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NIKOL'SKII-BESOV SPACES WITH A DOMINANT MIXED DERIVATIVE AND WITH A MIXED METRIC: INTERPOLATION PROPERTIES, EMBEDDING THEOREMS, TRACE AND EXTENSION THEOREMS

K.A. Bekmaganbetov, K.Ye. Kervenev, E.D. Nursultanov

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Key words: Nikol'skii-Besov spaces, dominant mixed derivative, mixed metric, anisotropic Lorentz spaces, interpolation of spaces, embedding theorems, trace theorems, extension theorems.

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Abstract. In this work, we define Nikol'skii-Besov spaces with a dominant mixed derivative and with a mixed metric. The interpolation properties of these spaces with respect to the anisotropic interpolation method are studied, sharp embedding theorems of different metrics are proved, and sharp trace and extension theorems are proved.

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1 Introduction

The theory of embeddings of spaces of differentiable functions originated in the work of S.L. Sobolev [26]. This theory studies important connections between differential (smoothness) properties of functions in various metrics. Further development of this theory is assosicated with new classes of function spaces introduced by S.M. Nikol'skii, O.V. Besov, P.I. Lizorkin, H. Triebel, and others. This development was driven both by intrinsic questions of the theory and by applications to the theory of boundary value problems of mathematical physics, approximation theory, and other areas of analysis (see, for example, monographs [13, 17, 22, 28]).

In the 1960s, the study of spaces with a dominant mixed derivative was initiated in the works of S.M. Nikol'skii [23], A.D. Dzhabrailov [20], and T.I. Amanov [3]. Further research of these spaces in connection with the theory of embeddings, interpolation, and approximation theory, is associated with the works of A.P. Uninskii, O.V. Besov, V.N. Temlyakov, E.D. Nursultanov, D.B. Bazarkhanov, A.S. Romanyuk, G.A. Akishev, K.A. Bekmaganbetov, Ye. Toleugazy, and others (see, for example, [29, 30, 15, 16, 27, 24, 5, 6, 25, 1, 2, 12]).

In Section 2, we define Nikol'skii-Besov spaces with a dominant mixed derivative and with a mixed metric, and study some elementary embedding properties. In Section 3, we study interpolation properties of these spaces with respect to the anisotropic interpolation method. In Section 4, we prove sharp embedding theorems of different metrics for the introduced spaces and anisotropic Lorentz spaces. In Section 5, we prove trace and extension theorems for the spaces under consideration.

2 Main definitions

By generalizing the construction in [22, Chapter 8], we define the Nikol'skii-Besov spaces with a dominant mixed derivative and with a mixed metric $S_{\mathbf{p}}^{\alpha\mathbf{q}}B(\mathbb{R}^n)$.

Let $\mathbf{1} \le \mathbf{p} = (p_1, \dots, p_n) \le \infty$. The Lebesgue space with a mixed metric $L_{\mathbf{p}}(\mathbb{R}^n)$ is the set of measurable functions for which the following norm is finite

$$\|f\|_{L_{\mathbf{p}}(\mathbb{R}^n)} = \left(\int_{-\infty}^{\infty} \left(\dots \left(\int_{-\infty}^{\infty} |f(x_1, \dots, x_n)|^{p_1} \, dx_1 \right)^{p_2/p_1} \dots \right)^{p_n/p_{n-1}} dx_n \right)^{1/p_n}.$$

Here, for $p = \infty$ the expression $\left(\int_{-\infty}^{\infty} |f(t)|^p \, dt \right)^{1/p}$ is understood as $\operatorname{esssup}_{t \in \mathbb{R}} |f(t)|$.

A generalized function f is called regular in the sense of $L_{\mathbf{p}}(\mathbb{R}^n)$ if for some $\rho_0 > 0$

$$I_{\rho_0} f = F \in L_{\mathbf{p}}(\mathbb{R}^n),$$

where

$$I_{\rho_0} f = \mathfrak{F}^{-1} \left(\left(1 + |\xi|^2\right)^{-\rho_0/2} \mathfrak{F}(f) \right),$$

and $\mathfrak{F}$ and $\mathfrak{F}^{-1}$ are the direct and inverse Fourier transforms, respectively.

Let f be a regular function in the sense of $L_{\mathbf{p}}(\mathbb{R}^n)$. A regular expansion of a function f in the sense of $L_{\mathbf{p}}(\mathbb{R}^n)$ over the Vallee-Poussin sums is the following representation

$$f = \sum_{\mathbf{s} \in \mathbb{Z}_+^n} Q_{\mathbf{s}}(f),$$

where

$$Q_{\mathbf{s}}(f) = \frac{1}{\pi^n} I_{-\rho} \left(\prod_{i=1}^{n} \left(V_{2^{s_i}}(x_i) - V_{\left[2^{s_i - 1}\right]}(x_i) \right) * I_\rho f \right),$$

where $\rho > 0$ is sufficiently large so that $I_\rho f \in L_{\mathbf{p}}(\mathbb{R}^n)$, $V_M(t) = \dfrac{1}{M} \displaystyle\int_M^{2M} \frac{\sin \lambda t}{t} d\lambda$ is an analogue of the Vallee-Poussin kernel for the parameter $M > 0$ and $V_0(t) \equiv 0$.

Let further $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ and $\mathbf{1} \le \mathbf{q} = (q_1, \dots, q_n) \le \infty$. The Nikol'skii-Besov space $S_{\mathbf{p}}^{\alpha \mathbf{q}} B(\mathbb{R}^n)$ with a dominant mixed derivative and with a mixed metric is the set of regular in the sense of $L_{\mathbf{p}}(\mathbb{R}^n)$ functions f for which the following norm is finite

$$\|f\|_{S_{\mathbf{p}}^{\alpha \mathbf{q}} B(\mathbb{R}^n)} = \left\| \left\{ 2^{(\alpha, \mathbf{s})} \|Q_{\mathbf{s}}(f)\|_{L_{\mathbf{p}}(\mathbb{R}^n)} \right\}_{\mathbf{s} \in \mathbb{Z}_+^{\mathbf{n}}} \right\|_{l_{\mathbf{q}}},$$

where $(\alpha, \mathbf{s}) = \displaystyle\sum_{j=1}^{n} \alpha_j s_j$ is the inner product and $\|\cdot\|_{l_{\mathbf{q}}}$ is the norm of the discrete Lebesgue space $l_{\mathbf{q}}$ with a mixed metric.

Remark 1. In the case in which $\alpha = (\alpha_1, \dots, \alpha_n) > \mathbf{0}$, these spaces with the parameter $\mathbf{q} = (\infty, \dots, \infty)$ were introduced and studied in the works [29, 30]. The case of $\mathbf{p} = (p, \dots, p)$ and $\mathbf{q} = (q, \dots, q)$ was considered in the works [23, 20, 3]. Periodic analogues of these spaces were studied in the series of work by K.A. Bekmaganbetov, K.Ye. Kervenev and Ye. Toleugazy [7, 8, 9].

