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ON THE SECOND COHOMOLOGY GROUPS OF EXCEPTIONAL LIE ALGEBRAS IN POSITIVE CHARACTERISTIC

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Abstract. The second cohomology groups of exceptional Lie algebras $E_n (n = 6, 7, 8)$, F_4, G_2 over an algebraically closed field of characteristic $p \geq h + 5$ with coefficients in modules, dual to Weyl modules, are explicitly described. Here h is the Coxeter number.

1 Introduction

Examples of modules, dual to Weyl modules, over classical modular Lie algebras with non-vanishing second cohomology, were found, for the first time, in [11]. In that paper, the non-triviality of the second cohomology groups with the coefficients in the module, dual to a Weyl module, associated with $L((p-2)\lambda_1 + \lambda_2)$ for Lie algebras of type A_n (Theorem on p. 324) was established. Here $L((p-2)\lambda_1 + \lambda_2)$ is an irreducible module with the highest weight $(p-2)\lambda_1 + \lambda_2$ over a Lie algebra of type A_n. However, the question of finding all dual to Weyl modules with non-vanishing second cohomology group for classical Lie algebras, remains open.

In the present paper we want to study this question. The aim is to find and calculate all non-vanishing second cohomology groups with coefficients in the modules, dual to Weyl modules, for simple exceptional Lie algebras $E_n (n = 6, 7, 8)$, F_4, G_2 over an algebraically closed field of characteristic $p \geq h + 5$. This restriction on p follows from the decomposability condition of the second cohomology groups. According to our result (Theorem 2.1), the number of the modules dual to Weyl modules with non-vanishing second cohomology is equal to $n + \frac{1}{2}(n+2)(n-1) = \frac{1}{2}(n^2 + 3n - 2)$, where n is the rank of the corresponding exceptional Lie algebra. In the non-vanishing case the structure of the second cohomology group is described as a rational decomposable module over an algebraic group of the corresponding exceptional Lie algebra.

In Section 2 we introduce the basic notation and formulate our main result. Section 3 is devoted to the proof of Theorem 2.1.

2 Notation and the main result

Let $\mathbf{g}$ be a classical Lie algebra of simple and simply connected algebraic group G over an algebraically closed field $k$of characteristic $p > 0$. We fix a maximal torus T and the Borel subgroup B of G corresponding to the negative roots. By G_1 we denote the kernel of the Frobenius morphism F of G.

Choose the root system R associated to (G,T) with the maximal short root α_0 and the maximal root $\widetilde{\alpha}$. The Weyl group W of R acts on the character group $X(T)$ of T by $s_\alpha(\lambda)=\lambda-\langle\lambda,\alpha^\vee\rangle\alpha$, where $s_\alpha\in W$, $\alpha\in R$ and $\alpha^\vee$ is the coroot of α. If ρ is the half of the sum of the positive roots, then the dot action is given by $w\cdot\lambda=w(\lambda+\rho)-\rho$, where $w\in W$, $\lambda\in X(T)$. We denote by $l(w)$ the length of $w\in W$, and by w_0 the longest element in W.

The affine Weyl group W_p is the group generated by all $s_{\alpha,np}$ for $\alpha\in R^+$ and $n\in\mathbb{Z}$, where R^+ is the set of positive roots. We will use the dot action of W_p on $X(T)$:

$$s_{\alpha,np}\cdot\lambda=\lambda-\langle\lambda+\rho,\alpha^\vee\rangle\alpha+np\alpha.$$

Let

$$X_+(T)=\{\lambda\in X(T)\,|\,\langle\lambda+\rho,\alpha^\vee\rangle\geq 0\ \text{ for all }\ \alpha\in S\}$$

is the set of dominant weights, where S is the set of simple roots, and

$$X_1(T)=\{\lambda\in X(T)\,|\,0\leq\langle\lambda+\rho,\alpha^\vee\rangle<p\ \text{ for all }\ \alpha\in S\}$$

is the set of restricted weights.

We denote by $\alpha_1,...,\alpha_n$ the simple roots (numbering corresponds to Bourbaki's tables [3]), and by $\lambda_1,...,\lambda_n$ the fundamental weights.

