EURASIAN MATHEMATICAL JOURNAL
volume 16 , number 2, 2025

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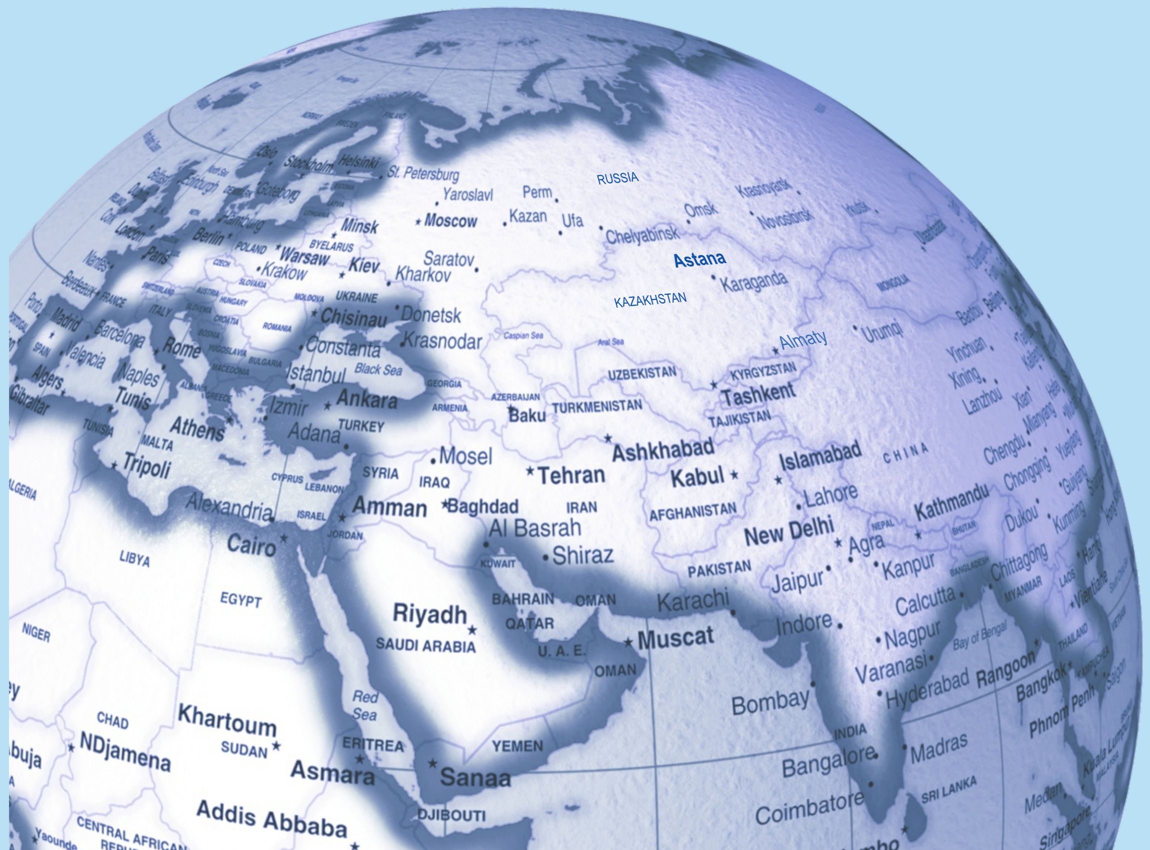
Events

EURASIAN NATIONAL UNIVERSITY

ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

EURASIAN MATHEMATICAL JOURNAL



ISSN 2077-9879

VOLUME 16, NUMBER 2 2025

ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2025, Volume 16, Number 2

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

EURASIAN MATHEMATICAL JOURNAL

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Subscription index of the EMJ 76090 via KAZPOST.