The following lemma shows some elementary embeddings of Nikol'skii-Besov spaces with a dominant mixed derivative and with a mixed metric.

Lemma 2.1. a) *Let* $\mathbf{1} \le \mathbf{q_0} = (q_1^0, \dots, q_n^0) \le \mathbf{q_1} = (q_1^1, \dots, q_n^1) \le \infty$, *then*

$$S_{\mathbf{p}}^{\alpha \mathbf{q_0}} B(\mathbb{R}^n) \hookrightarrow S_{\mathbf{p}}^{\alpha \mathbf{q_1}} B(\mathbb{R}^n).$$

b) *Let* $\alpha_{\mathbf{0}} = (\alpha_1^0, \dots, \alpha_n^0) < \alpha_{\mathbf{1}} = (\alpha_1^1, \dots, \alpha_n^1)$ *and* $\mathbf{1} \le \mathbf{q_0} = (q_1^0, \dots, q_n^0), \mathbf{q_1} = (q_1^1, \dots, q_n^1) \le \infty$, *then*

$$S_{\mathbf{p}}^{\alpha_{\mathbf{1}} \mathbf{q_1}} B(\mathbb{R}^n) \hookrightarrow S_{\mathbf{p}}^{\alpha_{\mathbf{0}} \mathbf{q_0}} B(\mathbb{R}^n).$$

Proof. The proof of statement a) follows from Jensen's inequality.

Let us prove statement b). According to paragraph a), for $\alpha_{\mathbf{0}} < \alpha_{\mathbf{1}}$ it suffices to prove the embedding

$$S_{\mathbf{p}}^{\alpha_{\mathbf{1}}\infty} B(\mathbb{R}^n) \hookrightarrow S_{\mathbf{p}}^{\alpha_{\mathbf{0}}\mathbf{1}} B(\mathbb{R}^n). \tag{2.1}$$

We have

$$\|f\|_{S_{\mathbf{p}}^{\alpha_{\mathbf{0}}\mathbf{1}} B(\mathbb{R}^n)} = \sum_{\mathbf{s}\in\mathbb{Z}_+^n} 2^{(\alpha_0,\mathbf{s})} \|Q_{\mathbf{s}}(f)\|_{L_{\mathbf{p}}(\mathbb{R}^n)}$$

$$\le \sup_{\mathbf{s}\in\mathbb{Z}_+^n} 2^{(\alpha_1,\mathbf{s})} \|Q_{\mathbf{s}}(f)\|_{L_{\mathbf{p}}(\mathbb{R}^n)} \sum_{\mathbf{s}\in\mathbb{Z}_+^n} 2^{(\alpha_0-\alpha_1,\mathbf{s})} = C_1 \|f\|_{S_{\mathbf{p}}^{\alpha_{\mathbf{1}}\infty} B(\mathbb{R}^n)}.$$

This inequality shows that embedding (2.1) holds. □

3 Interpolation

Let us give the definition of the anisotropic interpolation method. Let $E = \{0,1\}^n$, $\mathbf{A} = \{A_\varepsilon\}_{\varepsilon\in E}$ be a family of Banach spaces that are subspaces of some linear Hausdorff space. This family $\mathbf{A}$ is called a compatible family of Banach spaces (see [10, 21, 24]). For $\mathbf{t} \in \mathbb{R}_+^n$, we define the functional

$$K(\mathbf{t}, a; \mathbf{A}) = \inf_{a=\sum_{\varepsilon\in E} a_\varepsilon} \sum_{\varepsilon\in E} \mathbf{t}^\varepsilon \|a_\varepsilon\|_{A_\varepsilon},$$

where a is an element of the space $\sum_{\varepsilon\in E} A_\varepsilon$ and $\mathbf{t}^\varepsilon = t_1^{\varepsilon_1} \cdot \ldots \cdot t_n^{\varepsilon_n}$.

Let $\mathbf{0} < \theta = (\theta_1, \ldots, \theta_n) < \mathbf{1}$ and $\mathbf{0} < \mathbf{r} = (r_1, \ldots, r_n) \le \infty$. Let $\mathbf{A}_{\theta\mathbf{r}} = (A_\varepsilon; \varepsilon \in E)_{\theta\mathbf{r}}$ denote the linear subspace of the space $\sum_{\varepsilon\in E} A_\varepsilon$ such that

$$\|a\|_{\mathbf{A}_{\theta\mathbf{r}}} =$$

$$= \left(\int_0^\infty \left(\ldots \left(\int_0^\infty \left(t_1^{-\theta_1} \ldots t_n^{-\theta_n} K(\mathbf{t}, a; \mathbf{A}) \right)^{r_1} \frac{dt_1}{t_1} \right)^{r_2/r_1} \ldots \right)^{r_n/r_{n-1}} \frac{dt_n}{t_n} \right)^{1/r_n} < \infty.$$

Lemma 3.1 ([4, 24]). *Let $\mathbf{0} < \theta < \mathbf{1}$, $\mathbf{0} < \mathbf{r} \le \infty$, and let $\mathbf{A} = \{A_\varepsilon\}_{\varepsilon\in E}, \mathbf{B} = \{B_\varepsilon\}_{\varepsilon\in E}$ be two compatible families of Banach spaces. If there are two vectors $\mathbf{M_0} = (M_1^0, \ldots, M_n^0), \mathbf{M_1} = (M_1^1, \ldots, M_n^1)$ with positive components such that for a linear operator T holds $T : \mathbf{A}_\varepsilon \to \mathbf{B}_\varepsilon$ with the operator norm bounded by $C_\varepsilon \prod_{i=1}^n M_i^{\varepsilon_i}$ for any $\varepsilon \in E$, where $C_\epsilon > 0$ is independent of $M_i^{\varepsilon_i}$, $i = 1, \ldots, n$, then*

$$T : \mathbf{A}_{\theta\mathbf{r}} \to \mathbf{B}_{\theta\mathbf{r}},$$

with the norm

$$\|T\|_{\mathbf{A}_{\theta\mathbf{r}}\to\mathbf{B}_{\theta\mathbf{r}}} \le \max_{\varepsilon\in E} C_\varepsilon \prod_{i=1}^n \left(M_i^0\right)^{1-\theta_i} \left(M_i^1\right)^{\theta_i}.$$

Let multi-indices $\mathbf{p} = (p_1, \ldots, p_n)$, $\mathbf{r} = (r_1, \ldots, r_n)$ be such that if $1 \le p_i < \infty$, then $1 \le r_i \le \infty$, and if $p_i = \infty$, then $r_i = \infty$ $(i = 1, \ldots, n)$.

