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The Eurasian Mathematical Journal (EMJ)
The Astana Editorial Office
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Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
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The Moscow Editorial Office
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(RUDN University)
Room 473
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BOUNDARY VALUE PROBLEM FOR HYPERBOLIC INTEGRO-DIFFERENTIAL EQUATIONS OF MIXED TYPE

A.T. Assanova, Z.S. Kobeyeva, R.A. Medetbekova

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Abstract. The boundary value problem for a system of hyperbolic integro-differential equations of mixed type with degenerate kernels is considered on a rectangular domain. This problem is reduced to a family of boundary value problems for a system of integro-differential equations of mixed type and integral relations. The system of integro-differential equations of mixed type is transferred to a system of Fredholm integro-differential equations. For solving the family of boundary value problems for integro-differential equations Dzhumabaev's parametrization method is applied. A new concept of a general solution to a system of integro-differential equations with parameter is developed. The domain is divided into N subdomains by a temporary variable, the values of a solution at the interior lines of the subdomains are considered as additional functional parameters, and a system of integro-differential equations is reduced to a family of special Cauchy problems on the subdomains for Fredholm integro-differential equation with functional parameters. Using the solutions to these problems, a new general solutions to a system of Fredholm integro-differential equations with parameter is introduced and its properties are established. Based on a general solution, boundary conditions, and the continuity conditions of a solution at the interior lines of the partition, a system of linear functional equations with respect to parameters is composed. Its coefficients and right-hand sides are found by solving the family of special Cauchy problems for Fredholm integro-differential equations on the subdomains. It is shown that the solvability of the family of boundary value problems for Fredholm integro-differential equations is equivalent to the solvability of the composed system. Methods for solving boundary value problems are proposed, which are based on the construction and solving of these systems. Conditions for the existence and uniqueness of a solution to the boundary value problem for a system of hyperbolic integro-differential equations of mixed type with degenerate kernels are obtained.

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1 Introduction and statement of problem

Boundary value problems for systems of hyperbolic integro-differential equations of mixed type arise in various scientific and engineering fields when a phenomena exhibits both hyperbolic and integral characteristics.

Hyperbolic equations often model wave propagation, and the presence of integro-differential terms can account for the effects of heterogeneous media. Applications include seismology, acoustics, and electromagnetic wave propagation in complex environments [13, 31, 32].

Fluid flow problems involving memory effects, such as viscoelastic or non-Newtonian fluids, can be described using hyperbolic integro-differential equations. This is relevant in modeling flows with memory-dependent constitutive relationships [12]. Modeling the dynamic behavior of structures with distributed parameters, viscoelastic materials, or memory effects in the constitutive relations can lead to hyperbolic integro-differential equations. This is important in understanding the vibrations and responses of complex structures [14, 21, 24].

Hyperbolic integro-differential equations with mixed types can appear in the modeling of systems with time delays, which is common in control theory. These equations can be used to study the stability and control of systems with delays [22, 23].

The spread of infectious diseases, predator-prey interactions, or other ecological systems may be modeled using hyperbolic integro-differential equations. The integral terms can represent memory effects or non-local interactions within populations [25, 33, 34].

Modeling heat conduction in materials with complex structures, like composites or materials with memory effects, can lead to hyperbolic integro-differential equations. This is crucial in designing materials with specific thermal properties [5, 26].

In financial mathematics, models with memory effects, stochastic processes, or non-local interactions can be described using hyperbolic integro-differential equations. This is particularly relevant in option pricing and risk management [29, 30].

Non-local interactions in image processing, such as image denoising or inpainting, can be modeled using hyperbolic integro-differential equations. These equations allow for the consideration of information from distant pixels. Modeling biological systems involving neural dynamics, drug delivery, or reaction-diffusion processes can lead to hyperbolic integro-differential equations. These equations can help simulate and understand complex interactions in biological systems [10, 27].

Hyperbolic integro-differential equations are used to model various geophysical phenomena, including heat conduction in the Earth's crust, seismic wave propagation, and groundwater flow in heterogeneous media [11].

The solutions to these problems provide insights into the behavior of complex systems and aid in the design and optimization of processes in a wide range of scientific and engineering applications. Solving these equations often requires a combination of analytical and numerical techniques tailored to the specific characteristics of the problem at hand.

Therefore, the study of new methods for solving boundary value problems for hyperbolic integro-differential equations is driven by the need to address the complexities of real-world problems, improve computational efficiency, enhance accuracy, and adapt to diverse applications across various disciplines. It reflects the dynamic nature of scientific inquiry and the ongoing quest to develop more robust tools for understanding and manipulating complex systems.

This issue can be resolved by developing constructive methods. In present paper we propose an effective method for solving the boundary value problem for the second order system of hyperbolic integro-differential equations of mixed type. This method is based on the method of introducing new unknown functions [3, 7], the parametrization method [15] and a new concept of a general solution [17].

On the rectangular domain $\Omega = [0, T] \times [0, \omega]$, we consider the boundary value problem for the following second order system of hyperbolic integro-differential equations of mixed type:

$$\frac{\partial^2 u}{\partial x \partial t} = A(t,x)\frac{\partial u}{\partial x} + B(t,x)\frac{\partial u}{\partial t} + C(t,x)u + f(t,x) +$$

$$+\Phi_1(t,x)\int_0^T \Psi_1(s,x)\frac{\partial u(s,x)}{\partial x}ds + \Xi_1(t,x)\int_0^t \Theta_1(s,x)\frac{\partial u(s,x)}{\partial x}ds +$$

$$+\Phi_2(t,x)\int_0^T \Psi_2(s,x)u(s,x)ds + \Xi_2(t,x)\int_0^t \Theta_2(s,x)u(s,x)ds, \quad (t,x)\in\Omega, \tag{1.1}$$

$$P_1(x)\frac{\partial u(0,x)}{\partial x} + P_2(x)u(0,x) + S_1(x)\frac{\partial u(T,x)}{\partial x} + S_2(x)u(T,x) = \varphi(x), \quad x\in[0,\omega], \tag{1.2}$$

$$u(t,0) = \psi(t), \qquad t\in[0,T]. \tag{1.3}$$

Here $u = col(u_1, u_2, ..., u_n)$ is the unknown vector-function, the $n\times n$ matrices $A(t,x)$, $B(t,x)$, $C(t,x)$ and n-vector $f(t,x)$ are continuous on Ω; the $n\times n$ matrices $\Phi_i(t,x)$, $\Psi_1(t,x)$, $\Xi_i(t,x)$, $\Theta_i(t,x)$, $i=1,2$, are continuous on Ω; the $n\times n$ matrices $P_j(x)$, $S_j(x)$, $j=1,2$, and n-vector $\varphi(x)$ are continuous on $[0,\omega]$; the n-vector $\psi(t)$ is continuously differentiable on $[0,T]$.

