMW_3^{-1}\Psi(\widehat{A}^{-1}F), \tag{5.35}$$

$$\widehat{A}^{-1}F(t) = -Z(t)Z(t_0)\int_0^{t_0} Z^{-1}(s)F(s)ds + Z(t)\int_0^t Z^{-1}(s)F(s)ds, \tag{5.36}$$

$$\widehat{A}^{-1}Y(t) = -Z(t)Z(t_0)\int_0^{t_0} Z^{-1}(s)Y(s)ds + Z(t)\int_0^t Z^{-1}(s)Y(s)ds, \tag{5.37}$$

$Z = Z(t)$ *is a fundamental matrix of* $AX(t) = \vec{0}$, *satisfying* $Z(t_0) = I_m$, *and*

$$\widehat{A}X(t) = AX(t), \quad D(\widehat{A}) = \{X(t) \in D(A) : X(t_0) = \vec{0}\}. \tag{5.38}$$

*Proof.* (i) Consider the operator $B$ defined by (5.32), namely

$$BX(t) = AX(t) = X'(t) - \frac{1}{2}S(t)X(t), \quad X(t) \in D(B).$$

Then for $X(t) \in D(B^2) \cap D(B_2)$, since (5.10), the following formula is valid

$$B^2X(t) = A^2X(t) = X''(t) - S(t)X'(t) - \left[\frac{1}{2}S'(t) - \frac{1}{4}S^2(t)\right]X(t) = B_2X(t).$$

It remains to prove that $D(B^2) = D(B_2)$ for $T, N$ and $\Theta(X)$, satisfying (5.31). Indeed, because of the equality $BX = AX$, $X \in D(B)$, we obtain

$$\begin{aligned} D(B^2) &= \{X(t) \in D(B) : BX(t) \in D(B)\} \\ &= \{X(t) \in D(A^2) : X(t_0) = M\Psi(X),\ (BX)(t_0) = M\Psi(BX)\} \\ &= \{X(t) \in D(A^2) : X(t_0) = M\Psi(X),\ (AX)(t_0) = M\Psi(AX)\}, \end{aligned} \tag{5.39}$$

where

$$(AX)(t_0) = \Phi(AX) = \Phi\left(X'(t) - \frac{1}{2}S(t)X(t)\right) = X'(t_0) - \frac{1}{2}S(t_0)X(t_0) = X'(t_0) - TX(t_0).$$

Then from (5.39) we get

$$\begin{aligned} D(B^2) &= \{X(t) \in D(A^2) : X(t_0) = M\Psi(X),\ X'(t_0) = TX(t_0) + M\Psi(AX)\} \\ &= D(B_2). \end{aligned}$$

(ii) By Theorem 5.1, the operator $B_2$ is injective if and only if (5.13) is fulfilled, where $k = n$, $N = M$, $\Theta(Z) = \Psi(AZ)$ and $\Theta(\widehat{A}^{-1}Z) = \Psi(A\widehat{A}^{-1}Z)$, or if and only if

$$\det W_2 = \det \begin{pmatrix} I_{mn} - \Psi(Z)M & -\Psi(\widehat{A}^{-1}Z)M \\ -\Psi(AZ)M & I_{mn} - \Psi(A\widehat{A}^{-1}Z)M \end{pmatrix} \neq 0,$$

or

$$\det \begin{pmatrix} I_{mn} - \Psi(Z)M & -\Psi(\widehat{A}^{-1}Z)M \\ 0_{mn} & I_{mn} - \Psi(Z)M \end{pmatrix} = [\det(I_{mn} - \Psi(Z)M)]^2 = [\det W_3]^2 \neq 0.$$

Thus, $B_2 = B^2$ is injective if and only if $\det W_3 \neq 0$. The problem $B^2X(t) = F(t)$ by substituting $BX(t) = Y(t)$ is reduced to two systems $BY(t) = F(t)$ and $BX(t) = Y(t)$. By Theorem 3.1, a unique solution to the first system is given by (5.35), where $\widehat{A}^{-1}F(t)$ is given by (3.4) or (5.36). Substituting the value $Y(t)$ from (5.35) into the system $BX(t) = Y(t)$ and again using Theorem 3.1, we obtain (5.34). □