The anisotropic Lorentz space $L_{\mathbf{pr}}(\mathbb{R}^n)$ is the set of all functions $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ such that

$$\|f\|_{L_{\mathbf{pr}}(\mathbb{R}^n)} =$$

$$= \left(\int_0^\infty \left(\dots \left(\int_0^\infty \left(t_1^{1/p_1} \dots t_n^{1/p_n} f^{*_1,\dots,*_n}(t_1,\dots,t_n) \right)^{r_1} \frac{dt_1}{t_1} \right)^{r_2/r_1} \dots \right)^{r_n/r_{n-1}} \frac{dt_n}{t_n} \right)^{1/r_n} < \infty,$$

where $f^*(\mathbf{t}) = f^{*_1,\dots,*_n}(t_1,\dots,t_n)$ is the repeated non-increasing rearrangement of the function f (see [18]).

Let us denote $\mathbf{b}_\varepsilon = (b_1^{\varepsilon_1},\dots,b_n^{\varepsilon_n})$ for multi-indices $\mathbf{b}_0 = (b_1^0,\dots,b_n^0), \mathbf{b}_1 = (b_1^1,\dots,b_n^1)$, and $\varepsilon = (\varepsilon_1,\dots,\varepsilon_n) \in E$.

Lemma 3.2 ([24]). *Let* $\mathbf{1} \le \mathbf{p_0} = (p_1^0,\dots,p_n^0) \ne \mathbf{p_1} = (p_1^1,\dots,p_n^1) \le \infty$. *Then for* $\mathbf{0} < \theta = (\theta_1,\dots,\theta_n) < \mathbf{1}$ *and* $\mathbf{1} \le \mathbf{r} = (r_1,\dots,r_n) \le \infty$ *holds*

$$(L_{\mathbf{p}_\varepsilon}(\mathbb{R}^n); \varepsilon \in E)_{\theta\mathbf{r}} = L_{\mathbf{pr}}(\mathbb{R}^n),$$

where $\mathbf{1}/\mathbf{p} = (\mathbf{1} - \theta)/\mathbf{p_0} + \theta/\mathbf{p_1}$.

Let $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{R}^n$ and $\mathbf{1} \le \mathbf{q} = (\mathbf{q_1},\dots,\mathbf{q_n}) \le \infty$. We will denote by $l_{\mathbf{q}}^\alpha(A)$ the set of multi-sequences $\{a_\mathbf{k}\}_{\mathbf{k}\in\mathbb{Z}^n}$ with values in a Banach space A for which the following norm is finite:

$$\|a\|_{l_{\mathbf{q}}^\alpha(A)} = \left(\sum_{\mathbf{k}\in\mathbb{Z}^n} \left(2^{(\alpha,\mathbf{k})} \|a_\mathbf{k}\|_A \right)^{\mathbf{q}} \right)^{\mathbf{1/q}}.$$

Remark 2. The norm of the space $S_{\mathbf{p}}^{\alpha\mathbf{q}}B(\mathbb{R}^n)$ can be written as

$$\|f\|_{S_{\mathbf{p}}^{\alpha\mathbf{q}}B(\mathbb{R}^n)} = \left\| \{Q_\mathbf{s}(f)\}_{\mathbf{s}\in\mathbb{Z}_+^\mathbf{n}} \right\|_{l_{\mathbf{q}}^\alpha(L_\mathbf{p}(\mathbb{R}^n))}.$$

We will need this form of the norm when describing interpolation properties of the spaces $S_{\mathbf{p}}^{\alpha\mathbf{q}}B(\mathbb{R}^n)$.

Lemma 3.3 ([7]). *Let* $\alpha_0 = (\alpha_1^0,\dots,\alpha_n^0) \ne \alpha_1 = (\alpha_1^1,\dots,\alpha_n^1)$, $\mathbf{1} \le \mathbf{q_0} = (q_1^0,\dots,q_n^0), \mathbf{q_1} = (q_1^1,\dots,q_n^1) \le \infty$. *Then for* $\mathbf{0} < \theta = (\theta_1,\dots,\theta_n) < \mathbf{1}$, $\mathbf{1} \le \mathbf{q} = (q_1,\dots,q_n) \le \infty$

$$\left(l_{\mathbf{q}_\varepsilon}^{\alpha_\varepsilon}(A); \varepsilon \in E \right)_{\theta\mathbf{q}} = l_{\mathbf{q}}^\alpha(A),$$

where $\alpha = (\mathbf{1} - \theta)\alpha_\mathbf{0} + \theta\alpha_\mathbf{1}$.

Definition 1. Let A and B be Banach spaces. An operator $R \in L(A,B)$ is called a *retraction* if there exists an operator $S \in L(B,A)$ such that

$$RS = E \quad \text{(the identity operator in } L(B,B)).$$

In this case, the operator S is called a *coretraction* (corresponding to R).

Lemma 3.4. *Let* $-\infty < \alpha = (\alpha_1,\dots,\alpha_n) < \infty$, $\mathbf{1} \le \mathbf{p} = (p_1,\dots,p_n) < \infty$, *and* $\mathbf{1} \le \mathbf{q} = (q_1,\dots,q_n) \le \infty$. *Then the space* $S_{\mathbf{p}}^{\alpha\mathbf{q}}B(\mathbb{R}^n)$ *is a retraction of the space* $l_{\mathbf{q}}^\alpha(L_\mathbf{p}(\mathbb{R}^n))$.

Proof. First step. For a function $f \in S_{\mathbf{p}}^{\alpha\mathbf{q}}B(\mathbb{R}^n)$ we define the operator S by

$$Sf = \{Q_\mathbf{s}(f)\}_{\mathbf{s}\in\mathbb{Z}_+^n}.$$

Therefore, according to the definition, we have

$$\|Sf\|_{l_{\mathbf{q}}^\alpha(L_\mathbf{p}(\mathbb{R}^n))} = \|\{Q_\mathbf{s}(f)\}\|_{l_{\mathbf{q}}^\alpha(L_\mathbf{p}(\mathbb{R}^n))} = \|f\|_{S_{\mathbf{p}}^{\alpha\mathbf{q}}B(\mathbb{R}^n)},$$

which means that the S-property is satisfied.