A vector-function $u(t,x)\in C(\Omega,\mathbb{R}^n)$, which has partial derivatives $\frac{\partial u(t,x)}{\partial x}\in C(\Omega,\mathbb{R}^n)$, $\frac{\partial u(t,x)}{\partial t}\in C(\Omega,\mathbb{R}^n)$, $\frac{\partial^2 u(t,x)}{\partial x\partial t}\in C(\Omega,\mathbb{R}^n)$, is called a *solution* to problem (1.1)–(1.3) if it satisfies system (1.1) for all $(t,x)\in\Omega$, the nonlocal condition (1.2) for all $x\in[0,\omega]$ and the condition on the characteristics (1.3) for all $t\in[0,T]$.

2 Reduction to a family of problems for first order integro-differential equations

Previously, the relationship between nonlocal problems for hyperbolic equations and families of problems for ordinary differential equations was shown in [3, 4, 28]. With the help of new unknown functions, the problem under consideration was reduced to a family of problems for differential equations and integral relations. To solve a family of problems for differential equations, Dzhumabaev parametrization method was used [15] and criteria for the unique solvability of the problem under investigation were obtained in terms of coefficients and boundary data. This has made it possible to establish necessary and sufficient conditions for the well-posed solvability of nonlocal problems for hyperbolic equations in terms of the original data [3, 4]. These results were extended to nonlocal problems for loaded hyperbolic equations [19]. An application of this approach to problems for hyperbolic integro-differential equations leads to a new class of problems for integro-differential equations of mixed type. This, in turn, requires the development of new approaches and methods for solving them.

In this Section by method of introduction of new functions we transfer problem (1.1)–(1.3) to a family of problems for integro-differential equations of mixed type.

We introduce new functions $v(t,x)=\frac{\partial u(t,x)}{\partial x}$ and $w(t,x)=\frac{\partial u(t,x)}{\partial t}$ for all $(t,x)\in\Omega$ [4]. Problem (1.1)–(1.3) transfers to a family of boundary value problems for the following integro-differential equations of mixed type and integral relations

$$\frac{\partial v}{\partial t} = A(t,x)v(t,x) + F(t,x,u,w) +$$

$$+\Phi_1(t,x)\int_0^T \Psi_1(s,x)v(s,x)ds + \Xi_1(t,x)\int_0^t \Theta_1(s,x)v(s,x)ds, \qquad (t,x)\in\Omega, \tag{2.1}$$

$$P_1(x)v(0,x) + S_1(x)v(T,x) = \phi(x,u), \qquad x\in[0,\omega], \tag{2.2}$$

$$u(t,x) = \psi(t) + \int_0^x v(t,\xi)d\xi, \qquad w(t,x) = \dot{\psi}(t) + \int_0^x \frac{\partial v(t,\xi)}{\partial t}d\xi, \tag{2.3}$$

where

$$F(t,x,u,w)=f(t,x)+B(t,x)w(t,x)+C(t,x)u+$$
$$+\Phi_2(t,x)\int\limits_0^T\Psi_2(s,x)u(s,x)ds+\Xi_2(t,x)\int\limits_0^t\Theta_2(s,x)u(s,x)ds,$$
$$\phi(x,u)=\varphi(x)-P_2(x)u(0,x)-S_2(x)u(T,x).$$

A triple of functions $\{v(t,x),u(t,x),w(t,x)\}$, where $v(t,x)\in C(\Omega,\mathbb{R}^n)$, $u(t,x)\in C(\Omega,\mathbb{R}^n)$, $w(t,x)\in C(\Omega,\mathbb{R}^n)$ is called a solution to problem (2.1)–(2.3) if it satisfies integro-differential equations of mixed type with parameters (2.1), condition (2.2) and integral relations (2.3).

Let $u^*(t,x)$ be a classical solution to problem (1.1)–(1.3).

We construct a triple of functions $\{v^*(t,x),\ u^*(t,x),\ w^*(t,x)\}$, where $v^*(t,x)=\frac{\partial u^*(t,x)}{\partial x}$, $w^*(t,x)=\frac{\partial u^*(t,x)}{\partial t}$.

Then

$$u^*(t,x)=u^*(t,0)+\int\limits_0^x\frac{\partial u^*(t,\xi)}{\partial\xi}d\xi=\psi(t)+\int\limits_0^x v^*(t,\xi)d\xi$$

and taking into account that $u^*(t,x)$ is a solution to problem (1.1)–(1.3), we have

$$\frac{\partial^2u^*(t,x)}{\partial x\partial t}=\frac{\partial^2u^*(t,x)}{\partial t\partial x},$$

$$w^*(t,x)=\frac{\partial u^*(t,x)}{\partial t}=\frac{\partial u^*(t,0)}{\partial t}+\int\limits_0^x\frac{\partial^2u^*(t,\xi)}{\partial\xi\partial t}d\xi=$$
$$=\frac{\partial u^*(t,0)}{\partial t}+\int\limits_0^x\frac{\partial^2u^*(t,\xi)}{\partial t\partial\xi}d\xi=\dot{\psi}(t)+\int\limits_0^x\frac{\partial v^*(t,\xi)}{\partial t}d\xi,$$

$$\frac{\partial v^*}{\partial t}=\frac{\partial^2u^*}{\partial t\partial x}=A(t,x)\frac{\partial u^*}{\partial x}+B(t,x)\frac{\partial u^*}{\partial t}+C(t,x)u^*+f(t,x)+$$
$$+\Phi_1(t,x)\int\limits_0^T\Psi_1(s,x)\frac{\partial u^*(s,x)}{\partial x}ds+\Xi_1(t,x)\int\limits_0^t\Theta_1(s,x)\frac{\partial u^*(s,x)}{\partial x}ds+$$
$$+\Phi_2(t,x)\int\limits_0^T\Psi_2(s,x)u^*(s,x)ds+\Xi_2(t,x)\int\limits_0^t\Theta_2(s,x)u^*(s,x)ds=$$
$$=A(t,x)v^*+F(t,x,w^*(t,x),u^*(t,x))+$$
$$+\Phi_1(t,x)\int\limits_0^T\Psi_1(s,x)v^*(s,x)ds+\Xi_1(t,x)\int\limits_0^t\Theta_1(s,x)v^*(s,x)ds,$$

$$P_1(x)v^*(0,x)+S_1(x)v^*(T,x)=P_1(x)\frac{\partial u^*(0,x)}{\partial x}+S_1(x)\frac{\partial u^*(T,x)}{\partial x}=$$
$$=\varphi(x)-P_2(x)u^*(0,x)-S_2(x)u^*(T,x)=\phi(x,u^*),$$

i.e. the triple of functions $\{v^*(t,x),\ u^*(t,x),\ w^*(t,x)\}$ obtained in this way is a solution to problem (2.1)–(2.3).

