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INDEFINITE NEARLY KAEHLER MANIFOLDS OF CONSTANT TYPE

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Abstract. We generalize Cartan's Lemma for indefinite nearly Kaehler manifolds of constant type and as an application we obtain relationship between constancy of holomorphic sectional curvature and bisectional curvature of indefinite nearly Kaehler manifolds.

1 Introduction

Cartan [2] proved the following important result for Riemannian manifolds.

Theorem [A] (Cartan's Lemma). *Let (M, g) be a Riemannian manifold of dimension ≥ 3. Then M is a space of constant sectional curvature if and only if the Riemannian curvature tensor $R(X, Y, Z, X) = 0$ for all orthonormal vectors $X, Y, Z \in T_p(M)$, where $T_p(M)$ is a tangent space of all tangent vectors at any point p of M.*

The Kaehler version of this result has been proved by Nomizu [11].

Theorem [B]. *A Kaehler manifold M of dimension ≥ 6 is a complex space form if and only if $R(X, Y, JX, Y) = 0$ for all orthonormal vectors X and Y which span totally real plane section of $T_p(M)$ at an arbitrary point p of M.*

Graves and Nomizu [6] generalized Theorem [A] for an indefinite Riemannian manifold.

Theorem [C]. *Let (M, g) be an indefinite Riemannian manifold with indefinite metric g. If $R(X, Y, Z, X) = 0$ for orthonormal vectors X, Y and Z, then all non-degenerate planes have the same sectional curvature.*

Barros and Romero [1], Nagaich and Hussain [10] generalized Theorem [B] for an indefinite Kaehler manifold.

Theorem [D]. *Let (M^{2n}, g, J) be an indefinite Kaehler manifold with real dimension ≥ 6 . Then M is of constant holomorphic sectional curvature if and only if $R(X, Y, X, JX) = 0$ for every orthonormal set of vectors X , Y and JX.*

Nagaich and Hussain [9] have also proved the exact complex version of Cartan's Lemma for indefinite Kaehler manifolds.

Theorem [E]. *Let* (M^{2n}, g, J) *be an indefinite Kaehler manifold with real dimension* ≥ 6. *Then* M *is an indefinite complex space form if and only if* $R(X, Y, Z, X) = 0$ *for all orthonormal vectors* X, Y *and* Z *at any point* p *of* M *which span a totally real subspace of* $T_p(M)$.

In this paper, we shall generalize Cartan's Lemma for an indefinite nearly Kaehler manifold by proving the following

Main Theorem [F]. *Let* (M^{2n}, g, J) *be an indefinite nearly Kaehler manifold of constant type with dimension* ≥ 6. *Then* M *is of constant holomorphic sectional curvature if and only if* $R(X, Y, Z, X) = 0$ *for all orthonormal vectors* X, Y *and* Z.

2 Preliminaries

Nearly Kaehler geometry arises as one of the sixteen classes of almost Hermitian manifolds appearing in the celebrated Gray and Hervella classification [5] and defined as a Riemannian manifold (M^{2n}, g, J) with almost complex structure J such that

$$g(JX, JY) = g(X, Y) \qquad \text{and} \qquad (\nabla_X J)X = 0, \tag{2.1}$$

where $X, Y \in \chi(M)$ and ∇ is the Levi-Civita covariant operator defined by the metric tensor g. In this paper we consider non-Kaehlerian nearly Kaehler manifolds. Examples of such non-Kaehlerian nearly Kaehler manifolds are S^6 (with the canonical almost complex structure and metric) and G/K, where G is a compact semisimple Lie group and K is a fixed point set of an automorphism of G of order 3 (see [14]).

The metric g is said to be degenerate if there exists a non-zero vector $X \in \chi(M)$ such that $g(X, Y) = 0$ for all $Y \in \chi(M)$ and a vector field X is a space-like, time-like or null if $g(X, X) > 0$, $g(X, X) < 0$ or $g(X, X) = 0$ respectively for $X \neq 0$. A plane $p = sp\{X, JX\}$ is degenerate if and only if $g(X, X) = 0, X \neq 0$. For a non-degenerate plane $p = sp\{X, Y\}$, the sectional curvature is defined, as usual, by

$$K(X, Y) = \frac{R(X, Y, X, Y)}{g(X, X)g(Y, Y) - g(X, Y)^2}. \tag{2.2}$$

The holomorphic sectional curvature $H(X)$ for a unit vectors X is the sectional curvature $K(X, JX)$ and the holomorphic bisectional curvature $H(X, Y)$, for unit vectors X and Y is given as $R(X, JX, Y, JY)$. In a nearly Kaehler manifold, the following identities are well known [4]:

$$R(X, Y, X, Y) - R(X, Y, JX, JY) = \|(\nabla_X J)Y\|^2. \tag{2.3}$$

$$R(X, Y, Z, W) = R(JX, JY, JZ, JW). \tag{2.4}$$

$$R(X, JX, Y, JY) = R(X, Y, X, Y) + R(X, JY, X, JY) - 2\|(\nabla_X J)Y\|^2. \tag{2.5}$$