Second step. For a sequence $G = \{g_{\mathbf{s}}\}_{\mathbf{s}\in\mathbb{Z}_+^n}$, we define the operator R by

$$RG = \sum_{\mathbf{s}\in\mathbb{Z}_+^n} U_{\mathbf{s}} * g_{\mathbf{s}},$$

where

$$U_{\mathbf{s}}(\mathbf{x}) = \frac{1}{\pi^n}\prod_{i=1}^{n}\left(V_{2^{s_i+1}}(x_i) - V_{\left[2^{s_i-2}\right]}(x_i)\right).$$

Since $V_M \in L_1(\mathbb{R})$, we obtain

$$\|U_{\mathbf{s}} * g\|_{L_{\mathbf{p}}(\mathbb{R}^n)} \le C_2 \|g\|_{L_{\mathbf{p}}(\mathbb{R}^n)},$$

where C_2 is an absolute constant, and then

$$\|RG\|_{S_{\mathbf{p}}^{\alpha\mathbf{q}}B(\mathbb{R}^n)} = \|\{Q_{\mathbf{s}}(U_{\mathbf{s}} * g_{\mathbf{s}})\}\|_{l_{\mathbf{q}}^{\alpha}(L_{\mathbf{p}}(\mathbb{R}^n))} = \|\{Q_{\mathbf{s}}(g_{\mathbf{s}})\}\|_{l_{\mathbf{q}}^{\alpha}(L_{\mathbf{p}}(\mathbb{R}^n))} \le$$

$$\le C_2 \|\{g_{\mathbf{s}}\}\|_{l_{\mathbf{q}}^{\alpha}(L_{\mathbf{p}}(\mathbb{R}^n))} = C_2 \|G\|_{l_{\mathbf{q}}^{\alpha}(L_{\mathbf{p}}(\mathbb{R}^n))}.$$

The last inequality means that the R-property holds.

Third step. Let us show that $RS = E$. Indeed,

$$RSf = R(\{Q_{\mathbf{s}}(f)\}) = \sum_{\mathbf{s}\in\mathbb{Z}_+^n} U_{\mathbf{s}} * Q_{\mathbf{s}}(f) = \sum_{\mathbf{s}\in\mathbb{Z}_+^n} Q_{\mathbf{s}}(f) = f.$$

□

Theorem 3.1. *Let* $\mathbf{1} \le \mathbf{p} = (p_1, \ldots, p_n) < \infty$, $\alpha_{\mathbf{0}} = (\alpha_1^0, \ldots, \alpha_n^0) \ne \alpha_{\mathbf{1}} = (\alpha_1^1, \ldots, \alpha_n^1)$, $\mathbf{1} \le \mathbf{q_0} = (q_1^0, \ldots, q_n^0), \mathbf{q_1} = (q_1^1, \ldots, q_n^1) \le \infty$, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in E$. *Then for* $\mathbf{0} < \theta = (\theta_1, \ldots, \theta_n) < \mathbf{1}$ *and* $\mathbf{1} \le \mathbf{q} = (q_1, \ldots, q_n) \le \infty$

$$\left(S_{\mathbf{p}}^{\alpha_\varepsilon \mathbf{q}_\varepsilon}B(\mathbb{R}^n); \varepsilon \in E\right)_{\theta\mathbf{q}} = S_{\mathbf{p}}^{\alpha\mathbf{q}}B(\mathbb{R}^n),$$

where $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$.

Proof. The proof of the theorem follows by Lemmas 3.3 and 3.4. □

4 Embedding theorems

In this section, the sharp embedding theorems for Nikol'skii-Besov spaces with a dominant mixed derivative and with a mixed metric and for anisotropic Lorentz spaces are proved.

Lemma 4.1 (Inequality of different metrics, [22]). *Let* $Q_{\mathbf{s}}(\mathbf{x})$ *be an entire function of exponential type of order* $\mathbf{s} = (s_1, \ldots, s_n)$ *by* $\mathbf{x} = (x_1, \ldots, x_n)$. *Then for* $\mathbf{1} \le \mathbf{p}_0 = (p_1^0, \ldots, p_n^0) < \mathbf{p}_1 = (p_1^1, \ldots, p_n^1) < \infty$ *holds*

$$\|Q_{\mathbf{s}}\|_{L_{\mathbf{p_1}}(\mathbb{R}^n)} \le C_3 \prod_{i=1}^{n} s_i^{1/p_i^0 - 1/p_i^1} \|Q_{\mathbf{s}}\|_{L_{\mathbf{p_0}}(\mathbb{R}^n)},$$

where C_3 *is a positive constant independent of* $\mathbf{s}$.

Theorem 4.1. *Let* $-\infty < \alpha_0 = (\alpha_1^0, \ldots, \alpha_n^0) \le \alpha_1 = (\alpha_1^1, \ldots, \alpha_n^1) < \infty$, $\mathbf{1} \le \tau = (\tau_1, \ldots, \tau_n) \le \infty$, *and* $\mathbf{1} \le \mathbf{p}_0 = (p_1^0, \ldots, p_n^0), \mathbf{p}_1 = (p_1^1, \ldots, p_n^1) < \infty$. *Then*

$$S_{\mathbf{p_1}}^{\alpha_1\tau}B(\mathbb{R}^n) \hookrightarrow S_{\mathbf{p_0}}^{\alpha_0\tau}B(\mathbb{R}^n)$$

for $\alpha_0 - 1/\mathbf{p}_0 = \alpha_1 - 1/\mathbf{p}_1$.

Proof. Let $f \in S^{\alpha_1\tau}_{\mathbf{p}_1}B(\mathbb{R}^n)$. Then, according to the inequality of different metrics (Lemma 4.1), we obtain

$$\|f\|_{S^{\alpha_0\tau}_{\mathbf{p}_0}B(\mathbb{R}^n)} = \left\|\left\{2^{(\alpha_0,\mathbf{s})}\|Q_{\mathbf{s}}(f)\|_{L_{\mathbf{p}_0}(\mathbb{R}^n)}\right\}\right\|_{l_\tau}$$
$$\leq C_3\left\|\left\{2^{(\alpha_0+1/\mathbf{p}_1-1/\mathbf{p}_0,\mathbf{s})}\|Q_{\mathbf{s}}(f)\|_{L_{\mathbf{p}_1}(\mathbb{R}^n)}\right\}\right\|_{l_\tau}$$
$$= C_3\left\|\left\{2^{(\alpha_1,\mathbf{s})}\|Q_{\mathbf{s}}(f)\|_{L_{\mathbf{p}_1}(\mathbb{R}^n)}\right\}\right\|_{l_\tau} = C_3\|f\|_{S^{\alpha_1\tau}_{\mathbf{p}_1}B(\mathbb{R}^n)}.$$

□

Theorem 4.2. *Let $\mathbf{1} \leq \mathbf{p} = (p_1,\ldots,p_n) < \mathbf{q} = (q_1,\ldots,q_n) < \infty$ and $\mathbf{1} \leq \tau = (\tau_1,\ldots,\tau_n) \leq \infty$. Then*

$$S^{\alpha\tau}_{\mathbf{p}}B(\mathbb{R}^n) \hookrightarrow L_{\mathbf{q}\tau}(\mathbb{R}^n)$$

for $\alpha = 1/\mathbf{p} - 1/\mathbf{q}$.