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Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana, Republic of Kazakhstan

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EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 16, Number 2 (2025), 23 – 29

NOTES ON THE GENERALIZED GAUSS REDUCTION ALGORITHM

Y. Baissalov, R. Nauryzbayev

Communicated by J.A. Tussupov

Key words: lattice, well-ordered basis, reduced basis, generalized Gaussian algorithm.

AMS Mathematics Subject Classification: 68W40.

Abstract. The hypothetical possibility of building a quantum computer in the near future has forced a revision of the foundations of modern cryptography. The fact is that many difficult algorithmic problems, such as the discrete logarithm, factoring a (large) natural number into prime factors, etc., on the complexity of which many cryptographic protocols are based these days, have turned out to be relatively easy to solve using quantum algorithms.

Intensive research is currently underway to find problems that are difficult even for a quantum computer and have potential applications for cryptographic protocols. Our article contains notes related to the so-called generalized Gauss algorithm, which calculates the reduced basis of a two-dimensional lattice [8], [2]. Note that researchers are increasingly putting forward difficult algorithmic problems from lattice theory as candidates for the foundation of post-quantum cryptography. The majority of algorithmic problems related to lattice reduction become NP-hard as the lattice dimension increases [3], [1]. Fundamental problems such as the Shortest Vector Problem (SVP), the Closest Vector Problem (CVP), and Bounded Distance Decoding (BDD) are conjectured to remain hard even for quantum algorithms [4], [6]. Although the generalized Gauss reduction algorithm applies to two-dimensional lattices, where exact analysis is feasible (dimensions 3 and 4 are studied in [7], [5]), understanding such low-dimensional reductions provides important insights into the structure and complexity of lattice-based cryptographic constructions.

DOI: https://doi.org/10.32523/2077-9879-2025-16-2-23-29

1 Preliminaries

All necessary information on the basics of lattice theory can be found in [8]. For those who are familiar with the group theory, a *lattice* is a finitely generated subgroup of the additive group of the Euclidean space $\mathbb{R}^n$. In this note we will limit ourselves to considering the 2-generated lattice $L \in \mathbb{R}^n$. Any pair of vectors generating L is called a *basis* of the lattice.

The Euclidean space metric $\mathbb{R}^n$, obtained by the standard dot product, induces a metric on L. Let us clarify the notation associated with this metric: for vectors $a, b \in L$, let us denote by (a, b) their dot product, by $\|a\|$ the length of vector a, and by $[a]$ the square of this length, that is, $[a] = (a, a) = \|a\|^2$.

Definition 1. Vectors $a, b \in L$ will be called an *ordered basis* and denoted by $\langle a, b\rangle$ if the following conditions are satisfied:
(1) $\|a\| \leq \|b\|$;
(2) $\|a - b\| \leq \|a + b\|$.

Note that for any lattice basis it is easy to obtain an ordered basis: if the vectors $a, b \in L$ form a basis, then first we arrange them in increasing length, and if we already have $\|a\| \leq \|b\|$, and $\|a-b\| \leq \|a+b\|$ is not satisfied, then we change b to $-b$. Therefore, in what follows only ordered bases of the lattice L are considered.

Definition 2. (1) If $\|a\| \leq \|a-b\| < \|b\|$, then the ordered basis $\langle a, b\rangle$ is called *well-ordered.*
(2) An ordered basis $\langle a, b\rangle$ is called *reduced* if $\|b\| \leq \|a-b\|$.

In Sections 2 and 3 we present results that are valid for any normed lattices, that is, for lattices with a norm which their norm is obtained by restricting a certain norm on the space $\mathbb{R}^n$.

Definition 3. A function $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}_+$, where $\mathbb{R}_+$ is the set of all non-negative real numbers, is called a *norm* if it satisfies the following conditions for any vectors $x, y \in \mathbb{R}^n$ and for any real number $\alpha \in \mathbb{R}$:
(1) $\|x\| = 0$ if and only if x is the zero vector;
(2) $\|x+y\| \leq \|x\| + \|y\|$ (*the triangle inequality*);
(3) $\|\alpha x\| = |\alpha| \cdot \|x\|$.