Conversely, if a triple of functions $\{v^{**}(t,x), u^{**}(t,x), w^{**}(t,x)\}$ is a solution to problem (2.1)–(2.3), then from functional relations (2.3) we obtain that the function $u^{**}(t,x)$ satisfies the condition $u^{**}(t,0)=\psi(t)$ and has continuous partial derivatives of first order

$$\frac{\partial u^{**}(t,x)}{\partial x}=v^{**}(t,x), \quad \frac{\partial u^{**}(t,x)}{\partial t}=\dot{\psi}(t)+\int\limits_0^x \frac{\partial v^{**}(t,\xi)}{\partial t}d\xi=w^{**}(t,x),$$

and continuous partial derivatives of second order

$$\frac{\partial^2 u^{**}(t,x)}{\partial t\partial x}=\frac{\partial v^{**}(t,x)}{\partial t}, \qquad \frac{\partial^2 u^{**}(t,x)}{\partial x\partial t}=\frac{\partial v^{**}(t,x)}{\partial t}.$$

Substituting them into (2.1), (2.2), we obtain that the function $u^{**}(t,x)$ satisfies system of hyperbolic integro-differential equations of mixed type (1.1), boundary condition (1.2), respectively for all $(t,x)\in\Omega$, $x\in[0,\omega]$. Since it also satisfies initial condition (1.3), then $u^{**}(t,x)$ is a classical solution to problem (1.1)–(1.3).

Thus, the original problem for the second order system of hyperbolic integro-differential equations of mixed type (1.1)–(1.3) is reduced to an equivalent family of boundary value problems for integro-differential equations of mixed type and integral relations (2.1)–(2.3).

Here, the vector-function $v(t,x)$ is a solution to the family of boundary value problems for integro-differential equations of mixed type with parameters (2.1), (2.2), where the functional parameters $u(t,x)$ and $w(t,x)$ are related to $v(t,x)$ and $\frac{\partial v(t,x)}{\partial t}$ by integral relations (2.3).

Now, let us introduce the notations

$$z_{(1)}(t,x)=v(t,x), \qquad z_{(2)}(t,x)=\int\limits_0^t \Theta_1(s,x)v(s,x)ds, \qquad (t,x)\in\Omega.$$

Then we move on to a family of two-point boundary value problems for Fredholm integro-differential equations with unknown parameters:

$$\frac{\partial z}{\partial t}=\widetilde{A}(t,x)z(t,x)+\widetilde{\Phi}_1(t,x)\int\limits_0^T \widetilde{\Psi}_1(s,x)z(s,x)ds+\widetilde{F}(t,x,\widetilde{u},\widetilde{w}), \quad (t,x)\in\Omega, \tag{2.4}$$

$$\widetilde{P}_1(x)z(0,x)+\widetilde{S}_1(x)z(T,x)=\widetilde{\phi}(x,\widetilde{u}), \qquad x\in[0,\omega], \tag{2.5}$$

$$\widetilde{u}(t,x)=\widetilde{\psi}(t)+\int\limits_0^x z(t,\xi)d\xi, \qquad \widetilde{w}(t,x)=\dot{\widetilde{\psi}}(t)+\int\limits_0^x \frac{\partial z(t,\xi)}{\partial t}d\xi, \tag{2.6}$$

where $z(t,x)=\begin{pmatrix} z_{(1)}(t,x) \\ z_{(2)}(t,x)\end{pmatrix}$ is the unknown vector-function,

$$\widetilde{A}(t,x)=\begin{pmatrix} A(t,x) & \Xi_1(t,x) \\ \Theta_1(t,x) & O_n\end{pmatrix}, \ \widetilde{\Phi}_1(t,x)=\begin{pmatrix} \Phi_1(t,x) & O_n \\ O_n & O_n\end{pmatrix},$$

$$\widetilde{\Psi}_1(s,x)=\begin{pmatrix} \Psi_1(s,x) & O_n \\ O_n & O_n\end{pmatrix}, \ \widetilde{u}(t,x)=\begin{pmatrix} u(t,x) \\ O_n\end{pmatrix}, \ \widetilde{w}(t,x)=\begin{pmatrix} w(t,x) \\ O_n\end{pmatrix},$$

$$\widetilde{F}(t,x,\widetilde{u},\widetilde{w})=\begin{pmatrix} F(t,x,u,w) \\ O_n\end{pmatrix}, \widetilde{P}_1(x)=\begin{pmatrix} P_1(x) & O_n \\ O_n & I_n\end{pmatrix},$$

$$\widetilde{S}_1(x) = \begin{pmatrix} S_1(x) & O_n \\ O_n & O_n \end{pmatrix}, \ \widetilde{\phi}(x,\widetilde{u}) = \begin{pmatrix} \phi(x,u) \\ O_n \end{pmatrix}, \ \widetilde{\psi}(t) = \begin{pmatrix} \psi(t) \\ O_n \end{pmatrix},$$

O_n and I_n are the zero and identity matrices of dimension $n \times n$.

A triple of functions $\{z(t,x), \widetilde{u}(t,x), \widetilde{w}(t,x)\}$, where $z(t,x) \in C(\Omega, \mathbb{R}^{2n})$, $\widetilde{u}(t,x) \in C(\Omega, \mathbb{R}^{2n})$, $\widetilde{w}(t,x) \in C(\Omega, \mathbb{R}^{2n})$ is called a solution to problem (2.4)–(2.6) if it satisfies the family of Fredholm integro-differential equations with parameters (2.4), two-point condition (2.5) and integral relations (2.6).

For fixed $\widetilde{u}(t,x)$ and $\widetilde{w}(t,x)$ problem (2.4), (2.5) is the family of two-point boundary value problems for first order Fredholm integro-differential equations [8]. The unknown functions $\widetilde{u}(t,x)$ and $\widetilde{w}(t,x)$ are determined from integral relations (2.6).