For any $\lambda\in X(T)$ we define the one-dimensional B-module k_λ via the isomorphism $T\cong B/U$, and the induced G-module $H^0(\lambda)=Ind_B^G(k_\lambda)$. $H^0(\lambda)\neq 0$ if and only if $\lambda\in X_+(T)$. If $V(\lambda)$ is a Weyl module with the highest weight λ over G, then $H^0(\lambda)\cong V(-w_0\lambda)^*$. Hence, $H^0(\lambda)$ can also be considered as a dual to a Weyl module with the highest weight $-w_0(\lambda)$. A simple G-module $L(\lambda)$ with the highest weight λ can be defined in terms of the G-modules $H^0(\lambda)$ and $V(\lambda)$. It is a simple socle of $H^0(\lambda)$, and also a unique simple factor of $V(\lambda)$ ([4], 5.7).

Any classical Lie algebra is a restricted Lie algebra with the p-map $x\mapsto x^{[p]}$. Since for G_1, the theory of G_1-modules is the same with the theory of $U^{[p]}$-modules, where $U^{[p]}$ is a restricted enveloping algebra for $\mathbf{g}$, hence it is equal to the representation theory of $\mathbf{g}$ considered as a restricted Lie algebra ([9], I.8.6). Therefore $H^0(\lambda)$, $V(\lambda)$, $L(\lambda)$ can be considered as $\mathbf{g}$-modules, which are denoted by the same symbols.

The Hochschild cohomology group $H^k(G_1,V)$ of a restricted $\mathbf{g}$-module V coincides with the restricted cohomology group $H^k_*(\mathbf{g},V)$ ([4], 6.10).

The composition of a representation of G on a vector space V with the Frobenius morphism F defines a new representation on which G_1 (and, therefore, $\mathbf{g}$) acts trivially. The so obtained representation is denoted by $V^{(1)}$. On the other hand, if V is a G-module on which G_1 (and, therefore, $\mathbf{g}$) acts trivially, then there is a unique G-module $V^{(-1)}$ such that $V=(V^{(-1)})^{(1)}$.

Suppose now $\mathbf{g}=E_n$ $(n=6,7,8)$, F_4, or G_2. For each $i\in\{1,2,\cdots,n\}$, where n is a rank of $\mathbf{g}$, we introduce a set of the highest weights Λ_i^1 of the irreducible G-modules:

$\mathbf{g}=E_6$: $\Lambda_1^1=\{\lambda_1\}$, $\Lambda_2^1=\{\lambda_2,0\}$, $\Lambda_3^1=\{\lambda_3,\lambda_6\}$, $\Lambda_4^1=\{\lambda_4,\lambda_1+\lambda_6,\lambda_2\}$, $\Lambda_5^1=\{\lambda_5,\lambda_1\}$,$\Lambda_6^1=\{\lambda_6\}$;

$\mathbf{g} = E_7 : \Lambda_1^1 = \{\lambda_1, 0\}$, $\Lambda_2^1 = \{\lambda_2, \lambda_7\}$, $\Lambda_3^1 = \{\lambda_3, \lambda_6, \lambda_1\}$, $\Lambda_4^1 = \{\lambda_4, \lambda_1 + \lambda_6, \lambda_2 + \lambda_7, \lambda_2\}$, $\Lambda_5^1 = \{\lambda_5, \lambda_1 + \lambda_7, \lambda_2\}$, $\Lambda_6^1 = \{\lambda_6, \lambda_1\}$, $\Lambda_7^1 = \{\lambda_7\}$;

$\mathbf{g} = E_8 : \Lambda_1^1 = \{\lambda_1, \lambda_8\}$, $\Lambda_2^1 = \{\lambda_2, \lambda_7, \lambda_1\}$, $\Lambda_3^1 = \{\lambda_3, \lambda_6, \lambda_1 + \lambda_8, \lambda_2\}$, $\Lambda_4^1 = \{\lambda_4, \lambda_1 + \lambda_6, \lambda_2 + \lambda_7, \lambda_3 + \lambda_8, \lambda_1 + \lambda_2, \lambda_5\}$, $\Lambda_5^1 = \{\lambda_5, \lambda_1 + \lambda_7, \lambda_2 + \lambda_8, \lambda_3, \lambda_6\}$, $\Lambda_6^1 = \{\lambda_6, \lambda_1 + \lambda_8, \lambda_2, \lambda_7\}$, $\Lambda_7^1 = \{\lambda_7, \lambda_1, \lambda_8\}$, $\Lambda_8^1 = \{\lambda_8, 0\}$;

$\mathbf{g} = F_4 : \Lambda_1^1 = \{\lambda_1, 0\}$, $\Lambda_2^1 = \{\lambda_2, 2\lambda_4, \lambda_1\}$, $\Lambda_3^1 = \{\lambda_3, \lambda_4\}$, $\Lambda_4^1 = \{\lambda_4\}$;

$\mathbf{g} = G_2 : \Lambda_1^1 = \{\lambda_1\}$, $\Lambda_2^1 = \{\lambda_2, 0\}$.