We will call a norm *strict* if the equality in condition (2) is satisfied only when at least one of the vectors x, y is the zero vector or the vectors x, y are collinear and co-directional.

The following corollary of the triangle inequality is often useful.

Corollary 1.1. *For any $x, y \in \mathbb{R}^n$ we have $\big|\|x\| - \|y\|\big| \leq \|x-y\|$.*

Definition 4. (1) $\lambda_1 = \min\{\|a\| : 0 \neq a \in L\}$
(2) $\lambda_2 = \min\{\|b\| : \langle a, b\rangle$ is an ordered basis for some $a \in L\}$.

The numbers λ_1, λ_2 are always defined, since the lattice L is a discrete group: any ball of finite radius centered at the zero vector contains only a finite number of lattice elements [8].

The following theorem, the proof of which can be found in [8, Theorem 16] (see also [2, Theorem 4]), explains why a reduced basis is sometimes called a *minimal basis.*

Theorem 1.1. *An ordered basis $\langle a, b\rangle$ is reduced if and only if $\|a\| = \lambda_1$ and $\|b\| = \lambda_2$.*

The following useful lemma was also proven in [8, Lemma 17].

Lemma 1.1. *Consider three vectors on a line: x, $x+y$ and $x+\alpha y$, where $\alpha \in (1, \infty)$. For any norm $\|\cdot\|$ from the inequality $\|x\| \leq \|x+y\|$ it follows that $\|x+y\| \leq \|x+\alpha y\|$, and from the inequality $\|x\| < \|x+y\|$ it follows that $\|x+y\| < \|x+\alpha y\|$.*

Note that using Lemma 1.1 one can prove that if a basis $\langle a, b\rangle$ is well-ordered, then $\|a\| \leq \|a-b\| < \|b\| < \|a+b\|$ (see [2]).

2 About the function $l(\tau) = \|b - \tau a\|$

In this section, we study the properties of the function $l(\tau) = \|b - \tau a\|$, $\tau \in \mathbb{R}$, where a, b are vectors of some real space with the norm $\|\cdot\|$. If a is the zero vector, then $l(\tau) \equiv \|b\|$ is a constant function, and if b is the zero vector, then $l(\tau) = \|a\| \cdot |\tau|$ is the absolute value function multiplied by the constant $\|a\|$. A similar function will be obtained if the vectors a, b are linearly dependent: for example, if $b = \gamma a$, then $l(\tau) = \|a\| \cdot |\tau - \gamma|$. Therefore, the case is interesting when the vectors a, b are linearly independent.

Theorem 2.1. *Let a, b be linearly independent vectors of some real space with the norm $\|\cdot\|$. Then the function $l(\tau) = \|b - \tau a\|, \tau \in \mathbb{R}$, has the following properties:*
(1) l is continuous on the entire real line;
(2) l is not bounded from above: $\lim_{\tau \to -\infty} l(\tau) = +\infty$ and $\lim_{\tau \to +\infty} l(\tau) = +\infty$;
(3) there exists $\mu_0 \overset{def}{=} \min\{l(\tau) : \tau \in \mathbb{R}\} > 0$ and there exists a closed interval of minimality $[\tau_0, \tau_1] \overset{def}{=} \{\tau \in \mathbb{R} : l(\tau) = \mu_0\}$;
(4) on the interval $(-\infty, \tau_0)$ the function l strictly decreases, and on the interval $(\tau_1, +\infty)$ it strictly increases.

Proof. (1) Let us prove the continuity of the function l at an arbitrary point $\tau_0 \in \mathbb{R}$. By Corollary 1.1 we have

$$|l(\tau + \tau_0) - l(\tau_0)| = |\|b - (\tau + \tau_0)a\| - \|b - \tau_0 a\|| \leq \|\tau a\| = \|a\| \cdot |\tau|.$$

Therefore, $|l(\tau + \tau_0) - l(\tau_0)| < \varepsilon$ holds for $|\tau| < \delta = \frac{\varepsilon}{\|a\|}$.