It is well known that linear ordinary differential equations and Volterra integro-differential equations are solvable for any right-hand side and have classical general solutions. Note that there are linear Fredholm integro-differential equations that do not have classical general solutions [16]. An important problem arises: is it possible to construct general solutions that would exist for all differential and integro-differential equations and use them to solve boundary value problems? A new approach to defining a general solution was proposed in [17]. Based on Dzhumabaev's parametrization method [15], a new general solution is proposed, which, unlike the classical general solution, exists for all linear Fredholm integro-differential equations. Using a new general solution, criteria for the solvability of linear boundary value problems for Fredholm integro-differential equations were established and numerical and approximate methods for finding their solutions were constructed [18]. Further, these results were extended to problems with parameter for Fredholm integro-differential equations [2, 6, 9], problems for a system of differential equations with piecewise-constant argument of generalized type [1], problems for nonlinear Fredholm integro-differential equations [20].

3 Scheme of the parametrization method and Δ_N general solution

Consider the following family of problems for Fredholm integro-differential equations:

$$\frac{\partial z}{\partial t} = \widetilde{A}(t,x)z(t,x) + \widetilde{\Phi}_1(t,x)\int_0^T \widetilde{\Psi}_1(s,x)z(s,x)ds + F(t,x), \tag{3.1}$$

$$\widetilde{P}_1(x)z(0,x) + \widetilde{S}_1(x)z(T,x) = g(x), \qquad x \in [0,\omega], \tag{3.2}$$

where $z(t,x) = col(z_1(t,x), ..., z_{2n}(t,x))$ is the unknown vector-function, the $2n$ vector-function $F(t,x)$ is continuous on Ω, the $2n$ vector-function $g(x)$ is continuous on $[0,\omega]$.

A vector-function $z(t,x) = col(z_1(t,x), ..., z_{2n}(t,x)) \in C(\Omega, \mathbb{R}^n)$, which has a continuous partial derivative with respect to t is called a solution to the family of problems (3.1), (3.2), if it satisfies Fredholm integro-differential equations (3.1) for all $(t,x) \in \Omega$ and two-point conditions (3.2) for all $x \in [0,\omega]$.

The domain Ω is divided into subdomains and this partition is denoted by Δ_N:

$$\Omega = \bigcup_{r=1}^{N} \Omega_r, \quad \Omega_r = [t_{r-1}, t_r) \times [0,\omega], \quad r = \overline{1,N}, \quad 0 = t_0 < t_1 < ... < t_N = T.$$

Let $C(\Omega, \Delta_N, \mathbb{R}^{2nN})$ be the space of all vector-functions $z([t],x) = col(z_1(t,x), z_2(t,x), ..., z_N(t,x))$, where the notation $[t]$ means partition by t, the functions

$z_r : \Omega_r \to \mathbb{R}^{2n}$ are continuous and have finite left-sided limits $\lim\limits_{t\to t_r-0} z_r(t,x)$ uniformly with respect to $x \in [0,\omega]$ for all $r = \overline{1,N}$, with the norm

$$||v([\cdot],x)||_2 = \max_{r=\overline{1,N}} \sup_{t\to t_r-0} ||v_r(t,x)||.$$

We denote by $z_r(t,x)$ the restriction of the solution $z(t,x)$ to the subdomain Ω_r, i.e. $z_r(t,x) = z(t,x)$ for $(t,x) \in \Omega_r$, $r = \overline{1,N}$.

Then the vector-functions $z([t],x) = col(z_1(t,x),...,z_N(t,x)) \in C(\Omega,\Delta_N,\mathbb{R}^{2nN})$ with elements $z_r(t,x)$, $r = \overline{1,N}$, satisfy the following Fredholm integro-differential equations

$$\frac{\partial z_r}{\partial t} = \widetilde{A}(t,x)z_r(t,x) + \widetilde{\Phi}_1(t,x)\sum_{j=1}^{N}\int\limits_{t_{j-1}}^{t_j}\widetilde{\Psi}_1(s,x)z_j(s,x)ds + F(t,x), \tag{3.3}$$

$$(t,x) \in \Omega_r, \qquad r = \overline{1,N}.$$

Let us introduce functional parameters $\lambda_r(x) = z_r(t_{r-1},x)$, $r = \overline{1,N}$, $x \in [0,\omega]$. By replacing $\widetilde{z}_r(t,x) = z_r(t,x) - \lambda_r(x)$ on each r-th domain Ω_r, we obtain the following system Fredholm integro-differential equations with parameters

$$\frac{\partial \widetilde{z}_r}{\partial t} = \widetilde{A}(t,x)\widetilde{z}_r(t,x) + \widetilde{\Phi}_1(t,x)\sum_{j=1}^{N}\int\limits_{t_{j-1}}^{t_j}\widetilde{\Psi}_1(s,x)\widetilde{z}_j(s,x)ds + F(t,x)+$$

$$+\widetilde{A}(t,x)\lambda_r(x) + \widetilde{\Phi}_1(t,x)\sum_{j=1}^{N}\int\limits_{t_{j-1}}^{t_j}\widetilde{\Psi}_1(s,x)ds\lambda_j(x), \quad (t,x)\in\Omega_r,\ r=\overline{1,N}. \tag{3.4}$$

and initial conditions

$$\widetilde{z}_r(t_{r-1},x) = 0, \qquad x \in [0,\omega], \qquad r = \overline{1,N}. \tag{3.5}$$

For fixed $\lambda_r(x) \in C([0,\omega],\mathbb{R}^{2n})$, a special Cauchy problem for the system of Fredholm integro-differential equations (3.4), (3.5) is obtained. The family of problems (3.4), (3.5) has a unique solution is a system functions $\widetilde{z}([t],x,\lambda) = col(\widetilde{z}_1(t,x,\lambda_1),\widetilde{z}_2(t,x,\lambda_2),...,\widetilde{z}_N(t,x,\lambda_N))$ with elements $\widetilde{z}_r(t,x,\lambda_r)$ belongs to $C(\Omega,\Delta_N,\mathbb{R}^{2nN})$.

A vector-function $\widetilde{z}([t],x,\lambda)$ is called a solution special Cauchy problem with parameters (3.4), (3.5).

Let us now introduce a new general solution to the family of integro-differential equations (3.1).