Proof. According to Minkowski's inequality and the inequality of different metrics (Lemma 4.1), we obtain

$$\|f\|_{L_{\mathbf{q}}(\mathbb{R}^n)} = \left\|\sum_{\mathbf{s}=\mathbf{0}}^{\infty} Q_{\mathbf{s}}(f)\right\|_{L_{\mathbf{q}}(\mathbb{R}^n)}$$

$$\leq \sum_{\mathbf{s}=\mathbf{0}}^{\infty}\|Q_{\mathbf{s}}(f)\|_{L_{\mathbf{q}}(\mathbb{R}^n)} \leq C_3\sum_{\mathbf{s}=\mathbf{0}}^{\infty}2^{(1/\mathbf{p}-1/\mathbf{q},\mathbf{s})}\,\|Q_{\mathbf{s}}(f)\|_{L_{\mathbf{p}}(\mathbb{R}^n)} = C_3\|f\|_{S^{\alpha\mathbf{1}}_{\mathbf{p}}B(\mathbb{R}^n)},$$

where $\alpha = 1/\mathbf{p} - 1/\mathbf{q}$.

Therefore, for $\alpha = 1/\mathbf{p} - 1/\mathbf{q}$ we get

$$S^{\alpha\mathbf{1}}_{\mathbf{p}}B(\mathbb{R}^n) \hookrightarrow L_{\mathbf{q}}(\mathbb{R}^n).$$

Let us fix $\mathbf{p} = (p_1,\ldots,p_n)$ and let us choose $\alpha_i = (\alpha^i_1,\ldots,\alpha^i_n)$ and $\mathbf{q}_i = (q^i_1,\ldots,q^i_n)$ such that $\alpha^i_j = 1/p_j - 1/q^i_j$, where $i = 0,1$ and $j = 1,\ldots,n$. Then for every $\varepsilon \in E$ we have

$$S^{\alpha_\varepsilon\mathbf{1}}_{\mathbf{p}}B(\mathbb{R}^n) \hookrightarrow L_{\mathbf{q}_\varepsilon}(\mathbb{R}^n).$$

According to Lemma 3.2 and Theorem 3.1 we obtain

$$\left(S^{\alpha_\varepsilon\mathbf{1}}_{\mathbf{p}}B(\mathbb{R}^n);\varepsilon \in E\right)_{\theta\tau} \hookrightarrow \left(L_{\mathbf{q}_\varepsilon}(\mathbb{R}^n);\varepsilon \in E\right)_{\theta\tau}$$

or

$$S^{\alpha\tau}_{\mathbf{p}}B(\mathbb{R}^n) \hookrightarrow L_{\mathbf{q}\tau}(\mathbb{R}^n),$$

where $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $1/\mathbf{q} = (1-\theta)/\mathbf{q}_0 + \theta/\mathbf{q}_1$.

Let us check the relationship between the parameters α, $\mathbf{p}$ and $\mathbf{q}$

$$\alpha = (1-\theta)\alpha_0 + \theta\alpha_1 = (1-\theta)\left(1/\mathbf{p} - 1/\mathbf{q}_0\right) + \theta\left(1/\mathbf{p} - 1/\mathbf{q}_1\right) =$$

$$= ((1-\theta)/\mathbf{p} + \theta/\mathbf{p}) - ((1-\theta)/\mathbf{q}_0 + \theta/\mathbf{q}_1) = 1/\mathbf{p} - 1/\mathbf{q}.$$

□

Theorem 4.3. *Let $\mathbf{1} < \mathbf{q} = (q_1,\ldots,q_n) < \mathbf{p} = (p_1,\ldots,p_n) < \infty$ and $\mathbf{1} \leq \tau = (\tau_1,\ldots,\tau_n) \leq \infty$. Then*

$$L_{\mathbf{q}\tau}(\mathbb{R}^n) \hookrightarrow S^{\alpha\tau}_{\mathbf{p}}B(\mathbb{R}^n),$$

where $\alpha = 1/\mathbf{p} - 1/\mathbf{q}$.

Proof. According to the inequality of different metrics (Lemma 4.1) since $V_M \in L_1(\mathbb{R})$, we obtain

$$\|f\|_{S_{\mathbf{p}}^{\alpha\infty}B(\mathbb{R}^n)} = \sup_{\mathbf{s}\geq\mathbf{0}} 2^{(\alpha,\mathbf{s})}\left\|Q_{\mathbf{s}}(f)\right\|_{L_{\mathbf{p}}(\mathbb{R}^n)} \leq C_3 \sup_{\mathbf{s}\geq\mathbf{0}} 2^{(\alpha+1/\mathbf{q}-1/\mathbf{p},\mathbf{s})}\left\|Q_{\mathbf{s}}(f)\right\|_{L_{\mathbf{q}}(\mathbb{R}^n)}$$

$$= C_3 \sup_{\mathbf{s}\geq\mathbf{0}} \left\| \frac{1}{\pi^n} \prod_{i=1}^{n} \left(V_{2^{s_i}}(\cdot) - V_{\left[2^{s_i-1}\right]}(\cdot)\right) * f \right\|_{L_{\mathbf{q}}(\mathbb{R}^n)} \leq C_4 \left\|f\right\|_{L_{\mathbf{q}}(\mathbb{R}^n)},$$

for $\alpha = 1/\mathbf{p} - 1/\mathbf{q}$, where $C_4 > 0$ is independent of f.

Thus, for $\alpha = 1/\mathbf{p} - 1/\mathbf{q}$ we have

$$L_{\mathbf{q}}(\mathbb{R}^n) \hookrightarrow S_{\mathbf{p}}^{\alpha\infty}B(\mathbb{R}^n).$$

Let us fix $\mathbf{p} = (p_1, \ldots, p_n)$ and let us choose parameters $\alpha_i = (\alpha_1^i, \ldots, \alpha_n^i)$ and $\mathbf{q}_i = (q_1^i, \ldots, q_n^i)$ such that $\alpha_j^i = 1/p_j - 1/q_j^i$, where $i = 0, 1$ and $j = 1, \ldots, n$. Then for every $\varepsilon \in E$ we obtain

$$L_{\mathbf{q}_\varepsilon}(\mathbb{R}^n) \hookrightarrow S_{\mathbf{p}}^{\alpha_\varepsilon\infty}B(\mathbb{R}^n).$$

According to Lemma 3.2 and Theorem 3.1 we obtain

$$\left(L_{\mathbf{q}_\varepsilon}(\mathbb{R}^n); \varepsilon \in E\right)_{\theta\tau} \hookrightarrow \left(S_{\mathbf{p}}^{\alpha_\varepsilon\infty}B(\mathbb{R}^n); \varepsilon \in E\right)_{\theta\tau}$$

or

$$L_{\mathbf{q}\tau}(\mathbb{R}^n) \hookrightarrow S_{\mathbf{p}}^{\alpha\tau}B(\mathbb{R}^n),$$

where $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $1/\mathbf{q} = (1-\theta)/\mathbf{q}_0 + \theta/\mathbf{q}_1$.