(2) Using Corollary 1.1 again and property (3) of the norm, we obtain

$$l(\tau) = \|b - \tau a\| \geq \|a\| \cdot |\tau| - \|b\|,$$

which obviously implies $\lim_{\tau \to -\infty} l(\tau) = +\infty$ and $\lim_{\tau \to +\infty} l(\tau) = +\infty$.

(3) Let us choose numbers $\alpha_0 < 0 < \beta_0 \in \mathbb{R}$ so that $l(\tau) > l(0) = \|b\|$ for any real number τ lying outside the interval $[\alpha_0, \beta_0]$: this is possible according to (2). According to Weierstrass's theorem, the function l reaches its minimum at a point τ_0 of the interval $[\alpha_0, \beta_0]$, which we denote by $\mu_0 = l(\tau_0)$. Obviously, this μ_0 will be the minimum of the function over the entire $\mathbb{R}$.

Let us call $\tau \in \mathbb{R}$ a *point of monotonicity* (of the function l), if $l(\tau) > \mu_0$. Let $\gamma < \tau_0$ be a point of monotonicity. Then note that each $\delta < \gamma$ is a point of monotonicity, since $l(\delta) > l(\gamma) > \mu_0$ holds (apply Lemma 1.1 for the vectors $x = b - \tau_0 a$ and $y = (\tau_0 - \gamma)a$). So, the interval $(-\infty, \gamma]$ consists entirely of monotonicity points. In addition, due to the continuity of the function l, some neighborhood of the point γ will consist entirely of monotonicity points. This means that each monotonicity point $\gamma < \tau_0$ is contained in a certain interval of the form $(-\infty, \alpha)$, consisting entirely of monotonicity points. Since the union of intervals of this type again gives an open interval of the same type, we conclude that the monotonicity points located to the left of τ_0 form an interval of this type, which we will denote without loss of generality by $(-\infty, \tau_0)$. Similar reasoning shows that monotonicity points located to the right of τ_0 form an interval $(\tau_1, +\infty)$ for some $\tau_1 \geq \tau_0$.

(4) In the last paragraph of the proof of point (3), in fact, it was proven that $l(\delta) > l(\gamma)$ holds for $\delta < \gamma < \tau_0$, that is, that the function l strictly decreases on the interval $(-\infty, \tau_0)$. Similarly, using Lemma 1.1 we prove the second statement of this point, namely, that the function l is strictly increasing on the interval $(\tau_1, +\infty)$. □

Example. The norm defined for $\mathbb{R}^2$ as follows is not strict: for $(\alpha, \beta) \in \mathbb{R}^2$ we set

$$\|(\alpha, \beta)\| \overset{def}{=} \max\{|\alpha|, |\beta|\}.$$

With $a = (0, 1), b = (1, 0)$ for the function $l(\tau) = \|b - \tau a\|$ we have $\mu_0 \overset{def}{=} \min\{l(\tau) : \tau \in \mathbb{R}\} = 1$, and the interval of minimality is $[-1, 1]$. □

Note that it may well be $\tau_0 = \tau_1$, that is, the interval $[\tau_0, \tau_1]$ can consist of only one point. This situation occurs if the norm $\|\cdot\|$ on the subspace generated by the vectors a, b is strict. Indeed, if

$\tau_0 \neq \tau_1$ and the norm $\|\cdot\|$ is strict, then the vectors $b - \tau_0 a$, $b - \tau_1 a$ are not collinear, therefore the sum of their lengths is strictly greater than the length of their sum:

$$2\mu_0 = \|b - \tau_0 a\| + \|b - \tau_1 a\| > \|2b - (\tau_0 + \tau_1)a\| = 2\left\|b - \frac{\tau_0 + \tau_1}{2}a\right\|.$$

We obtain $l(\frac{\tau_0+\tau_1}{2}) = \|\frac{\tau_0+\tau_1}{2}a + b\| < \mu_0$, which contradicts the minimality of the value μ_0.