Definition 1. Let $\widetilde{z}([t],x,\lambda) = col(\widetilde{z}_1(t,x,\lambda_1),\widetilde{z}_2(t,x,\lambda_2),\ldots,\widetilde{z}_N(t,x,\lambda_N))$ be a solution to a special Cauchy problem (3.4), (3.5) for the parameter $\lambda(x) = (\lambda_1(x),\lambda_2(x),...,\lambda_N(x)) \in C([0,\omega],\mathbb{R}^{2nN})$. Then the function $z(\Delta_N,t,x,\lambda)$, given by the equalities

$z(\Delta_N,t,x,\lambda) = \lambda_r(x) + \widetilde{z}_r(t,x,\lambda_r)$, for $(t,x) \in \Omega_r$, $r = \overline{1,N}$,

and

$z(\Delta_N,T,x,\lambda) = \lambda_N(x) + \lim\limits_{t\to T-0}\widetilde{z}_N(t,x,\lambda_N)$,

is called a Δ_N general solution to family of Fredholm integro-differential equations (3.1).

From Definition 3.1 it is clear that a Δ_N general solution depends on N arbitrary functions $\lambda_r(x) \in C([0,\omega],\mathbb{R}^{2n})$, $x \in [0,\omega]$, $r = \overline{1,N}$, and satisfies family of integro-differential equations (3.1) for all $(t,x) \in (0,T)\backslash\{t_p, p = \overline{1,N-1}\} \times [0,\omega]$.

Using a fundamental matrix $U_r(t,x)$ of the family of differential equations

$$\frac{\partial z_r}{\partial t} = \widetilde{A}(t,x)z_r(t,x), \qquad (t,x)\in\Omega_r, \quad r=\overline{1,N},$$

we write the solution to the family of special Cauchy problems with parameters (3.4), (3.5) in the following form

$$\widetilde{z}_r(t,x) = U_r(t,x)\int\limits_{t_{r-1}}^{t} U_r^{-1}(\tau,x)\widetilde{\Phi}_1(\tau,x)\sum_{j=1}^{N}\int\limits_{t_{j-1}}^{t_j}\widetilde{\Psi}_1(s,x)\widetilde{z}_j(s,x)ds d\tau +$$

$$+U_r(t,x)\int\limits_{t_{r-1}}^{t} U_r^{-1}(\tau,x)F(\tau,x)d\tau + U_r(t,x)\int\limits_{t_{r-1}}^{t} U_r^{-1}(\tau,x)\widetilde{A}(\tau,x)d\tau\lambda_r(x)+$$

$$+U_r(t,x)\int\limits_{t_{r-1}}^{t} U_r^{-1}(\tau,x)\widetilde{\Phi}_1(\tau,x)\sum_{j=1}^{N}\int\limits_{t_{j-1}}^{t_j}\widetilde{\Psi}_1(s,x)ds\lambda_j(x), \quad (t,x)\in\Omega_r, \; r=\overline{1,N}. \tag{3.6}$$

Consider the following family of Cauchy problems on subdomains

$$\frac{\partial z_r}{\partial t} = \widetilde{A}(t,x)z_r(t,x) + P(t,x), \quad z(t_{r-1},x)=0, \quad (t,x)\in\Omega_r, \quad r=\overline{1,N}, \tag{3.7}$$

where $P(t,x)$ is a square matrix or a vector of dimension $2n$, continuous on Ω.

Let us denote by $a_r(P,t,x)$ the unique solution to family of Cauchy problems (3.7) on each r-th domain. It has the following form

$$a_r(P,t,x) = U_r(t,x)\int\limits_{t_{r-1}}^{t} U_r^{-1}(\tau,x)P(\tau,x)d\tau, \quad (t,x)\in\Omega_r, \quad r=\overline{1,N}.$$

We introduce the notation $\mu(x)=\sum\limits_{j=1}^{N}\int\limits_{t_{j-1}}^{t_j}\widetilde{\Psi}_1(s,x)\widetilde{z}_j(s,x)ds$. Then we can rewrite (3.7) in the form

$$\widetilde{z}_r(t,x) = a_r(\widetilde{\Phi}_1,t,x)\mu(x) + a_r(F,t,x) + a_r(\widetilde{A},t,x)\lambda_r(x)+$$

$$+a_r(\widetilde{\Phi}_1,t,x)\sum_{j=1}^{N}\int\limits_{t_{j-1}}^{t_j}\widetilde{\Psi}_1(s,x)ds\lambda_j(x), \quad (t,x)\in\Omega_r, \; r=\overline{1,N}. \tag{3.8}$$

First multiplying both sides by $\widetilde{\Psi}_1(t,x)$, integrating over t from t_{r-1} to t_r, summing over $r=\overline{1,N}$, we obtain from (3.8) the following system of equations:

$$[I-G(N,x)]\mu(x) = \sum_{r=1}^{N}\int\limits_{t_{r-1}}^{t_r}\widetilde{\Psi}_1(t,x)a_r(F,t,x)dt + \sum_{r=1}^{N}\int\limits_{t_{r-1}}^{t_r}\widetilde{\Psi}_1(t,x)a_r(\widetilde{A},t,x)dt\lambda_r(x)+$$

$$+\sum_{r=1}^{N}\int\limits_{t_{r-1}}^{t_r}\widetilde{\Psi}_1(t,x)a_r(\widetilde{\Phi}_1,t,x)dt\sum_{j=1}^{N}\int\limits_{t_{j-1}}^{t_j}\widetilde{\Psi}_1(s,x)ds\lambda_j(x), \quad (t,x)\in\Omega_r, \; r=\overline{1,N}, \tag{3.9}$$

where I is a unit matrix on dimension $2n$, $G(N,x)=\sum_{r=1}^{N}\int_{t_{r-1}}^{t_r}\widetilde{\Psi}_1(t,x)a_r(\widetilde{\Phi}_1,t,x)dt$.

Assuming the invertibility of the $2n\times 2n$ matrix $I-G(N,x)$, from (3.9) for all $x\in[0,\omega]$ we uniquely define $\mu(x)$ in terms of $\lambda_r(x)$, $r=\overline{1,N}$, and $F(t,x)$. Then, substituting the found expression instead of $\mu(x)$ in (3.8), we obtain a representation of $\widetilde{z}_r(t,x)$ via $\lambda_r(x)$, $(t,x)\in\Omega_r$, $r=\overline{1,N}$.

Corollary 3.1. *Let $z^*(t,x)$ be a solution to system of equations (3.1) and $z(\Delta_N,t,x,\lambda)$ be a Δ_N general solution to family integro-differential equations (3.1).*

Then there exists a unique $\lambda^*(x)=col(\lambda_1^*(x),\lambda_2^*(x),\ldots,\lambda_N^*(x))\in C([0,\omega],\mathbb{R}^{2nN})$ such that the equality $z(\Delta_N,t,x,\lambda^*)=z^*(t,x)$ holds for all $(t,x)\in\Omega$.