Let us check the relationship between the parameters α, $\mathbf{p}$ and $\mathbf{q}$

$$\alpha = (1-\theta)\alpha_0 + \theta\alpha_1 = (1-\theta)\left(1/\mathbf{p} - 1/\mathbf{q}_0\right) + \theta\left(1/\mathbf{p} - 1/\mathbf{q}_1\right) =$$

$$= ((1-\theta)/\mathbf{p} + \theta/\mathbf{p}) - ((1-\theta)/\mathbf{q}_0 + \theta/\mathbf{q}_1) = 1/\mathbf{p} - 1/\mathbf{q}.$$

□

Remark 3. It is possible to show that the conditions of Theorems 4.1 – 4.3 are sharp. The proof of these facts can be carried out by analogy with the corresponding proofs in the papers [8, 11].

5 The theorems about trace and extension

In this section, trace and extension theorems for functions belonging to Nikol'skii-Besov spaces with a dominant mixed derivative and with a mixed metric are proved.

Let $1 \leq m < n$. For $\mathbf{a} = (a_1, \ldots, a_m, a_{m+1}, \ldots, a_n)$, we denote $\bar{\mathbf{a}} = (a_1, \ldots, a_m)$ and $\tilde{\mathbf{a}} = (a_{m+1}, \ldots, a_n)$.

Lemma 5.1 (Inequality of different dimensions, [22]). *Let $\mathbf{1} \leq \mathbf{p} = (p_1, \ldots, p_m, p_{m+1}, \ldots, p_n) < \infty$ and let $Q_{\mathbf{s}}(\mathbf{x})$ be an entire function of exponential type of order $\mathbf{s} = (s_1, \ldots, s_m, s_{m+1}, \ldots, s_n)$ by $\mathbf{x} = (x_1, \ldots, x_m, x_{m+1}, \ldots, x_n)$. Then for an arbitrary fixed point $\tilde{\mathbf{x}} \in \mathbb{R}^{n-m}$ holds the inequality*

$$\left\|Q_{\mathbf{s}}(\cdot, \tilde{\mathbf{x}})\right\|_{L_{\bar{\mathbf{p}}}(\mathbb{R}^m)} \leq C_5 \prod_{i=m+1}^{n} s_i^{1/p_i} \left\|Q_{\mathbf{s}}\right\|_{L_{\mathbf{p}}(\mathbb{R}^n)},$$

where C_5 is a positive constant independent of $\mathbf{s}$ and $\tilde{\mathbf{x}}$.

Theorem 5.1. *Let* $\mathbf{1} \leq \mathbf{p} = (p_1, \dots, p_m, p_{m+1}, \dots, p_n) < \infty$, $\alpha = (\alpha_1, \dots, \alpha_m, \alpha_{m+1}, \dots, \alpha_n)$, *and* $\mathbf{1} \leq \tau = (\tau_1, \dots, \tau_m, \tau_{m+1}, \dots, \tau_n) \leq \infty$ *with* $\alpha_i = 1/p_i$ *and* $\tau_i = 1$ *for* $i = m+1, \dots, n$. *Then the trace operator* $T : f \mapsto f(\cdot, \tilde{\mathbf{0}})$ *is well-defined and satisfies*

$$T : S^{\alpha\tau}_{\mathbf{p}} B(\mathbb{R}^n) \to S^{\bar{\alpha}\bar{\tau}}_{\bar{\mathbf{p}}} B(\mathbb{R}^m).$$

Proof. Fix any $f \in S^{\alpha\tau}_{\mathbf{p}} B(\mathbb{R}^n)$. We will show that

$$f \in L_\infty(\mathbb{R}^{n-m}; S^{\bar{\alpha}\bar{\tau}}_{\bar{\mathbf{p}}} B(\mathbb{R}^m)) \tag{5.1}$$

and

$$\|f(\cdot, \cdot + \tilde{\mathbf{h}}) - f(\cdot, \cdot)\|_{L_\infty(\mathbb{R}^{n-m}; S^{\bar{\alpha}\bar{\tau}}_{\bar{\mathbf{p}}} B(\mathbb{R}^m))} \to 0 \tag{5.2}$$

as $\tilde{\mathbf{h}} \to \tilde{\mathbf{0}}$. By properties (5.1) and (5.2) it follows that f coincides almost everywhere with a unique bounded uniformly continuous function $g : \mathbb{R}^{n-m} \to S^{\bar{\alpha}\bar{\tau}}_{\bar{\mathbf{p}}} B(\mathbb{R}^m)$. The trace operator is then well-defined by $Tf := g(\tilde{\mathbf{0}})$.

We now show (5.1). According to the inequality of different dimensions (Lemma 5.1) and Minkowski's inequality, for almost everywhere $\tilde{\mathbf{x}} \in \mathbb{R}^{n-m}$ holds

$$\begin{aligned}
\|f(\cdot, \tilde{\mathbf{x}})\|_{S^{\bar{\alpha}\bar{\tau}}_{\bar{\mathbf{p}}} B(\mathbb{R}^m)} &= \left\| \left\{ 2^{(\bar{\alpha}, \bar{\mathbf{s}})} \left\| \sum_{\tilde{\mathbf{s}} \in \mathbb{Z}^{n-m}_+} Q_{(\bar{\mathbf{s}}, \tilde{\mathbf{s}})}(f)(\cdot, \tilde{\mathbf{x}}) \right\|_{L_{\bar{\mathbf{p}}}(\mathbb{R}^m)} \right\} \right\|_{l_{\bar{\tau}}} \\
&\leq \left\| \left\{ \sum_{\tilde{\mathbf{s}} \in \mathbb{Z}^{n-m}_+} 2^{(\bar{\alpha}, \bar{\mathbf{s}})} \left\| Q_{(\bar{\mathbf{s}}, \tilde{\mathbf{s}})}(f)(\cdot, \tilde{\mathbf{x}}) \right\|_{L_{\bar{\mathbf{p}}}(\mathbb{R}^m)} \right\} \right\|_{l_{\bar{\tau}}} \\
&\leq \sum_{\tilde{\mathbf{s}} \in \mathbb{Z}^{n-m}_+} \left\| \left\{ 2^{(\bar{\alpha}, \bar{\mathbf{s}})} \left\| Q_{(\bar{\mathbf{s}}, \tilde{\mathbf{s}})}(f)(\cdot, \tilde{\mathbf{x}}) \right\|_{L_{\bar{\mathbf{p}}}(\mathbb{R}^m)} \right\} \right\|_{l_{\bar{\tau}}} \\
&= \sum_{\tilde{\mathbf{s}} \in \mathbb{Z}^{n-m}_+} \left\| \left\{ 2^{(\bar{\alpha}, \bar{\mathbf{s}})} \left\| Q_{\mathbf{s}}(f)(\cdot, \tilde{\mathbf{x}}) \right\|_{L_{\bar{\mathbf{p}}}(\mathbb{R}^m)} \right\} \right\|_{l_{\bar{\tau}}} \\
&\leq C_5 \sum_{\tilde{\mathbf{s}} \in \mathbb{Z}^{n-m}_+} 2^{(\mathbf{1}/\tilde{\mathbf{p}}, \tilde{\mathbf{s}})} \left\| \left\{ 2^{(\bar{\alpha}, \bar{\mathbf{s}})} \left\| Q_{\mathbf{s}}(f) \right\|_{L_{\mathbf{p}}(\mathbb{R}^n)} \right\} \right\|_{l_{\bar{\tau}}} \\
&= C_5 \left\| \left\{ 2^{(\alpha, \mathbf{s})} \left\| Q_{\mathbf{s}}(f) \right\|_{L_{\mathbf{p}}(\mathbb{R}^n)} \right\} \right\|_{l_\tau} = C_5 \|f\|_{S^{\alpha\tau}_{\mathbf{p}} B(\mathbb{R}^n)}.
\end{aligned} \tag{5.3}$$