It is known that if M is of constant holomorphic sectional curvature c at every point $m \in M$, then the Riemannian curvature tensor of M is of the following form

$$\begin{aligned} R(X,Y,Z,W) &= \frac{c}{4}\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W) \\ &\quad + g(X,JW)g(Y,JZ) - g(X,JZ)g(Y,JW) \\ &\quad - 2g(X,JY)g(Z,JW)\} \\ &\quad + \frac{1}{4}\{g((\nabla_X J)W, (\nabla_Y J)Z) - g((\nabla_X J)Z, (\nabla_Y J)W) \\ &\quad - 2g((\nabla_X J)Y, (\nabla_Z J)W)\}, \end{aligned} \tag{2.6}$$

for all $X, Y, Z, W \in \chi(M)$.

Definition ([4]). *Let M be an almost Hermitian manifold. Then M is said to be of constant type at $m \in M$ if for all $x \in T_m(M)$ we have*

$$[\|\nabla_x(J)(y)\| = \|\nabla_x(J)(z)\|]$$

whenever

$$[< x, y >=< Jx, y >=< x, z >=< Jx, z >= 0 \quad and \quad \|y\| = \|z\|.]$$

If this holds for all $m \in M$ we say that M has (pointwise) constant type. Finally, if M has pointwise constant type and for $X, Y \in \chi(M)$ with $< X, Y >=< JX, Y >= 0$, the function $\|\nabla_X(J)(Y)\|$ is constant whenever $\|X\| = \|Y\| = 1$, then we say that M has global constant type.

Lemma [G] ([4]). *Let M be a nearly Kaehler manifold. Then M has (pointwise) constant type if and only if there exists $\alpha \in F(M)$ such that*

$$\|(\nabla_X J)Y\|^2 = \alpha\{g(X,X)g(Y,Y) - g(X,Y)^2 - g(X,JY)^2\}, \tag{2.7}$$

for all $X, Y \in \chi(M)$ and $F(M)$ is the set of real valued C^∞ functions on M. Furthermore, M has global constant type if and only if (2.7) holds with a constant function α.

Also for orthonormal vectors, from (2.5) and (2.7), the holomorphic bisectional curvature $H(X,Y)$ satisfies

$$H(X,Y) = K(X,Y) + K(X,JY) - 2\alpha. \tag{2.8}$$

3 Proof of the Main Theorem

Let (M^{2n}, g, J) be an indefinite nearly Kaehler manifold of constant type and $\{X, Y, Z, JX, JY, JZ\}$ be a set of orthonormal vectors, then by (2.7) we have

$$\begin{aligned} 2\alpha &= \|(\nabla_X J)(Y+Z)\|^2 \\ &= \|(\nabla_X J)Y\|^2 + \|(\nabla_X J)Z\|^2 + 2g((\nabla_X J)Y, (\nabla_X J)Z), \end{aligned} \tag{3.1}$$

this implies

$$g((\nabla_X J)Y, (\nabla_X J)Z) = 0. \tag{3.2}$$

Let M be of constant holomorphic sectional curvature, then from (2.6), replacing W by X, we have

$$\begin{aligned} R(X,Y,Z,X) &= \frac{c}{4}\{g(X,X)g(Y,Z) - g(X,Z)g(Y,X) \\ &\qquad + g(X,JX)g(Y,JZ) - g(X,JZ)g(Y,JX) \\ &\qquad - 2g(X,JY)g(Z,JX)\} \\ &\quad + \frac{1}{4}\{g((\nabla_X J)X,(\nabla_Y J)Z) - g((\nabla_X J)Z,(\nabla_Y J)X) \\ &\qquad - 2g((\nabla_X J)Y,(\nabla_Z J)X)\}. \end{aligned} \tag{3.3}$$

Since $\{X, Y, Z, JX, JY, JZ\}$ is a set of orthonormal vectors, then we have

$$R(X,Y,Z,X) = \frac{3}{4}g((\nabla_X J)Y, (\nabla_X J)Z). \tag{3.4}$$

Using (3.2) we get

$$R(X,Y,Z,X) = 0. \tag{3.5}$$

Now, we shall discuss the converse part of the theorem in two cases:

Case I: when $g(X,X) = g(Y,Y)$.