In particular, we have $\tau_0 = \tau_1$, when the norm $\|\cdot\|$ is generated by the dot product in $\mathbb{R}^n$. In addition, in this case the value of τ_0 is explicitly calculated. Indeed, we have

$$l(\tau)^2 = [b - \tau a] = (b - \tau a, b - \tau a) = [a]\tau^2 - 2(a,b)\tau + [b],$$

and this quadratic function reaches a minimum at point $\tau_0 = \frac{(a,b)}{(a,a)} = \frac{(a,b)}{[a]}$.

In the next section we use an oracle that solves the following problem.

Problem. For a given ordered basis $\langle a, b\rangle$, find an integer $\mu = \mu(a,b)$ such that $\|b - \mu a\| = \min\{\|b - na\| : n \in \mathbb{Z}\}$, where $\mathbb{Z}$ is the set of integers.

By Theorem 2.1 for the function $l(\tau) = \|b - \tau a\|$ it follows that the problem is correct, that is, it always has a solution. In general, if the interval $[\tau_0, \tau_1]$ contains an integer, then any integer from it will be a solution, if not, then $\mu = \lfloor \tau_0 \rfloor$ or $\mu = \lceil \tau_1 \rceil$, where $\lfloor x \rfloor$ ($\lceil x \rceil$) is the largest (smallest) integer from the interval $(-\infty, x]$ ($[x, +\infty)$. Thus, this problem can be solved effectively if we can efficiently calculate an approximate value of some number from $[\tau_0, \tau_1]$. This is the case when, for example, the norm $\|\cdot\|$ is defined by the scalar product in $\mathbb{R}^n$, in this case $\tau_0 = \tau_1 = \frac{(a,b)}{(a,a)} = \frac{(a,b)}{[a]}$.

As noted in [8], if we know a not very large interval of real numbers containing $[\tau_0, \tau_1]$, then the above problem can be effectively solved using the binary search algorithm. It is also proved there that $\mu(a,b) \in [1, 2\|b\|/\|a\|)$ provided $\|b\| > \|b - a\|$.

3 On the generalized Gauss reduction algorithm

In this section we will give some notes about the generalized Gaussian reduction algorithm, which allows to find a minimal lattice basis from an initial ordered basis. This algorithm is described in sufficient detail in [8] and [2].

First, we will describe the introductory part of the algorithm, during which we obtain from a given ordered basis, in the worst case, some well-ordered basis, and in the best case, a solution to our problem, i.e. we find some reduced basis.

Let us assume that an ordered basis $\langle a, b\rangle$ is given. Recall that by the definition of an ordered basis we have $\|a\| \leq \|b\|$ and $\|a - b\| \leq \|a + b\|$. Let us consider possible cases:

(1)$\|b\| \leq \|a - b\|$.

In this case, the basis $\langle a, b\rangle$ is reduced and our problem is solved.

(2) $\|a - b\| < \|a\|$.

If $\|a\| = \|b\|$, then $\langle a - b, a\rangle$ is a reduced basis and our problem is solved again:

$$\|a - b\| < \|a\| = \|-b\| = \|(a - b) - a\|$$

$$= 2\|a\| - \|b\| \leq \|2a - b\| = \|(a - b) + a\|.$$

If $\|a\| < \|b\|$, then $\langle b - a, b\rangle$ is a well-ordered basis:

$$\|b - a\| = \|a - b\| < \|a\| = \|-a\| = \|(b - a) - b\|$$

$$< \|b\| < 2\|b\| - \|a\| \leq \|2b - a\| = \|(b - a) + b\|.$$

(3) $\|a\| \le \|a-b\| < \|b\|$.
In this case, the basis $\langle a, b\rangle$ is well-ordered. □

We would like to evaluate the complexity of the generalized Gaussian algorithm, so we must consider worst-case scenarios in all stages of the algorithm. We assume that having received an ordered basis at the input, after the introductory part of the algorithm described above, we obtain a well-ordered basis at the output. The time spent on the introductory part will be short, since the main operations in it are to compare the lengths of some specific vectors.