We now show (5.2).

Since $f \in S^{\alpha\tau}_{\mathbf{p}} B(\mathbb{R}^n)$, for any $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that

$$I^2_{N(\varepsilon)} = \left\| \left\{ 2^{(\alpha, \mathbf{s})} \|Q_{\mathbf{s}}(f)\|_{L_{\mathbf{p}}(\mathbb{R}^n)} \right\}_{\{\mathbf{s} : (\mathbf{s}, \mathbf{1}) > N(\varepsilon)\}} \right\|_{l_\tau} < \frac{\varepsilon}{3C_5}. \tag{5.4}$$

Applying inequality (5.3) and the Minkowski inequality, according to estimate (5.4) we obtain

$$\|f(\cdot, \cdot + \tilde{\mathbf{h}}) - f(\cdot, \cdot)\|_{L_\infty(\mathbb{R}^{n-m}; S^{\bar{\alpha}\bar{\tau}}_{\bar{\mathbf{p}}} B(\mathbb{R}^m))} \leq C_5 \|f(\cdot, \cdot + \tilde{\mathbf{h}}) - f(\cdot, \cdot))\|_{S^{\alpha\tau}_{\mathbf{p}} B(\mathbb{R}^n)}$$

$$\leq C_5 \left(\left\| \left\{ 2^{(\alpha, \mathbf{s})} \left\| Q_{\mathbf{s}}(f(\cdot, \cdot + \tilde{\mathbf{h}})) - Q_{\mathbf{s}}(f(\cdot, \cdot)) \right\|_{L_{\mathbf{p}}(\mathbb{R}^n)} \right\}_{\{\mathbf{s} : (\mathbf{s}, \mathbf{1}) \leq N(\varepsilon)\}} \right\|_{l_\tau} \right.$$

$$+\left\|\left\{2^{(\alpha,\mathbf{s})}\left\|Q_{\mathbf{s}}(f(\cdot,\cdot+\tilde{\mathbf{h}}))-Q_{\mathbf{s}}(f(\cdot,\cdot))\right\|_{L_{\mathbf{p}}(\mathbb{R}^n)}\right\}_{\{\mathbf{s}:(\mathbf{s},\mathbf{1})>N(\varepsilon)\}}\right\|_{l_\tau}\Bigg)$$

$$\leq C_5\Bigg(\left\|\left\{2^{(\alpha,\mathbf{s})}\left\|Q_{\mathbf{s}}(f(\cdot,\cdot+\tilde{\mathbf{h}}))-Q_{\mathbf{s}}(f(\cdot,\cdot))\right\|_{L_{\mathbf{p}}(\mathbb{R}^n)}\right\}_{\{\mathbf{s}:(\mathbf{s},\mathbf{1})\leq N(\varepsilon)\}}\right\|_{l_\tau}$$

$$+2\left\|\left\{2^{(\alpha,\mathbf{s})}\left\|Q_{\mathbf{s}}(f)\right\|_{L_{\mathbf{p}}(\mathbb{R}^n)}\right\}_{\{\mathbf{s}:(\mathbf{s},\mathbf{1})>N(\varepsilon)\}}\right\|_{l_\tau}\Bigg)$$

$$=C_5\left(I^1_{N(\varepsilon)}+2I^2_{N(\varepsilon)}\right)<C_5I^1_{N(\varepsilon)}+\frac{2\varepsilon}{3}. \tag{5.5}$$

In order to evaluate $I^1_{N(\varepsilon)}$, we will use the following inequality (see [3])

$$\left\|Q_{\mathbf{s}}(f(\cdot,\cdot+\tilde{\mathbf{h}}))-Q_{\mathbf{s}}(f(\cdot,\cdot))\right\|_{L_{\mathbf{p}}(\mathbb{R}^n)}\leq C_6 2^{(\tilde{\mathbf{s}},\tilde{\mathbf{1}})}\max_{i=m+1,\dots,n}|h_i|\left\|Q_{\mathbf{s}}(f)\right\|_{L_{\mathbf{p}}(\mathbb{R}^n)},$$

where $C_6>0$ is independent of f.

Hence, we get

$$I_1\left(N(\varepsilon)\right)\leq C_6 2^{N(\varepsilon)}\max_{i=m+1,\dots,n}|h_i|\left\|\left\{2^{(\alpha,\mathbf{s})}\left\|Q_{\mathbf{s}}(f)\right\|_{L_{\mathbf{p}}(\mathbb{R}^n)}\right\}_{\{\mathbf{s}:(\mathbf{s},\mathbf{1})\leq N(\varepsilon)\}}\right\|_{l_\tau}$$

$$\leq C_6 2^{N(\varepsilon)}|\tilde{\mathbf{h}}|\|f\|_{S^{\alpha\tau}_{\mathbf{p}}B(\mathbb{R}^n)}.$$

We now choose $|\tilde{\mathbf{h}}|<\dfrac{\varepsilon}{3C_5C_6 2^{N(\varepsilon)}\|f\|_{S^{\alpha\tau}_{\mathbf{p}}B(\mathbb{R}^n)}}$, then

$$I^1_{N(\varepsilon)}<\frac{\varepsilon}{3C_5}. \tag{5.6}$$

Plugging estimate (5.6) into (5.5), we obtain

$$\|f(\cdot,\cdot+\tilde{\mathbf{h}})-f(\cdot,\cdot)\|_{L_\infty(\mathbb{R}^{n-m};S^{\bar{\alpha}\bar{\tau}}_{\bar{\mathbf{p}}}B(\mathbb{R}^m))}<\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, (5.2) is proved. □

Remark 4. Trace theorems for Nikol'skii-Besov spaces with a dominant mixed derivative were previously obtained in [23, 20, 3, 30] under the condition $\alpha_i>1/p_i$ for $i=m+1,\dots,n$. Compared to the above mentioned works, in Theorem 5.1 we allow a weaker condition $\alpha_i=1/p_i$ with $\tau_i=1$ (this effect was previously seen, for instance, in [14, 15] and [11]).