## 6 Examples

**Example 1** In the function space $C^1[0,1]$, the following system of four first-order differential equations with four homogeneous initial conditions

$$\begin{aligned} y_1'(t) &+ 2\pi y_2(t) + \pi^2 y_3(t) = \cos\pi t, \\ y_2'(t) &- 2\pi y_1(t) - \pi^2 y_4(t) = \sin\pi t, \\ y_3'(t) &- y_1(t) = 2\sin\pi t, \\ y_4'(t) &- y_2(t) = -\cos\pi t, \\ y_1(0) &= y_2(0) = y_3(0) = y_4(0) = 0, \end{aligned} \tag{6.1}$$

has the unique solution

$$\begin{aligned} y_1(t) &= \frac{1}{4}\left[t(\pi+4)\cos\pi t + \left(\pi t^2(3\pi-2) - 1\right)\sin\pi t\right], \\ y_2(t) &= \frac{1}{4}\left[\pi t^2(2-3\pi)\cos\pi t + t(\pi+4)\sin\pi t\right], \\ y_3(t) &= \frac{1}{4}\left[t^2(2-3\pi)\cos\pi t + 7t\sin\pi t\right], \\ y_4(t) &= \frac{1}{4\pi}\left[\left(3 - \pi t^2(3\pi-2)\right)\sin\pi t - 7\pi t\cos\pi t\right]. \end{aligned} \tag{6.2}$$

*Proof.* Let $Y = Y(t) = \operatorname{col}(y_1(t), y_2(t), y_3(t), y_4(t))$ and write (6.1) in the matrix form

$$Y'(t) - D_0Y(t) = \mathbf{F}, \tag{6.3}$$

where

$$D_0 = \begin{pmatrix} 0 & -2\pi & \pi^2 & 0 \\ 2\pi & 0 & 0 & \pi^2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} \cos \pi t \\ \sin \pi t \\ 2\sin \pi t \\ -\cos \pi t \end{pmatrix}.$$

Note that $D_0$ can be written as

$$D_0 = \begin{pmatrix} 2A_0 & -A_0^2 \\ I_2 & 0_2 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0 & -\pi \\ \pi & 0 \end{pmatrix}, \quad A_0^2 = \begin{pmatrix} -\pi^2 & 0 \\ 0 & -\pi^2 \end{pmatrix}.$$

Let $\mathcal{X} = C[0,1]$, $\mathcal{X}^1 = C^1[0,1]$, $m = 2$, $X = \operatorname{col}(x_1(t), x_2(t))$. Consider the homogeneous system

$$X'(t) - A_0 X(t) = \vec{0}.$$

It can be easily shown that the fundamental matrix of this system is

$$Z = \begin{pmatrix} \cos \pi t & -\sin \pi t \\ \sin \pi t & \cos \pi t \end{pmatrix}.$$

Then from (4.2) it follows that the fundamental matrix of the homogeneous system

$$Y'(t) - D_0 Y(t) = \vec{0}$$

is

$$\mathbf{Z}(t) = \begin{pmatrix} \cos \pi t & -\sin \pi t & t\cos \pi t & -t\sin \pi t \\ \sin \pi t & \cos \pi t & t \sin \pi t & t\cos \pi t \\ \frac{1}{\pi}\sin \pi t & \frac{1}{\pi}\cos \pi t & \frac{1}{\pi^2}\cos \pi t + \frac{t}{\pi}\sin \pi t & -\frac{1}{\pi^2}\sin \pi t + \frac{t}{\pi}\cos \pi t \\ -\frac{1}{\pi}\cos \pi t & \frac{1}{\pi}\sin \pi t & -\frac{t}{\pi}\cos \pi t + \frac{1}{\pi^2}\sin \pi t & \frac{1}{\pi^2}\cos \pi t + \frac{t}{\pi}\sin \pi t \end{pmatrix}. \tag{6.4}$$

Since $\mathbf{M} \equiv \mathbf{0}$ it follows from (4.5) and (6.4) that $\det \mathbf{W} = \det \mathbf{Z}(0) = 1/\pi^4 \neq 0$ and hence by Theorem 4.1 problem (6.3) is uniquely solvable and its solution is given by (4.7), i.e.