In this case, define $\acute{X} = X\cos\theta + Y\sin\theta$ and $\acute{Y} = -X\sin\theta + Y\cos\theta$. Then clearly $\acute{X}$, $\acute{Y}$ and $J\acute{X}$ form an orthonormal set of vectors, using (3.5) we have

$$R(\acute{X}, J\acute{X}, \acute{Y}, J\acute{X}) = 0. \tag{3.6}$$

From this we get

$$\begin{aligned} 0 &= -\sin\theta\cos^3\theta H(X) + \sin^3\theta\cos\theta H(Y) - \sin^3\theta\cos\theta R(Y,JY,X,JX) \\ &\quad - \sin^3\theta\cos\theta R(X,JY,X,JY) - \sin^3\theta\cos\theta R(Y,JX,X,JY) \\ &\quad + \sin\theta\cos^3\theta R(X,JY,Y,JX) + \sin\theta\cos^3\theta R(Y,JX,Y,JX) \\ &\quad + \sin\theta\cos^3\theta R(X,JX,Y,JY). \end{aligned} \tag{3.7}$$

Choosing $\theta = \frac{\pi}{4}$, we get

$$H(X) = H(Y). \tag{3.8}$$

Case II: when $g(X,X) = -g(Y,Y)$.

In this case, we define $\acute{X} = X\cosh\theta + Y\sinh\theta$ and $\acute{Y} = X\sinh\theta + Y\cosh\theta$. Again $\acute{X}$, $\acute{Y}$ and $J\acute{X}$ form an orthonormal set of vectors, using (3.5) we have

$$R(\acute{X}, J\acute{X}, \acute{Y}, J\acute{X}) = 0. \tag{3.9}$$

This gives

$$\begin{aligned} 0 &= \sinh\theta\cosh^3\theta H(X) + \sinh^3\theta\cosh\theta H(Y) + \sinh^3\theta\cosh\theta R(Y,JY,X,JX) \\ &\quad + \sinh^3\theta\cosh\theta R(X,JY,X,JY) + \sinh^3\theta\cosh\theta R(Y,JX,X,JY) \\ &\quad + \sinh\theta\cosh^3\theta R(X,JY,Y,JX) \\ &\quad + \sinh\theta\cosh^3\theta R(Y,JX,Y,JX) \\ &\quad + \sinh\theta\cosh^3\theta R(X,JX,Y,JY). \end{aligned} \tag{3.10}$$

Consequently, we have

$$\begin{aligned}0 &= \cosh^2\theta H(X)+\sinh^2\theta H(Y)+(\cosh^2\theta+\sinh^2\theta)R(X,JX,Y,JY)\\&\quad+(\cosh^2\theta+\sinh^2\theta)R(X,JY,X,JY)+(\cosh^2\theta\\&\quad+sinh^2\theta)R(X,JY,Y,JX).\end{aligned}\tag{3.11}$$

This implies

$$\begin{aligned}0 &= \cos^2\theta H(X)-\sin^2\theta H(Y)+(\cos^2\theta-\sin^2\theta)\{R(X,JX,Y,JY)\\&\quad R(X,JY,X,JY)+R(X,JY,Y,JX)\}.\end{aligned}\tag{3.12}$$

Choosing $\theta=\frac{\pi}{4}$ in above relation, we get

$$H(X)=H(Y).\tag{3.13}$$

Thus, from (3.8) and (3.13) the result follows.

Corollary [H]. *Let M be an indefinite nearly Kaehler manifold of constant type with dimension ≥ 6. Then M is of constant holomorphic sectional curvature if and only if $R(X,Y,X,JX)=0$ for every orthonormal set of vectors X, Y and JX.*

Theorem [I] ([12])**.** *There does not exist a nearly Kaehler manifold of constant curvature provided that $n\neq 6$.*

Theorem [J] ([13])**.** *A 6-dimensional nearly Kaehler manifold of constant holomorphic sectional curvature is a space of constant curvature.*

In [7, 8] Iwatani showed that an 8-dimensional and a 10-dimensional nearly Kaehler manifolds of constant holomorphic sectional curvature are Kaehler manifolds. Therefore a nearly Kaehler manifold of constant holomorphic sectional curvature, satisfying above result, which is not kaehlerian is S^6.