If $z(t,x)$ is a solution to system (3.1), and $z([t],x)=col(z_1(t,x),z_2(t,x),...,z_N(t,x))$ is the vector-function composed of its restrictions to the subdomains Ω_r, $r=\overline{1,N}$, then the following equalities

$$\lim_{t\to t_p-0} z_p(t,x)=z_{p+1}(t_p,x),\qquad x\in[0,\omega],\qquad p=\overline{1,N-1},\tag{3.10}$$

hold. These equalities are the continuity conditions for the solution to system (3.1) at the interior lines of the partition Δ_N.

Theorem 3.1. *Let a vector-function $z([t],x)=col(z_1(t,x),z_2(t,x),...,z_N(t,x))$ belong to $C(\Omega,\Delta_N,\mathbb{R}^{2nN})$. Assume that the functions $z_r(t,x)$, $r=\overline{1,N}$, satisfy system (3.1) and continuity conditions (3.10). Then the function $z^*(t,x)$, given by the equalities*

$z^*(t,x)=z_r(t,x)$ *for* $t\in(t,x)\in\Omega_r,\quad r=\overline{1,N}$,

and

$z^*(T,x)=\lim_{t\to T-0} z_N(t,x),\quad x\in[0,\omega]$,

is continuously differentiable on Ω and satisfies system (3.1).

Now, we consider family of problems for systems of $2n$ Fredholm integro-differential equations (3.1), (3.2). Using notations above, we obtain the following family of problems

$$\frac{\partial\widetilde{z}_r}{\partial t}=\widetilde{A}(t,x)\widetilde{z}_r(t,x)+\widetilde{\Phi}_1(t,x)\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\widetilde{\Psi}_1(s,x)\widetilde{z}_j(s,x)ds+F(t,x)+$$

$$+\widetilde{A}(t,x)\lambda_r(x)+\widetilde{\Phi}_1(t,x)\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\widetilde{\Psi}_1(s,x)ds\lambda_j(x),\quad (t,x)\in\Omega_r,\ r=\overline{1,N}.\tag{3.11}$$

$$\widetilde{z}_r(t_{r-1},x)=0,\qquad x\in[0,\omega],\qquad r=\overline{1,N}.\tag{3.12}$$

$$\widetilde{P}_1(x)\lambda_1(x)+\widetilde{S}_1(x)\lim_{t\to T-0}\widetilde{z}_N(t,x)+\widetilde{S}_1(x)\lambda_N(x)=g(x),\qquad x\in[0,\omega],\tag{3.13}$$

$$\lim_{t\to t_p-0}\widetilde{z}_p(t,x)+\lambda_p(x)=\lambda_{p+1}(x),\qquad x\in[0,\omega],\qquad p=\overline{1,N-1}.\tag{3.14}$$

A solution to problem (3.11)–(3.14) is the pair $\{\widetilde{z}([t],x),\lambda(x)\}$, where the vector-functions $\widetilde{z}([t],x)=col(\widetilde{z}_1(t,x),\widetilde{z}_2(t,x),...,\widetilde{z}_N(t,x))\in C(\Omega,\Delta_N,\mathbb{R}^{2nN})$, $\lambda(x)=col(\lambda_1(x),\lambda_2(x),...,\lambda_N(x))\in C([0,\omega],\mathbb{R}^{2nN})$ with the elements $\widetilde{z}_r(t,x)$, $\lambda_r(x)$, $r=\overline{1,N}$, satisfy system (3.11), initial conditions (3.12), boundary conditions (3.13), continuity conditions (3.14).

Using the results of this section, we obtain a representation of $\widetilde{z}_r(t,x)$ in terms of $\lambda_r(x)$, $(t,x)\in\Omega_r$, $r=\overline{1,N}$. From the resulting representation, determining the values of the left-hand limits

$\lim_{t\to T-0} \widetilde{z}_N(t,x)$, $\lim_{t\to t_p-0} \widetilde{z}_p(t,x)$, $p=\overline{1,N-1}$, and substituting into conditions (3.13), (3.14), we obtain a system of functional equations of the form

$$Q(\Delta_N,x)\lambda(x) = -E(\Delta_N,x,g,F), \qquad \lambda(x)\in C([0,\omega],\mathbb{R}^{2nN}), \tag{3.15}$$

where $Q(\Delta_N,x)$ is a $2nN\times 2nN$ matrix, composed of the functions $\lambda_r(x)\in C([0,\omega],\mathbb{R}^{2n})$, $r=\overline{1,N}$, and $E(\Delta_N,x,g,F)$ contains the right-hand sides F and g.

Theorem 3.2. *Let the $2n\times 2n$ matrix $I-G(N,x)$ and the $2nN\times 2nN$ matrix $Q(\Delta_N,x)$ be invertible for all $x\in[0,\omega]$. Then family of problems (3.11) − (3.14) has a unique solution $\{\widetilde{z}^*([t],x),\lambda^*(x)\}$.*

From the equivalence of problems (3.1), (3.2) and (3.11)–(3.14) follows

Theorem 3.3. *Let the $2n\times 2n$ matrix $I-G(N,x)$ and the $2nN\times 2nN$ matrix $Q(\Delta_N,x)$ be invertible for all $x\in[0,\omega]$. Then family of boundary value problems for system of Fredholm integro-differential equations (3.1), (3.2) has a unique solution $z^*(t,x)$.*

The proofs of these theorems are similar to the proofs of the corresponding theorems in [8].

4 Algorithm and main results

Based on the results of Section 3, we offer the following algorithm for finding a solution to the family of two-point boundary value problems for system of Fredholm integro-differential equations with functional parameters (2.4)–(2.6).

Algorithm.

Step 1. i) Assume that $\widetilde{u}^{(0)}(t,x)=\widetilde{\psi}(t)$, $\widetilde{w}^{(0)}(t,x)=\dot{\widetilde{\psi}}(t)$ in the left-hand side of (2.4), (2.5). Solving the family of two-point boundary value problems for system of Fredholm integro-differential equations, we find of a function $z^{(1)}(t,x)$ for all $(t,x)\in\Omega$. ii) From integral relations (2.6) we determine $\widetilde{u}^{(1)}(t,x)$ and $\widetilde{w}^{(1)}(t,x)$ for $z(t,x)=z^{(1)}(t,x)$ and $\frac{\partial z(t,x)}{\partial t}=\frac{\partial z^{(1)}(t,x)}{\partial t}$ for all $(t,x)\in\Omega$.

And so on.