$$Y(t) = \mathbf{Z}\int_0^t \mathbf{Z}^{-1}(s)\mathbf{F}(s)ds,$$

where

$$\mathbf{Z}^{-1}(t) = \begin{pmatrix} \cos \pi t - \pi t \sin \pi t & \pi t \cos \pi t + \sin \pi t & -\pi^2 t\cos \pi t & -\pi^2 t \sin \pi t \\ -\pi t\cos \pi t - \sin \pi t & \cos \pi t - \pi t \sin \pi t & \pi^2 t \sin \pi t & -\pi^2 t\cos \pi t \\ \pi \sin \pi t & -\pi\cos \pi t & \pi^2 \cos \pi t & \pi^2 \sin \pi t \\ \pi \cos \pi t & \pi \sin \pi t & -\pi^2 \sin \pi t & \pi^2\cos \pi t \end{pmatrix}.$$

After performing the calculations, we get solution (6.2). □

**Example 2** Let $X(t) = \operatorname{col}(x(t), y(t))$, $F(t) = \operatorname{col}(f_1(t), f_2(t))$. Find the unique solution of the problem $B_2 X(t) = F(t)$ on $C[0,1]$ defined by

$$\begin{gathered} x''(t) - 2x'(t) - 4y'(t) + 9x(t) + 8y(t) = f_1(t), \\ y''(t) - 8x'(t) - 6y'(t) + 16x(t) + 17y(t) = f_2(t), \\ x(0) = 3x(1), \quad y(0) = -2y(1), \\ x'(0) = x(0) + 2y(0) + 3x'(1) - 3x(1) - 6y(1), \\ y'(0) = 4x(0) + 3y(0) - 2y'(1) + 8x(1) + 6y(1). \end{gathered} \tag{6.5}$$

*Proof.* First we rewrite problem (6.5) in the matrix form

$$B_2\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x''(t) \\ y''(t) \end{pmatrix} - \begin{pmatrix} 2 & 4 \\ 8 & 6 \end{pmatrix}\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} + \begin{pmatrix} 9 & 8 \\ 16 & 17 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}, \tag{6.6}$$

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x(1) \\ y(1) \end{pmatrix},$$

$$\begin{pmatrix} x'(0) \\ y'(0) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x'(1) - x(1) - 2y(1) \\ y'(1) - 4x(1) - 3y(1) \end{pmatrix}.$$

If we compare problem (6.6) with (5.30), it is natural to take $\mathcal{X} = C = C[0,1]$, $\mathcal{X}^1 = C^1[0,1] = C_1$, $\mathcal{X}^2 = C^2[0,1]$, $\mathcal{X}_2^1 = C_2^1[0,1] = C_2^1$, $m = 2$, $n = 1$, $t_0 = 0$,

$S(t) = \begin{pmatrix} 2 & 4 \\ 8 & 6 \end{pmatrix}$, $Q(t) = -\begin{pmatrix} 9 & 8 \\ 16 & 17 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$,

$M = N = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$, $X(t_0) = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$, $X'(t_0) = \begin{pmatrix} x'(0) \\ y'(0) \end{pmatrix}$,

$\Psi(X) = X(1) = \begin{pmatrix} x(1) \\ y(1) \end{pmatrix}$, $\Theta(X) = \begin{pmatrix} x'(1) - x(1) - 2y(1) \\ y'(1) - 4x(1) - 3y(1) \end{pmatrix} = \begin{pmatrix} x'(1) \\ y'(1) \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x(1) \\ y(1) \end{pmatrix}$.

By Remark 1, it follows that $\Psi \in [C_2[0,1]]^*$ and $\Theta \in [C_2^1[0,1]]^*$, since $\Psi_i, \Theta_i$, $i = 1,2$ are linear and $|\Psi(X)| \leq ||X(t)||_{C_2}$ and

$$|\Theta(X)| \leq 5(||x'(t)||_C + ||y'(t)||_C + ||x(t)||_C + ||y(t)||_C) = 5(||X'(t)||_{C_2} + ||X(t)||_{C_2}) = 5||X(t)||_{C_2^1}.$$

It is easy to verify that $Q$ and $S$ satisfy (5.10), then by Theorem 5.1, there exists the operator $A$ defined by (5.12), namely

$$AX(t) = X'(t) - \frac{1}{2}S(t)X(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Note that $\Theta(X) = \Psi(AX) = (AX)(1)$, $M = N$, $T = \frac{1}{2}S(0)$, i.e. conditions (5.31) are fulfilled. Then, by Corollary 5.1, problem (6.6) is uniquelly solved if and only if (5.33) holds, namely $\det W_3 = \det[I_2 - \Psi(Z)M] \neq 0$. It is easy to verify that the fundamental matrix $Z = Z(t)$, $Z(0) = I_2$ for the system $AX(t) = \vec{0}$ has the form

