Short communications

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ON THE SINGULAR NUMBERS OF CORRECT RESTRICTIONS OF NON-SELFADJOINT ELLIPTIC DIFFERENTIAL OPERATORS

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Abstract. Conditions are established on a correct restriction of an elliptic differential operator of order $2l$ defined on a bounded domain in $\mathbb{R}^n$ with sufficiently smooth boundary, ensuring that its singular numbers s_k are of order $k^{-\frac{2l}{n}}$. As an application certain estimates are obtained for the deviation upon domain perturbation of singular numbers of such correct restrictions.

Let $l, n \in \mathbb{N}$ and $\mathcal{L}$ be an elliptic differential expression of the following form: for $u \in C^\infty(\mathbb{R}^n)$

$$(\mathcal{L}u)(x) = \sum_{|\alpha|,|\beta|\le l} (-1)^{|\alpha|+|\beta|} D^\alpha \left(A_{\alpha\beta}(x) D^\beta u\right), \quad x \in \mathbb{R}^n,$$

where $A_{\alpha\beta} \in C^l(\mathbb{R}^n)$ are real-valued functions for all multi-indices α, β satisfying $|\alpha|, |\beta| \le l$. Moreover, let, for a domain $\Omega \subseteq \mathbb{R}^n$, $L_\Omega : D(L_\Omega) \to L_2(\Omega)$ be a linear operator closed in $L_2(\Omega)$ generated by $\mathcal{L}$ on Ω.

A restriction $A : D(A) \to L_2(\Omega)$ of L_Ω is *correct* if the equation $Au = f$ has a unique solution $u \in D(A)$ for any $f \in L_2(\Omega)$ and the corresponding inverse operator $A^{-1} : L_2(\Omega) \to D(A)$ is bounded. For the properties of correct restrictions see [5], [3].

Definition 1. Let A and B be compact linear operators in a Hilbert space H. If there exist $0 < a < b$ and $c_1, c_2 > 0$ such that for singular numbers[1] $s_k(A)$ and $s_k(B)$ the condition

$$c_1 k^{-a} \le s_k(A), \quad s_k(B) \le c_2 k^{-b}.$$

holds, we say that in the representation $C = A+B$ the operator A is a *leading* operator and the operator B is a *non-leading* operator.

[1] As usual it is assumed that $s_1(A) \ge s_2(A) \ge \cdots \ge s_k(A) \ge \cdots$ where each singular number is repeated as many times as its multiplicity. The same refers to the operator B.

Theorem 1. *Let A and B be compact linear operators in a Hilbert space H. If in the representation $C = A + B$ the operator A is a leading operator and the operator B is a non-leading operator and for all $0 < \sigma < 1$*

$$\lim_{k\to\infty} \frac{s_{k+[k^\sigma]}(A)}{s_k(A)} = 1\,,$$

then

$$\lim_{k\to\infty} \frac{s_k(A+B)}{s_k(A)} = 1\,.$$

Idea of the proof. The well-known inequality $s_k(A+B) \le s_m(A)+s_n(B)$ if $m+n = k+1$ implies that $s_k(A+B) \le s_{k-[k^\theta]+1}(A)(1 + c_3 k^{\alpha-\theta\beta})$ for all $k \in \mathbb{N}$, where $c_3 = c_1^{-1}c_2 2^\beta$ and $\frac{\alpha}{\beta} < \theta < 1$. Replacing here A by $A+B$, B by $-A$, and $k - [k^\theta] + 1$ by k one can prove that for each $\gamma > 1$ there exists $c_4 > 0$ and $\varkappa_\gamma \in \mathbb{N}$ such that $s_k(A+B) \ge s_{k+[\gamma k^\theta]}(A)(1 - c_4 k^{\alpha-\theta\beta})$ for all $k \ge \varkappa_\gamma$, which implies the result.

Theorem 2. *Let $l, n \in \mathbb{N}$, $n \ge 2$, $2l(1-\frac{1}{n}) < s \le 2l$, and Ω be a bounded domain in $\mathbb{R}^n$ with the boundary $\partial\Omega$ of class C^{2l}. Moreover, let A and B be correct restrictions of the operator L_Ω such that*

$$D(A) \subseteq W_2^{2l}(\Omega)\,, \quad D(B) \subseteq W_2^s(\Omega)$$

and the operators $A^{-1} : L_2(\Omega) \to W_2^{2l}(\Omega)\,,\quad B^{-1} : L_2(\Omega) \to W_2^s(\Omega)$ are bounded. Then in the representation $B^{-1} = A^{-1} + K$ the operator A^{-1} is a leading operator and the operator $K = B^{-1} - A^{-1}$ is a non-leading operator.