We move on to describe the next, main stage of the algorithm, which consists of cyclically repeating the same procedure. Let us assume that before the start of this stage we have a well-ordered basis $\langle a, b\rangle$. A cyclically repeated procedure updates this basis as follows. First, using the oracle described in section 2, we find $\mu = \mu(a, b)$ and consider the basis consisting of the vectors a and $b - \mu a$. We correct the second vector of this basis, multiplying it by $\varepsilon \in \{-1, +1\}$ so that the sum of vectors a and $\varepsilon(b - \mu a)$ has a norm no less than the norm of their difference. Further,

(1) if $\|a\| \le \|b-\mu a\|$, then $\langle a, \varepsilon(b-\mu a)\rangle$ is a reduced basis and the algorithm terminates,

(2) if $\|b - \mu a\| < \|a\|$, then according to the analysis from the introductory part of the algorithm, the ordered basis $\langle \varepsilon(b-\mu a), a\rangle$ will be either reduced or well-ordered, since case (2) from the introductory part of the algorithm for the basis $\langle \varepsilon(b-\mu a), a\rangle$ is impossible.

So, the procedure, having obtained a well-ordered basis $\langle a, b\rangle$ at the input, produces a new well-ordered basis $\langle \varepsilon(b-\mu a), a\rangle$ at the output (in an unsuccessful scenario). Since each time the procedure is executed, the length of one of the vectors of the well-ordered basis decreases, after a certain finite number of steps the procedure, due to the discreteness of the lattice, will produce the reduced basis and the algorithm completes its work.

Finally, let us move on to estimating the number of repetitions of the procedure of the main stage of the algorithm. Let k be the number of repetitions and $\langle a, b\rangle = \langle a_k,\ a_{k+1}\rangle$ be a well-ordered basis at the beginning of the stage. Let us represent the results of cyclic procedures as a sequence of ordered bases

$$\langle a_k,\ a_{k+1}\rangle\,,\ \langle a_{k-1},\ a_k\rangle\,, \dots, \langle a_1,\ a_2^0\rangle\,,$$

where $\langle a_1,\ a_2^0\rangle$ is a reduced basis. Then the following lemma, proven in [8], is true.

Lemma 3.1. *For $i \ge 3$, the inequality $2\|a_i\| < \|a_{i+1}\|$ holds.*

The notation a_2^0 is introduced due to the following circumstances. There are two possibilities for completing the algorithm by obtaining the reduced basis $\langle a_1,\ a_2^0\rangle$ from the well-ordered basis $\langle a_2,\ a_3\rangle$. It may well be $a_1 = \varepsilon(a_3 - \mu a_2)$, $a_2^0 = a_2$, if case (2) occurred during the last update of the basis by the main stage procedure. But there could also be case (1), then $a_1 = a_2$, $a_2^0 = \varepsilon(a_3 - \mu a_2)$.

Note that in any case we have $\|a_2^0\| = \lambda_2 < \|a_3\|$. Therefore, we get

$$\frac{\|b\|}{\lambda_2} = \frac{\|a_{k+1}\|}{\lambda_2} > \frac{2^{k-2}\|a_3\|}{\lambda_2} > 2^{k-2},$$

which implies the estimate $k < 2 + \log_2\left(\frac{\|b\|}{\lambda_2}\right)$. □

Finally, the last remark concerns the minimality intervals of the functions $l(\tau) = \|b - \tau a\|$, $\tau \in \mathbb{R}$, for well-ordered bases $\langle a, b\rangle$. It is clear that long minimality intervals can significantly reduce the running time of the Gaussian reduction algorithm. Without going into complex computational analysis, we will limit ourselves to just one simple example confirming this fact.