$$Z = \frac{1}{3}\begin{pmatrix} e^{5t} + 2e^{-t} & e^{5t} - e^{-t} \\ 2e^{5t} - 2e^{-t} & 2e^{5t} + e^{-t} \end{pmatrix}, \quad Z^{-1} = \frac{1}{3}\begin{pmatrix} e^{-5t} + 2e^{t} & e^{-5t} - e^{t} \\ 2e^{-5t} - 2e^{t} & 2e^{-5t} + e^{t} \end{pmatrix},$$

$$\Phi(Z) = I_2, \ \det W_3 \neq 0,$$

$$W_3^{-1} = \frac{1}{e^6 - 18e^5 + 3e - 4}\begin{pmatrix} 4e^6 + 3e + 2) & 2(1 - e^6) \\ 6(e^6 - 1) & -3(e^6 - e + 2) \end{pmatrix}.$$

By Corollary 5.1, problem (6.5) has the unique solution which is given by (5.34), where $\widehat{A}^{-1}F(t) = Z(t)\int_0^t Z^{-1}(s)F(s)ds$, $\quad \Psi(\widehat{A}^{-1}F) = (\widehat{A}^{-1}F)(1)$,

$$Y(t) = \widehat{A}^{-1}F(t) + ZMW_3^{-1}(\widehat{A}^{-1}F)(1), \quad \widehat{A}^{-1}Y(t) = Z(t)\int_0^t Z^{-1}(s)Y(s)ds,$$

$$\Psi(\widehat{A}^{-1}Y) = (\widehat{A}^{-1}Y)(1).$$

Substituting these values into (5.34), we obtain the unique solution to (6.5)

$$X(t) = \widehat{A}^{-1}Y(t) + ZMW_3^{-1}(\widehat{A}^{-1}Y)(1).$$

□

**Example 3** The following system of two second-order differential equations with nonlocal boundary conditions

$$\begin{aligned} y''(t) &+ 2\pi x'(t) - \pi^2 y(t) = \sin \pi t, \\ x''(t) &- 2\pi y'(t) - \pi^2 x(t) = \cos \pi t, \\ y(0) &= -2y(1) + 2x(1), \\ x(0) &= x(1), \\ y'(0) &= -\pi x(0) - 2y'(1) - 2\pi x(1) + 2x'(1) - 2\pi y(1), \\ x'(0) &= \pi y(0) + x'(1) - \pi y(1) \end{aligned} \tag{6.7}$$

has the unique solution

$$\begin{aligned} y(t) &= \frac{t-2}{2\pi} \cos \pi t - \frac{1}{2\pi^2} \sin \pi t, \\ x(t) &= \frac{t-2}{2\pi} \sin \pi t. \end{aligned} \tag{6.8}$$

*Proof.* First we write problem (6.7) in the matrix form

$$\begin{aligned} \begin{pmatrix} y''(t) \\ x''(t) \end{pmatrix} &- \begin{pmatrix} 0 & -2\pi \\ 2\pi & 0 \end{pmatrix} \begin{pmatrix} y'(t) \\ x'(t) \end{pmatrix} - \begin{pmatrix} \pi^2 & 0 \\ 0 & \pi^2 \end{pmatrix} \begin{pmatrix} y(t) \\ x(t) \end{pmatrix} = \begin{pmatrix} \sin \pi t \\ \cos \pi t \end{pmatrix}, \\ \begin{pmatrix} y(0) \\ x(0) \end{pmatrix} &= \begin{pmatrix} -2 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y(1) \\ x(1) \end{pmatrix}, \\ \begin{pmatrix} y'(0) \\ x'(0) \end{pmatrix} &= \begin{pmatrix} 0 & -\pi \\ \pi & 0 \end{pmatrix} \begin{pmatrix} y(0) \\ x(0) \end{pmatrix} + \begin{pmatrix} -2 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y'(1) + \pi x(1) \\ x'(1) - \pi y(1) \end{pmatrix}. \end{aligned} \tag{6.9}$$