Now, as an application of above result, we prove the following theorem.

Theorem [K]. *Let (M^{2n},g,J) be an indefinite nearly Kaehler manifold of constant type with dimension ≥ 6. Then M is of constant holomorphic sectional curvature if and only if M has constant bisectional curvature.*

Proof. One part of the theorem is obvious by (2.8). Conversely, since M is of constant type, therefore

$$R(X,Y,Z,X)=0,\tag{3.14}$$

where X, Y and Z are orthonormal vectors. Let M be of constant bisectional curvature, then

$$H(X,Y)=c.\tag{3.15}$$

Therefore using (2.8) we have

$$K(X,Y)+K(X,JY)=c+2\alpha.\tag{3.16}$$

We prove the theorem in two different cases, as defined earlier.

Case I: It is clear that $U = X\cos\theta + Y\sin\theta$ and $V = -X\sin\theta + Y\cos\theta$ are orthonormal vectors, therefore from (3.15) we have

$$H(U, V) = c, \tag{3.17}$$

this gives

$$\begin{aligned} c &= \sin^2\theta\cos^2\theta[H(X) + H(Y)] + (\sin^4\theta + \cos^4\theta)H(X, Y) \\ &\quad -2\sin^2\theta\cos^2\theta K(X, JY) - 2\sin^2\theta\cos^2\theta R(X, JY, Y, JX), \end{aligned} \tag{3.18}$$

replace Y by JY in (3.18) we get

$$\begin{aligned} c &= \sin^2\theta\cos^2\theta[H(X) + H(Y)] + (\sin^4\theta + \cos^4\theta)H(X, Y) \\ &\quad -2\sin^2\theta\cos^2\theta K(X, Y) + 2\sin^2\theta\cos^2\theta R(X, Y, JY, JX). \end{aligned} \tag{3.19}$$

Adding (3.18) and (3.19) using (3.16) with Bianchi's first identity, we get

$$\begin{aligned} 2c &= 2\sin^2\theta\cos^2\theta[H(X) + H(Y)] + 2(\sin^4\theta + \cos^4\theta)H(X, Y) \\ &\quad -4\sin^2\theta\cos^2\theta[H(X, Y) + \alpha]. \end{aligned} \tag{3.20}$$

Choosing $\theta = \frac{\pi}{4}$, we get

$$H(X) + H(Y) = 4c + 2\alpha. \tag{3.21}$$

Case II: It is also clear that $U = X\cosh\theta + Y\sinh\theta$ and $V = X\sinh\theta + Y\cosh\theta$ are orthonormal vectors, therefore from (3.15) we have

$$H(U, V) = c, \tag{3.22}$$

from this, we get

$$\begin{aligned} c &= \sinh^2\theta\cosh^2\theta[H(X) + H(Y)] + (\sinh^4\theta + \cosh^4\theta)H(X, Y) \\ &\quad +2\sinh^2\theta\cosh^2\theta K(X, JY) + 2\sinh^2\theta\cosh^2\theta R(X, JY, Y, JX). \end{aligned} \tag{3.23}$$

Replacing Y by JY in (3.23) we get

$$\begin{aligned} c &= \sinh^2\theta\cosh^2\theta[H(X) + H(Y)] + (\sinh^4\theta + \cosh^4\theta)H(X, Y) \\ &\quad -2\sinh^2\theta\cosh^2\theta K(X, Y) - 2\sinh^2\theta\cosh^2\theta R(X, Y, JY, JX). \end{aligned} \tag{3.24}$$

Adding (3.23) and (3.24), using (3.16) and Bianchi's first identity, we get

$$\begin{aligned} 2c &= \sinh^2\theta\cosh^2\theta[H(X) + H(Y)] + 2(\sinh^4\theta + cosh^4\theta)H(X, Y) \\ &\quad +4\sinh^2\theta\cosh^2\theta[H(X, Y) + \alpha]. \end{aligned} \tag{3.25}$$

Choosing $\theta = \dfrac{\pi}{4}$, we get

$$H(X) + H(Y) = -[4c + 2\alpha]. \tag{3.26}$$

Therefore from (3.21) and (3.26) it is obvious that M is of constant holomorphic sectional curvature. □

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