Lemma 3.2. *If the minimality interval of the function $l(\tau) = \|b - \tau a\|$, $\tau \in \mathbb{R}$, for the basis $\langle a, b\rangle$ contains an integer n_0, then $\|b - n_0 a\| = \lambda_1$ or $\|a\| = \lambda_1$.*

Proof. So, assume that $\|b - n_0 a\| = \mu_0 \stackrel{def}{=} \min\{l(\tau) : \tau \in \mathbb{R}\}$. On the other hand, for some $\alpha, \beta \in \mathbb{Z}$ we have $\|\alpha a + \beta b\| = \lambda_1$. If $\beta = 0$, then obviously $|\alpha| = 1$ and $\|a\| = \lambda_1$. Therefore, let us assume that $\beta \neq 0$. Then, $\lambda_1 = |\beta| \cdot \|\frac{\alpha}{\beta} a + b\| = |\beta| \cdot l(-\frac{\alpha}{\beta}) \geq |\beta| \cdot \mu_0 = |\beta| \cdot \|b - n_0 a\|$, which implies $|\beta| = 1$ and $\|b - n_0 a\| = \lambda_1$. □

Thus, if during the execution of the procedure of the main stage of the algorithm, a well-ordered basis $\langle a, b \rangle$ is given as input, satisfying the condition of Lemma 3.2, then at the output we obtain an ordered basis $\langle c, d \rangle$ with $\|c\| = \lambda_1$, and, if $\langle c, d \rangle$ is not a reduced basis, then at the next step the result of the procedure falling into case (1) will be a reduced basis. Therefore, the number k of repetitions of the procedure will not exceed 2.

Acknowledgments

The research of the first author is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP19677451).

References

[1] M. Ajtai, *The shortest vector problem in L_2 is NP-hard for randomized reductions (extended abstract)*, Proceedings of the thirtieth annual ACM symposium on Theory of computing, (1998), 10-19. https://doi.org/10.1145/276698.276705

[2] M. Kaib, C.P. Schnorr, *The generalized gauss reduction algorithm*, Journal of Algorithms, 21 (1996), 565-578. https://doi.org/10.1006/jagm.1996.0059

[3] D. Micciancio, *The hardness of the shortest vector problem*, SIAM Journal on Computing, 30(6), (2001), 2008-2035. https://doi.org/10.1137/S0097539700373039

[4] D. Micciancio, S. Goldwasser, *Complexity of lattice problems: a cryptographic perspective*, Springer (2002). https://doi.org/10.1007/978-1-4615-0897-7

[5] P.Q. Nguyen, D. Stehlé, *Low-dimensional lattice basis reduction revisited*, ACM Transactions on Algorithms 5 (2004), no. 4, 1 - 48. https://doi.org/10.1145/1597036.1597050

[6] O. Regev, *On lattices, learning with errors, random linear codes, and cryptography*, Journal of the ACM, 56(6), (2009), 1-40. https://doi.org/10.1145/1568318.15683

[7] I. Semaev, *A 3-dimensional lattice reduction algorithm*, In: Silverman, J.H. (eds) Cryptography and Lattices. CaLC 2001. Lecture Notes in Computer Science, vol. 2146, 181-193, Springer, Berlin, Heidelberg (2001). https://doi.org/10.1007/3-540-44670-2_13

[8] A.V. Shokurov, N.N. Kuzyurin, S.A. Fomin, *Lattices, algorithms and modern cryptography, electronic textbook.* January 12, 2023 (in Russian). Access mode: `https://discopal-lab.0x1.tv/share/raw/daa0d349f9f28831eb97affb2ff02ff0ab5314f3/lectures-cs/books/cryptography/book-lattice-cryptography.pdf` (date of access 01/05/2025).

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Received: 19.07.2024