The main result is following

Theorem 2.1. *Let $\mathbf{g}$ be a classical Lie algebra over an algebraically closed field k of characteristics p, and $H^0(\lambda)$ be a dual to a Weyl module. If $\mathbf{g} = E_n$ $(n = 6, 7, 8)$, F_4, G_2 and $p \geq h + 5$, then $H^2(\mathbf{g}, H^0(\lambda))^{(-1)}$ is trivial, except in the following cases:*

(a) $H^2(\mathbf{g}, H^0(p\nu + w_2 \cdot 0))^{(-1)} \cong H^0(\nu)$, for all $w_2 \in \{w \in W \mid l(w) = 2\}$;

(b) $H^2(\mathbf{g}, H^0(p\lambda_i - \alpha_i))^{(-1)} \cong \bigoplus_{\nu \in \Lambda_i^1} H^0(\nu)$, for all $i \in \{1, 2, \cdots, n\}$.

By Theorem 2.1 the number of peculiar duals to Weyl modules is equal to the sum the rank of $\mathbf{g}$ and the number of elements with length 2 in the Weyl group. The Weyl group of $\mathbf{g}$ has exactly $\frac{1}{2}(n+2)(n-1)$ elements with length 2 hence a number of a peculiar dual to Weyl modules is equal to $n + \frac{1}{2}(n+2)(n-1) = \frac{1}{2}(n^2 + 3n - 2)$.

3 The proof of Theorem 2.1

The proof is based on connection between Hochschild cohomology groups of G_1 and Chevalley-Eilenberg cohomology groups of the Lie algebra $\mathbf{g}$. For a restricted module V, this connection is defined by the following exact sequence [8], [9], [5]:

$$0 \to H^1(G_1, V) \to H^1(\mathbf{g}, V) \to H^0(\mathbf{g}(V) \otimes \mathbf{g}^* \to$$
$$H^2(G_1, V) \to H^2(\mathbf{g}, V) \to H^1(G_1, V) \otimes \mathbf{g}^* \to H^3(G_1, V). \tag{3.1}$$

Obviously that $H^2(\mathbf{g}, H^0(0)) \cong H^2(\mathbf{g}, k) = 0$.

Now, let $V = H^0(\lambda)$ and $\lambda \neq 0$. Since $H^0(\lambda)^G = 0$ at $\lambda \neq 0$, then $H^0(\mathbf{g}, H^0(\lambda)) \otimes \mathbf{g}^* = 0$. It is well known that the first Hochschild cohomology group of G_1 coincides with the first usual Lie algebra cohomology group of $\mathbf{g}$ [8]. Then, from (3.1) we get the following exact sequence

$$0 \to H^2(G_1, H^0(\lambda)) \to H^2(\mathbf{g}, H^0(\lambda)) \xrightarrow{f}$$
$$H^1(G_1, H^0(\lambda)) \otimes \mathbf{g}^* \to H^3(G_1, H^0(\lambda)). \tag{3.2}$$

Thus, the calculation of $H^2(\mathbf{g}, H^0(\lambda))$ is reduced to the calculations of $H^2(G_1, H^0(\lambda))$ and the image of the map f in the exact sequence (3.2). The cohomology groups $H^1(G_1, H^0(\lambda))$ and $H^2(G_1, H^0(\lambda))$ are well known. We use these results [1], [10], [2].

Lemma 3.1. *If $p > 3$ and $\lambda \in X_1(T)$. Then*

$$H^1(G_1, H^0(\lambda))^{(-1)} \cong \begin{cases} H^0(\lambda_i), & \text{for all } \lambda = p\lambda_i - \alpha_i \text{ with } i \in \{1, 2, \cdots, n\}; \\ 0, & \text{in other cases.} \end{cases}$$

Lemma 3.2. *Let $p > 5$ and $\lambda \in X_1(T)\backslash\{0\}$. Then*

$$H^2(G_1, H^0(\lambda))^{(-1)} \cong \begin{cases} H^0(\nu), & \text{for all } \lambda = p\nu + w_2 \cdot 0 \text{ with } w_2 \in \{w \in W \mid l(w) = 2\} \\ 0, & \text{in other cases.} \end{cases}$$

Now we prove the following

Proposition 3.1. *Let $p \geq h + 5$ and $\lambda \in X_1(T)\backslash\{0\}$. Then*

$$Im\, f \cong \begin{cases} \bigoplus_{\nu \in \Lambda_i^1} H^0(\nu)^{(1)}, & \text{for all } \lambda = p\lambda_i - \alpha_i \text{ with } i \in \{1, 2, \cdots, n\}; \\ 0, & \text{in other cases.} \end{cases}$$

Proof. First, we prove that the G-modules $H^1(G_1, H^0(\lambda))^{(-1)} \otimes \mathbf{g}^*$ and $H^3(G_1, H^0(\lambda))^{(-1)}$ are decomposable.