If we compare problem (6.9) with (5.30), it is natural to take $\mathcal{X} = C[0,1]$, $\mathcal{X}^1 = C^1[0,1]$, $\mathcal{X}^2 = C^2[0,1]$, $m = 2$, $n = 1$, $t_0 = 0$,
$S(t) = \begin{pmatrix} 0 & -2\pi \\ 2\pi & 0 \end{pmatrix}$, $Q(t) = \begin{pmatrix} \pi^2 & 0 \\ 0 & \pi^2 \end{pmatrix}$, $T = \begin{pmatrix} 0 & -\pi \\ \pi & 0 \end{pmatrix}$,
$M = N = \begin{pmatrix} -2 & 2 \\ 0 & 1 \end{pmatrix}$, $X(t) = \begin{pmatrix} y(t) \\ x(t) \end{pmatrix}$, $X(t_0) = \begin{pmatrix} y(0) \\ x(0) \end{pmatrix}$, $X'(t_0) = \begin{pmatrix} y'(0) \\ x'(0) \end{pmatrix}$,
$\Psi(X) = \begin{pmatrix} y(1) \\ x(1) \end{pmatrix}$, $\Theta(X) = \begin{pmatrix} y'(1) + \pi x(1) \\ x'(1) - \pi y(1) \end{pmatrix}$, $F(t) = \begin{pmatrix} \sin \pi t \\ \cos \pi t \end{pmatrix}$.
By Remark 1, it follows that $\Psi \in [C_2^1[0,1]]^*$ and $\Theta \in [C_2^1[0,1]]^*$, since $\Psi_i, \Theta_i$, $i = 1, 2$ are linear and

$$|\Psi(X)| \leq ||X(t)||_{C_2}, \quad |\Theta(X)| \leq \pi ||X(t)||_{C_2^1}.$$

It is easy to verify that $Q$ and $S$ satisfy (5.10), then, by Theorem 5.1, there exists the operator $A$ defined by (5.12), namely

$$AX(t) = X'(t) - \frac{1}{2} S(t) X(t) = \begin{pmatrix} y'(t) \\ x'(t) \end{pmatrix} - \begin{pmatrix} 0 & -\pi \\ \pi & 0 \end{pmatrix} \begin{pmatrix} y(t) \\ x(t) \end{pmatrix}.$$

Let $Z = Z(t)$, $Z(0) = I_2$ be a fundamental matrix to the system $AX(t) = \vec{0}$. Note that $\Theta(X) = \Psi(AX) = (AX)(1)$, $M = N$, $T = \frac{1}{2} S(0)$, i.e. conditions (5.31) are fulfilled. Then, by Corollary 5.1, problem (6.7) is uniquelly solved if and only if (5.33) holds, namely $\det W_3 = \det[I_2 - \Psi(Z) M] \neq 0$. It is easy to verify that

$$Z = \begin{pmatrix} \cos \pi t & -\sin \pi t \\ \sin \pi t & \cos \pi t \end{pmatrix}, \; Z^{-1} = \begin{pmatrix} \cos \pi t & \sin \pi t \\ -\sin \pi t & \cos \pi t \end{pmatrix}, \; \Psi(Z) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\Phi(Z) = Z(0) = I_2, \quad \det W_3 = \det \begin{pmatrix} -1 & 2 \\ 0 & -2 \end{pmatrix} \neq 0, \quad W_3^{-1} = -\begin{pmatrix} 1 & 1 \\ 0 & 1/2 \end{pmatrix}.$$

By Corollary 5.1, problem (6.7) has solution given by (5.34), where $\widehat{A}^{-1}F(t) = Z(t)\int_0^t Z^{-1}(s)F(s)ds = \begin{pmatrix} 0 \\ \frac{1}{\pi}\sin\pi t \end{pmatrix}$, $\Psi(\widehat{A}^{-1}F) = (\widehat{A}^{-1}F)(1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,
$Y(t) = \widehat{A}^{-1}F(t) + ZMW_3^{-1}\Psi(\widehat{A}^{-1}F) = \begin{pmatrix} 0 \\ \frac{1}{\pi}\sin\pi t \end{pmatrix}$,
$\widehat{A}^{-1}Y(t) = Z(t)\int_0^t Z^{-1}(s)Y(s)ds = \begin{pmatrix} \frac{t}{2\pi}\cos\pi t - \frac{1}{2\pi^2}\sin\pi t \\ \frac{t}{2\pi}\sin\pi t \end{pmatrix}$,
$\Psi(\widehat{A}^{-1}Y) = (\widehat{A}^{-1}Y)(1) = \begin{pmatrix} -\frac{1}{2\pi} \\ 0 \end{pmatrix}$.
Substituting these values into (5.34) we obtain the unique solution to (6.7)

$$X(t) = \widehat{A}^{-1}Y(t) + ZMW_3^{-1}\Psi(\widehat{A}^{-1}Y) = \begin{pmatrix} \frac{t-2}{2\pi}\cos\pi t - \frac{1}{2\pi^2}\sin\pi t \\ \frac{t-2}{2\pi}\sin\pi t \end{pmatrix},$$

which gives (6.8). □

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## References

[1] V.M. Abdullaev, *Solution of differential equations with nonseparated multipoint and integral conditions*, Sib. Zh. Ind. Mat. 15 (2012), no. 3, 3–15 (in Russian).