Step k. i) Assume that $\widetilde{u}(t,x)=\widetilde{u}^{(k-1)}(t,x)$, $\widetilde{w}(t,x)=\widetilde{w}^{(k-1)}(t,x)$ in the left-hand side of (2.4), (2.5). Solving the family of two-point boundary value problems for system of Fredholm integro-differential equations, we find the function $z^{(k)}(t,x)$ for all $(t,x)\in\Omega$. ii) From integral relations (2.6) we determine the functions $\widetilde{u}^{(k)}(t,x)$ and $\widetilde{w}^{(k)}(t,x)$ for $z(t,x)=z^{(k)}(t,x)$ and $\frac{\partial z(t,x)}{\partial t}=\frac{\partial z^{(k)}(t,x)}{\partial t}$ for all $(t,x)\in\Omega$.

$k=1,2,...,$

The algorithm for finding a solution to the family of two-point boundary value problems for system of Fredholm integro-differential equations with functional parameters (2.4)–(2.6) consists of two stages: 1) the family of two-point boundary value problems for system of Fredholm integro-differential equations (2.4), (2.5) is solved and the unknown function $z(t,x)$ is found for fixed $\widetilde{u}(t,x)$ and $\widetilde{w}(t,x)$; 2) $\widetilde{u}(t,x)$ and $\widetilde{w}(t,x)$ are determined from integral relations (2.6) by using $z(t,x)$ and $\frac{\partial z(t,x)}{\partial t}$.

We show that the conditions for unique solvability of the family of two-point boundary value problems for system of Fredholm integro-differential equations (3.1), (3.2) are the convergence conditions of the proposed algorithm.

For fixed $\widetilde{u}(t,x)$ and $\widetilde{w}(t,x)$ the family of two-point boundary value problems for system of Fredholm integro-differential equations with functional parameters (2.4)–(2.6) is the family of boundary value problems for system of Fredholm integro-differential equations (3.1), (3.2) with

$F(t,x) = \widetilde{F}(t,x,\widetilde{u},\widetilde{w})$, $g(x) = \widetilde{\phi}(x,\widetilde{u})$. From Theorem 3.1 it follows that the family of boundary value problems for system of Fredholm integro-differential equations (3.1), (3.2) has a unique solution $z^*(t,x)$. Moreover, similarly to Theorem 2.1 in [16], the conditions of Theorem 3.1 ensure that the estimate

$$\max_{t\in[0,T]} ||z^*(t,x)|| \leq \mathcal{N}(x)\max\Big(||g(x)||, \max_{t\in[0,T]}||F(t,x)||\Big), \tag{4.1}$$

holds, where

$$\mathcal{N}(x) =$$
$$e^{\alpha(x)\theta}\Big\{\widetilde{\Phi}_1^*(x)\Big[||[I-G(N,x)]^{-1}||\widetilde{\Psi}_1^*(x)\Big(e^{\alpha(x)\theta}-1+e^{\alpha(x)\theta}\widetilde{\Phi}_1^*(x)\widetilde{\Psi}_1^*(x)\Big)+\widetilde{\Psi}_1^*(x)\Big]+1\Big\}$$
$$\times||[Q(\Delta_N,x)]^{-1}||(1+||\widetilde{S}_1(x)||)\max\Big\{1,\theta e^{\alpha(x)\theta}\Big[1+e^{\alpha(x)\theta}\widetilde{\Phi}_1^*(x)||[I-G(N,x)]^{-1}||\widetilde{\Psi}_1^*(x)\Big]\Big\}$$
$$+e^{\alpha(x)\theta}\Big[\widetilde{\Phi}_1^*(x)||[I-G(N,x)]^{-1}||\widetilde{\Psi}_1^*(x)e^{\alpha(x)\theta}+1\Big],$$
$$\alpha(x) = \max_{t\in[0,T]}||\widetilde{A}(t,x)||, \qquad \theta = \max_{r=\overline{1,N}}(t_r - t_{r-1}),$$
$$\widetilde{\Phi}_1^*(x) = \max_{r=\overline{1,N}}\int_{t_{r-1}}^{t_r}||\widetilde{\Phi}_1(t,x)||dt, \qquad \widetilde{\Psi}_1^*(x) = \int_0^T||\widetilde{\Psi}_1(t,x)||dt.$$

Suppose $\widetilde{u}^{(k-1)}(t,x)$ and $\widetilde{w}^{(k-1)}(t,x)$ are known. According to the Step k of the algorithm, we have

$$\max_{t\in[0,T]}||z^{(k)}(t,x)|| \leq \mathcal{N}(x)\max\Big(||\widetilde{\phi}(x,\widetilde{u}^{(k-1)})||, \max_{t\in[0,T]}||\widetilde{F}(t,x,\widetilde{u}^{(k-1)},\widetilde{w}^{(k-1)})||\Big), \tag{4.2}$$

$$\max_{t\in[0,T]}\Big|\Big|\frac{\partial z^{(k)}(t,x)}{\partial t}\Big|\Big| \leq \Big\{\max\Big(\alpha(x)+\max_{t\in[0,T]}||\widetilde{\Phi}_1(t,x)||\widetilde{\Psi}_1^*(x)\Big)\mathcal{N}(x)+1\Big\}$$
$$\times\max\Big(||\widetilde{\phi}(x,\widetilde{u}^{(k-1)})||, \max_{t\in[0,T]}||\widetilde{F}(t,x,\widetilde{u}^{(k-1)},\widetilde{w}^{(k-1)})||\Big), \tag{4.3}$$

$k = 1,2,\ldots$.

Once $z^{(k)}(t,x)$ is found the successive approximations for $\widetilde{u}(t,x)$ and $\widetilde{u}(t,x)$ are found from relations (2.6):

$$\widetilde{u}^{(k)}(t,x) = \widetilde{\psi}(t) + \int_0^x z^{(k)}(t,\xi)d\xi, \qquad \widetilde{w}^{(k)}(t,x) = \dot{\widetilde{\psi}}(t) + \int_0^x \frac{\partial z^{(k)}(t,\xi)}{\partial t}d\xi, \tag{4.4}$$