By Lemma 3.1, $H^1(G_1, H^0(\lambda))^{(-1)} \otimes \mathbf{g}^*$ is not trivial if and only if

$$\lambda \in \{p\lambda_i - \alpha_i \in X_1(T) | i \in \{1, 2, \cdots, n\}\},$$

and in this case $H^1(G_1, H^0(p\lambda_i - \alpha_i))^{(-1)} \cong H^0(\lambda_i)$ for all $i \in I$. The isomorphism $\mathbf{g}^* \cong H^0(\widetilde{\alpha})$ yields the isomorphism

$$H^1(G_1, H^0(p\lambda_i - \alpha_i))^{(-1)} \otimes \mathbf{g}* \cong H^0(\lambda_i) \otimes H^0(\widetilde{\alpha}).$$

Then, assuming the usual partial order on the set of weight, the last tensor product and $H^3(G_1, H^0(p\lambda_i - \alpha_i))^{(-1)}$ have the same greatest weights, and it is equal to $\widetilde{\alpha} + \lambda_i$. Since

$$\max_i \{\langle \widetilde{\alpha} + \lambda_i + \rho, \alpha_0^\vee \rangle\} = h + 5 \leq p,$$

then the highest weights of all composition factors of $H^0(\lambda_i) \otimes H^0(\widetilde{\alpha})$ and of $H^3(G_1, H^0(p\lambda_i - \alpha_i))^{(-1)}$ lie in the bottom p-alcove of the affine Weyl group. So, they are decomposable as G-modules [6, 7].

By Lemma 3.2, $H^2(G_1, H^0(p\lambda_i - \alpha_i)) = 0$ for all $i \in \{1, 2, \cdots, n\}$.

Thus, it follows from the exactness of (3.2) and from Lemma 3.2, that to establish the isomorphisms claimed in Proposition 3.1, it is enough to compare the composition factors of $H^0(\lambda_i) \otimes H^0(\widetilde{\alpha})$ with the composition factors of $H^3(G_1, H^0(p\lambda_i - \alpha_i))^{(-1)}$.

We will determine the composition factors of $H^0(\lambda_i) \otimes H^0(\widetilde{\alpha})$ using the table 5 in [12], because $H^0(\lambda_i) \otimes H^0(\widetilde{\alpha})$ is decomposable as a G-module. For the calculation of $H^3(G_1, H^0(p\lambda_i - \alpha_i))^{(-1)}$ we use the general Andersen-Jantzen's formula [1]:

$$H^k(G_1, H^0(w \cdot 0 + p\nu))^{(-1)} \cong \begin{cases} H^0(S^{(k-l(w))/2}(\mathbf{u}^*) \otimes k_\nu), & \text{if } k - l(w) \text{ is even;} \\ 0, \text{ in other cases,} \end{cases} \tag{3.3}$$

where $S(\mathbf{u}^*)$ is the symmetric algebra of $\mathbf{u}^*$, $\mathbf{u}$ is a maximal nilpotent subalgebra of $\mathbf{g}$ corresponding to the negative roots.

The results of calculations are gathered in the following tables.

Table 1. *Weights of the composition factors for* $\mathbf{g} = E_6$.