[2] V.M. Abdullaev, K.R. Aida-Zade, *On the numerical solution of loaded systems of ordinary differential equations*, Zurnal Vychislitel'noj Matematiki i Matematicheskoj Fiziki 44 (2004), no. 9, 1585–1595 (in Russian).

[3] A. Abildayeva, A. Assanova, A. Imanchiyev, *A multi-point problem for a system of differential equations with piecewise-constant argument of generalized type as a neural network model*, Eurasian Mathematical Journal 13 (2022), no. 2, 8–17.

[4] A.A. Abramov, L.F. Yukhno, *Solving a system of linear ordinary differential equations with redundant conditions*, Computational Mathematics and Mathematical Physics, Pleiades Publishing, Ltd. 54 (2014), no. 4, 598–603.

[5] M.M. Baiburin, *On multipoint boundary value problems for first order linear differential equations, system*, Applied Mathematics and Control Science (2018), no. 6, 16–30 (in Russian).

[6] J. Biazar, E. Babolian, R. Islam, *Solution of the system of ordinary differential equations by Adomian decomposition method*, Applied Mathematics and Computation 147 (2004), 713–719.

[7] V.A. Churikov, *Solution of a multipoint boundary-value problems for a system of linear ordinary differential equations with holomorphic coefficients*, Ukrainian Mathematical Journal 42 (1990), 113–115.

[8] A.M. Fink, J.A. Gatica, *Positive solutions of second order system of boundary value problems*, J. Math. Anal. Appl. 180 (1993), 93–108.

[9] G.L. Karakostas, P.Ch. Tsamatos, *Nonlocal boundary value problems for ordinary differential systems of higher order*, Nonlin. Anal. 51 (2002), 1421–1427.

[10] I.T. Kiguradze, *Boundary value problems for systems of ordinary differential equations*, Itogi Nauki i Tekhniki. Ser. Sovrem. Probl. Mat. Nov. Dostizh. 30 (1987), 3–103 (in Russian).

[11] B.K. Kokebaev, M. Otelbaev, A.N. Shynybekov, *About restrictions and extensions of operators*, DAN SSSR 271 (1983), no. 6, 1307–1310 (in Russian).

[12] A.M. Krall, *The development of general differential and general differential-boundary systems*, Rocky Mountain Journal of Mathematics 5 (1975), no. 4, 493–542.

[13] N.T. Orumbayeva, A.T. Assanova, A.B. Keldibekova, *On an algorithm of finding an approximate solution of a periodic problem for a third-order differential equation*, Eurasian Mathematical Journal 13 (2022), no. 1, 69–85.

[14] I.N. Parasidis, P.C. Tsekrekos, *Some quadratic correct extensions of minimal operators in Banach space*, Operators and Matrices. Zagreb 4 (2010), no. 2, 225–243.

[15] Pham Ky Anh, *Multipoint boundary-value problems for transferable differential-algebraic equations. I-Linear case*, Vietnam Journal of Mathematics 25 (1997), no. 4, 347–358.

[16] Zh.A. Sartabanov, G.M. Aitenova, G.A. Abdikalikova, *Multiperiodic solutions of quasilinear systems of integro-differential equations with $D_\epsilon$-operator and $\epsilon$-period of hereditarity*, Eurasian Math. J. 13 (2022), no. 1, 86–100.

[17] S. Sekar, M. Nalini, *Numerical analysis of different second order systems using Adomian decomposition method*, Applied Mathematical Sciences 8 (2014), no. 77, 3825–3832.

[18] H.L. Smith, *Systems of ordinary differential equations which generate an order preserving flow*, A survey of results. SIAM review (1988), 87–113.

[19] S. Timoshenko, *Theory of elastic stability,* McGraw-Hill, New-York. 1961.

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