Theorem 5.2. *Let $\alpha=(\alpha_1,\dots,\alpha_m,\alpha_{m+1},\dots,\alpha_n)$, $\mathbf{1}\leq\tau=(\tau_1,\dots,\tau_m,\tau_{m+1},\dots,\tau_n)\leq\infty$ with $\alpha_i=1/p_i,\tau_i=1$ for $i=m+1,\dots,n$, and $\mathbf{1}\leq\mathbf{p}=(p_1,\dots,p_m,p_{m+1},\dots,p_n)<\infty$. Then for any function $\varphi(\bar{\mathbf{x}})\in S^{\bar{\alpha}\bar{\tau}}_{\bar{\mathbf{p}}}B(\mathbb{R}^m)$ there exists a function $f(\bar{\mathbf{x}},\tilde{\mathbf{x}})$ having the following properties:*

$$f\in S^{\alpha\tau}_{\mathbf{p}}B(\mathbb{R}^n);$$

$$\|f\|_{S^{\alpha\tau}_{\mathbf{p}}B(\mathbb{R}^n)}\leq C_7\left\|\varphi\right\|_{S^{\bar{\alpha}\bar{\tau}}_{\bar{\mathbf{p}}}B(\mathbb{R}^m)},$$

where $C_7>0$ is independent of φ;

$$f(\bar{\mathbf{x}},\tilde{\mathbf{0}})=\varphi(\bar{\mathbf{x}}),\ \bar{\mathbf{x}}\in\mathbb{R}^m.$$

Proof. Let $\varphi \in S_{\bar{\mathbf{p}}}^{\bar{\alpha}\bar{\tau}} B(\mathbb{R}^m)$. This function can be represented as a series

$$\varphi(\bar{\mathbf{x}}) = \sum_{\bar{\mathbf{s}}=0}^{\infty} Q_{\bar{\mathbf{s}}}(\varphi)(\bar{\mathbf{x}})$$

and

$$\|\varphi\|_{S_{\bar{\mathbf{p}}}^{\bar{\alpha}\bar{\tau}} B(\mathbb{R}^m)} = \left\| \left\{ 2^{(\bar{\alpha},\bar{\mathbf{s}})} \left\| Q_{\bar{\mathbf{s}}}(\varphi) \right\|_{L_{\bar{\mathbf{p}}}(\mathbb{R}^m)} \right\} \right\|_{l_{\bar{\tau}}}.$$

Fix any functions $f_i(x_i) \in C_0^\infty(\mathbb{R})$ with $f_i(0) = 1$, $i = m+1, \ldots, n$. We introduce a new function $f(\mathbf{x})$ by

$$f(\bar{\mathbf{x}}, \tilde{\mathbf{x}}) = \varphi(\bar{\mathbf{x}}) \cdot \prod_{i=m+1}^{n} f_i(x_i).$$

Clearly, $Q_{\mathbf{s}}(f) = Q_{\bar{\mathbf{s}}}(\varphi) \prod_{i=m+1}^{n} Q_{s_i}(f_i)$. Therefore,

$$\begin{aligned} \|f\|_{S_{\mathbf{p}}^{\alpha\tau} B(\mathbb{R}^n)} &= \left\| \left\{ 2^{(\alpha,\mathbf{s})} \left\| Q_{\mathbf{s}}(f) \right\|_{L_{\mathbf{p}}(\mathbb{R}^n)} \right\} \right\|_{l_\tau} \\ &= \left\| \left\{ 2^{(\bar{\alpha},\bar{\mathbf{s}})} \left\| Q_{\bar{\mathbf{s}}}(\varphi) \right\|_{L_{\bar{\mathbf{p}}}(\mathbb{R}^m)} \right\} \right\|_{l_{\bar{\tau}}} \prod_{i=m+1}^{n} \left\| \left\{ 2^{s_i/p_i} \left\| Q_{s_i}(f_i) \right\|_{L_{p_i}(\mathbb{R})} \right\} \right\|_{l_1} \\ &= C_7 \|\varphi\|_{S_{\bar{\mathbf{p}}}^{\bar{\alpha}\bar{\tau}} B(\mathbb{R}^m)}. \end{aligned}$$

Here $C_7 < \infty$ since the norm $\left\| \left\{ 2^{s_i/p_i} \| Q_{s_i}(\cdot) \|_{L_{p_i}(\mathbb{R})} \right\} \right\|_{l_1}$ is equivalent to the Besov norm $\|\cdot\|_{B_{p_i}^{1/p_i,1}(\mathbb{R})}$ (see [22]), and $f_i \in C_0^\infty(\mathbb{R}) \subset B_{p_i}^{1/p_i,1}(\mathbb{R})$.

Further, we have

$$\lim_{\tilde{\mathbf{h}} \to \tilde{\mathbf{0}}} \left\| f(\cdot, \tilde{\mathbf{h}}) - \varphi(\cdot) \right\|_{S_{\bar{\mathbf{p}}}^{\bar{\alpha}\bar{\tau}} B(\mathbb{R}^m)} = \lim_{\tilde{\mathbf{h}} \to \tilde{\mathbf{0}}} \left\| \varphi(\cdot) \left(\prod_{i=m+1}^{n} f_i(h_i) - 1 \right) \right\|_{S_{\bar{\mathbf{p}}}^{\bar{\alpha}\bar{\tau}} B(\mathbb{R}^m)}$$

$$= \|\varphi\|_{S_{\bar{\mathbf{p}}}^{\bar{\alpha}\bar{\tau}} B(\mathbb{R}^m)} \cdot \lim_{\tilde{\mathbf{h}} \to \tilde{\mathbf{0}}} \left| \prod_{i=m+1}^{n} f_i(h_i) - 1 \right| = 0.$$

These arguments show that φ is the trace of the function f. □

Remark 5. The extension operator constructed in the proof of Theorem 5.2 is linear. It should be noted that in the work of V.I. Burenkov and M.L. Gol'dman [19] it was shown that in the limiting case for Nikol'skii-Besov spaces it is possible to construct only a nonlinear extension operator, but this effect is not observed for Nikol'skii-Besov spaces with a dominant mixed derivative.

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