Idea of the proof. Given $f \in L_2(\Omega)$ we consider Kf as a solution of the Dirichlet boundary value problem $L_\Omega u = 0$ with the boundary data

$$\mathrm{tr}_{\partial\Omega}\Big(\frac{\partial^m u}{\partial n^m}\Big) = \mathrm{tr}_{\partial\Omega}\Big(\frac{\partial^m Kf}{\partial n^m}\Big) \in W_2^{s-m-\frac{1}{2}}(\partial\Omega), \quad m = 0, ..., l-1,$$

where $s - m - \frac{1}{2} > 0$. Next we use the facts that for any $0 < \mu < s$ the embedding operator $E_1 : W_2^\mu(\Omega) \to L_2(\Omega)$ is compact and its singular numbers $s_k(E_1)$ have the order $k^{-\frac{\mu}{n}}$, and that the embedding operators

$$E_{2j} : W_2^{s-j-\frac{1}{2}}(\partial\Omega) \to W_2^{\mu-j-\frac{1}{2}}(\partial\Omega)\,, \quad j = 0, 1, ..., l-1\,.$$

are compact and their singular numbers $s_k(E_{2j})$ have the same order $k^{-\frac{s-\mu}{n-1}}$. (See [6].) Finally we apply deep results from [4] on solvability of this Dirichlet problem for the case in which the boundary data belongs to Sobolev spaces of negative order.

Theorem 3. *Let $l, n \in \mathbb{N}$, $n \ge 2$, $2l(1-\frac{1}{n}) < s \le 2l$ and Ω be a bounded domain in $\mathbb{R}^n$ with the boundary $\partial\Omega$ of class C^{2l}. Then there exists $b > 0$ such that the singular numbers [2] $s_k(B)$ of all correct restrictions B of the operator L_Ω satisfying the condition $D(B) \subseteq W_2^s(\Omega)$ with the bounded inverse $B^{-1} : L_2(\Omega) \to W_2^{2l}(\Omega)$*

$$\lim_{k\to\infty} s_k(B) k^{-\frac{2l}{n}} = b\,.$$

[2] As usual it is assumed that $s_1(B) \le s_2(B) \le \cdots \le s_k(B) \le \cdots$ where each singular number is repeated as many times as its multiplicity.

Idea of the proof. We consider a correct restriction A of the operator L_Ω satisfying the condition $D(A) \subseteq W_2^{2l}(\Omega)$ with the bounded inverse $A^{-1} : L_2(\Omega) \to W_2^{2l}(\Omega)$ whose asymptotics of singular numbers is known, represent the inverse B^{-1} in the form $B^{-1} = A^{-1} + K$, where by Theorem 2 A^{-1} and K are a leading operator, a non-leading operator respectively, and apply Theorem 1.

Let now, for $u \in C^\infty(\mathbb{R}^n)$, $\mathcal{L}u$ be a second order elliptic differential expression without lower terms with symmetric $A_{\alpha\beta}$, namely

$$\mathcal{L}u = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big), \quad x \in \mathbb{R}^n,$$

where $a_{ij} \in C^1(\mathbb{R}^n)$ are real-valued functions satisfying $a_{ij} = a_{ji}$ for all $i, j = 1, ..., n$. Moreover, let $\mathcal{A}$ be a fixed atlas in $\mathbb{R}^n$, $M > 0$, and $C_M^2(\mathcal{A})$ be a family of bounded domains $\Omega \subseteq \mathbb{R}^n$ with boundaries of class C^2, whose precise description is given in [1], [2].

Theorem 4. *Let $n \in \mathbb{N}$, $n \geq 2$, $2 - \frac{2}{n} < s \leq 2$. Moreover, let $\mathfrak{B}(\mathcal{A}) = \{B_\Omega\}_{\Omega \in C_M^2(\mathcal{A})}$ be a family of correct restrictions B_Ω of the operator L_Ω such that $D(B_\Omega) \subseteq W_2^s(\Omega)$ and $\sup_{B_\Omega \in \mathfrak{B}(\mathcal{A})} \|B_\Omega^{-1}\|_{L_2(\Omega)\to W_2^s(\Omega)} < \infty$. Then there exist $\delta, c_5 > 0$ and for each $\varepsilon \in (0, \delta]$ there exists $k(\varepsilon) \in \mathbb{N}$ such that*

$$|s_k(B_{\Omega_1}) - s_k(B_{\Omega_2})| \leq c_5\, k^{\frac{2}{n}}\, \varepsilon$$

for all $k \geq k(\varepsilon)$ and for all $\Omega_1, \Omega_2 \in C_M^2(\mathcal{A})$ satisfying

$$(\Omega_1)_\varepsilon \subseteq \Omega_2 \subseteq (\Omega_1)^\varepsilon \quad \text{or} \quad (\Omega_2)_\varepsilon \subseteq \Omega_1 \subseteq (\Omega_2)^\varepsilon .$$

Idea of the proof. We consider the family $\mathfrak{A}(\mathcal{A}) = \{A_\Omega\}_{\Omega \in C_M^2(\mathcal{A})}$ of correct restriction A_Ω of the operators L_Ω defined by the homogeneous Dirichlet boundary condition. For this family $\mathfrak{A}(\mathcal{A})$, which satisfies conditions $D(A_\Omega) \subseteq W_2^2(\Omega)$ and $\sup_{A_\Omega \in \mathfrak{A}(\mathcal{A})} \|A_\Omega^{-1}\|_{L_2(\Omega)\to W_2^2(\Omega)} < \infty$, the estimate of the above type is proved in [2]. Next we apply an appropriate corollary of Theorem 3 stating that the ratio $\frac{s_k(B_\Omega)}{s_k(A_\Omega)}$ converges to 1 uniformly with respect the families $\mathfrak{A}(\mathcal{A})$ and $\mathfrak{B}(\mathcal{A})$ which allows reducing the desired estimate to a similar estimate for $s_k(A_\Omega)$ for sufficiently large $k \geq k(\varepsilon)$.

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