We construct the differences $\Delta z^{(k)}(t,x) = z^{(k)}(t,x) - z^{(k-1)}(t,x)$, $\Delta\widetilde{u}^{(k)}(t,x) = \widetilde{u}^{(k)}(t,x) - \widetilde{u}^{(k-1)}(t,x)$, $\Delta\widetilde{w}^{(k)}(t,x) = \widetilde{w}^{(k)}(t,x) - \widetilde{w}^{(k-1)}(t,x)$, and by using the unique solvability of family problems (3.1), (3.2), and estimates (4.2), (4.3), we establish estimates

$$\max\Big\{\max_{t\in[0,T]}||\Delta z^{(k+1)}(t,x)||, \max_{t\in[0,T]}\Big|\Big|\frac{\partial\Delta z^{(k+1)}(t,x)}{\partial t}\Big|\Big|\Big\}$$
$$\leq \max\Big\{\mathcal{N}(x), \max\Big(\alpha(x)+\max_{t\in[0,T]}||\widetilde{\Phi}_1(t,x)||\widetilde{\Psi}_1^*(x)\Big)\mathcal{N}(x)+1\Big\}\mathcal{N}_1(x)$$
$$\times\max\Big\{\max_{t\in[0,T]}||\Delta\widetilde{w}^{(k)}(t,x)||, \max_{t\in[0,T]}||\Delta\widetilde{u}^{(k)}(t,x)||\Big\}, \tag{4.5}$$

$$\max\Big\{\max_{t\in[0,T]}||\Delta w^{(k)}(t,x)||, \max_{t\in[0,T]}||\Delta u^{(k)}(t,x)||\Big\}$$

$$\leq \int_0^x \max\Big\{\max_{t\in[0,T]}||\Delta z^{(k)}(t,\xi)||, \max_{t\in[0,T]}\Big|\Big|\frac{\partial \Delta z^{(k)}(t,\xi)}{\partial t}\Big|\Big|\Big\}d\xi, \tag{4.6}$$

where

$$\mathcal{N}_1(x) = \max\Big\{||P_2(x)|| + ||S_2(x)||, \max_{t\in[0,T]}||B(t,x)|| + \max_{t\in[0,T]}||C(t,x)||$$

$$+ \max_{t\in[0,T]}||\Phi_2(t,x)||T\max_{t\in[0,T]}||\Psi_2(t,x)|| + \max_{t\in[0,T]}||\Xi_2(t,x)||T\max_{t\in[0,T]}||\Theta_2(t,x)||\Big\}.$$

This implies the main inequality

$$\max\Big\{\max_{t\in[0,T]}||\Delta z^{(k+1)}(t,x)||, \max_{t\in[0,T]}\Big|\Big|\frac{\partial \Delta z^{(k+1)}(t,x)}{\partial t}\Big|\Big|\Big\}$$

$$\leq \max\Big\{\mathcal{N}(x), \max\Big(\alpha(x) + \max_{t\in[0,T]}||\widetilde{\Phi}_1(t,x)||\widetilde{\Psi}_1^*(x)\Big)\mathcal{N}(x)+1\Big\}\mathcal{N}_1(x)$$

$$\times\int_0^x \max\Big\{\max_{t\in[0,T]}||\Delta z^{(k)}(t,\xi)||, \max_{t\in[0,T]}\Big|\Big|\frac{\partial \Delta z^{(k)}(t,\xi)}{\partial t}\Big|\Big|\Big\}d\xi. \tag{4.7}$$

From (4.7) it follows that the sequences $\{z^{(k)}(t,x)\}$ and $\{\frac{\partial z^{(k)}(t,x)}{\partial t}\}$ are convergent in the space $C(\Omega,\mathbb{R}^{2n})$ as $k\to\infty$. Then the uniform convergence on Ω of the sequences $\{\widetilde{u}^{(k)}(t,x)\}$ and $\{\widetilde{w}^{(k)}(t,x)\}$ follows from estimate (4.6).

In this case, the limit functions $z^*(t,x)$, $\frac{\partial z^*(t,x)}{\partial t}$, $\widetilde{u}^*(t,x)$ and $\widetilde{w}^*(t,x)$ are continuous on Ω, and the triple $\{z^*(t,x),\ \widetilde{u}^*(t,x),\ \widetilde{w}^*(t,x)\}$ is a solution to problem (2.4)-(2.6).

The uniqueness of a solution to problem (2.4)-(2.6) is proved assuming the contrary.

Now, using the constructed solution to the family of problems (2.4)–(2.6), the triple of functions $\{z^*(t,x), \widetilde{u}^*(t,x), \widetilde{w}^*(t,x)\}$, we verify the validity of the following equalities:

$$z^*(t,x) = col(z^*_{(1)}(t,x), z^*_{(2)}(t,x)),$$

$$\widetilde{u}^*(t,x) = \widetilde{\psi}(t) + \int_0^x z^*(t,\xi)d\xi, \qquad \widetilde{w}^*(t,x) = \dot{\widetilde{\psi}}(t) + \int_0^x \frac{\partial z^*(t,\xi)}{\partial t}d\xi,$$

$$u^*(t,x) = \psi(t) + \int_0^x z^*_{(1)}(t,\xi)d\xi \qquad \text{for all} \qquad (t,x)\in\Omega.$$

The function $u^*(t,x)$ is the desired solution to problem (1.1)–(1.3).

Theorem 4.1. *Let the $2n\times 2n$ matrix $I-G(N,x)$ and the $2nN\times 2nN$ matrix $Q(\Delta_N,x)$ be invertible for all $x\in[0,\omega]$. Then boundary value problem for system of hyperbolic integro-differential equations of mixed type* (3.1) $-$ (3.3) *has the unique solution $u^*(t,x)$.*

The proof of this theorem follows from the above algorithm and is similar to the proof of Theorem 3.2 in [4].

Conclusion. In the paper, we propose an effective method of solving the boundary value problem for a second order system of hyperbolic integro-differential equations of mixed type with degenerate kernels. This method is based on the method of introducing new functions, Dzhumabaev's parametrization method and a new concept of a general solution to a family Fredholm integro-differential equations. New general solution enables us to establish qualitative properties of the boundary value problems for second order systems of hyperbolic integro-differential equations and to develop algorithms for solving them. The algorithms are based on constructing and solving systems of linear functional equations with respect to the new general solution and integral equations. Further, we will study the boundary value problem for second order systems of hyperbolic integro-differential equations of mixed type in general case. The obtained results can be used to solve the boundary value problems for impulsive hyperbolic integro-differential equations of mixed type.

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Anar Turmaganbetkyzy Assanova
Department of Differential Equations and Dynamical Systems
Institute of Mathematics and Mathematical Modeling
28 Shevchenko St,
050010 Almaty, Republic of Kazakhstan
E-mails: assanova@math.kz; anartasan@gmail.com

Zagira Saparbekovna Kobeyeva, Ryskul Ashimalievna Medetbekova
Department of Mathematics and Informatics
Shymkent University
131 Jibek Joly St,
160023, Shymkent, Republic of Kazakhstan
E-mail: ryskulmedetbekova@mail.ru

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