i	$H^0(\lambda_i) \otimes H^0(\widetilde{\alpha})$	$H^3(G_1, H^0(p\lambda_i - \alpha_i))^{(-1)}$
1	$\lambda_1 + \lambda_2, \lambda_5, \lambda_1$	$\lambda_1 + \lambda_2, \lambda_5$
2	$2\lambda_2, \lambda_4, \lambda_1 + \lambda_6, \lambda_2, 0$	$2\lambda_2, \lambda_4, \lambda_1 + \lambda_6$
3	$\lambda_2 + \lambda_3, \lambda_1 + \lambda_5, \lambda_2 + \lambda_6$ $2\lambda_1, \lambda_3, \lambda_6$	$\lambda_2 + \lambda_3, \lambda_1 + \lambda_5,$ $\lambda_2 + \lambda_6, 2\lambda_1$
4	$\lambda_2 + \lambda_4, \lambda_3 + \lambda_5,$ $\lambda_1 + \lambda_2 + \lambda_6, \lambda_1 + \lambda_3,$ $\lambda_5 + \lambda_6, 2\lambda_2\lambda_4, \lambda_1 + \lambda_6, \lambda_2$	$\lambda_2 + \lambda_4, \lambda_3 + \lambda_5,$ $\lambda_1 + \lambda_2 + \lambda_6, \lambda_1 + \lambda_3$ $\lambda_5 + \lambda_6, 2\lambda_2$
5	$\lambda_2 + \lambda_5, \lambda_3 + \lambda_6,$ $\lambda_1 + \lambda_2, 2\lambda_6, \lambda_5, \lambda_1$	$\lambda_2 + \lambda_5, \lambda_3 + \lambda_6,$ $\lambda_1 + \lambda_2, 2\lambda_6$
6	$\lambda_2 + \lambda_6, \lambda_3, \lambda_6$	$\lambda_2 + \lambda_6, \lambda_3$

Table 2. *Weights of the composition factors for* $\mathbf{g} = E_7$.

i	$H^0(\lambda_i) \otimes H^0(\widetilde{\alpha})$	$H^3(G_1, H^0(p\lambda_i - \alpha_i))^{(-1)}$
1	$2\lambda_1, \lambda_3, \lambda_6, \lambda_1, 0$	$2\lambda_1, \lambda_3, \lambda_6$
2	$\lambda_1 + \lambda_2, \lambda_5, \lambda_1 + \lambda_7, \lambda_2, \lambda_7$	$\lambda_1 + \lambda_2, \lambda_5, \lambda_1 + \lambda_7$
3	$\lambda_1 + \lambda_3, \lambda_4, \lambda_1 + \lambda_6,$ $\lambda_2 + \lambda_7, 2\lambda_1, \lambda_3, \lambda_6, \lambda_1$	$\lambda_1 + \lambda_3, \lambda_4, \lambda_1 + \lambda_6,$ $\lambda_2 + \lambda_7, 2\lambda_1$
4	$\lambda_1 + \lambda_4, \lambda_2 + \lambda_5,$ $\lambda_3 + \lambda_6, \lambda_1 + \lambda_2 + \lambda_7,$ $\lambda_5 + \lambda_7, \lambda_1 + \lambda_3, 2\lambda_2$ $\lambda_4, \lambda_1 + \lambda_6, \lambda_2 + \lambda_7, \lambda_3$	$\lambda_1 + \lambda_4, \lambda_2 + \lambda_5,$ $\lambda_1 + \lambda_2 + \lambda_7, \lambda_3 + \lambda_6,$ $\lambda_5 + \lambda_7, \lambda_1 + \lambda_3$ $2\lambda_2$
5	$\lambda_1 + \lambda_5, \lambda_2 + \lambda_6,$ $\lambda_3 + \lambda_7, \lambda_1 + \lambda_2,$ $\lambda_6 + \lambda_7, \lambda_5, \lambda_1 + \lambda_7, \lambda_2$	$\lambda_1 + \lambda_5, \lambda_2 + \lambda_6,$ $\lambda_3 + \lambda_7, \lambda_1 + \lambda_2$ $\lambda_6 + \lambda_7$
6	$\lambda_1 + \lambda_6, \lambda_2 + \lambda_7, \lambda_3, 2\lambda_7, \lambda_6, \lambda_1$	$\lambda_1 + \lambda_6, \lambda_2 + \lambda_7, \lambda_3, 2\lambda_7$
7	$\lambda_1 + \lambda_7, \lambda_2, \lambda_7$	$\lambda_1 + \lambda_7, \lambda_2$

Table 3. *Weights of the composition factors for* $\mathbf{g} = E_8$.

i	$H^0(\lambda_i) \otimes H^0(\widetilde{\alpha})$	$H^3(G_1, H^0(p\lambda_i - \alpha_i))^{(-1)}$
1	$\lambda_1 + \lambda_8, \lambda_2, \lambda_7, \lambda_1, \lambda_8$	$\lambda_1 + \lambda_8, \lambda_2, \lambda_7$
2	$\lambda_2 + \lambda_8, \lambda_3, \lambda_6, \lambda_1 + \lambda_8, \lambda_2, \lambda_7, \lambda_1$	$\lambda_2 + \lambda_8, \lambda_3, \lambda_6, \lambda_1 + \lambda_8$
3	$\lambda_3 + \lambda_8, \lambda_1 + \lambda_2, \lambda_5, \lambda_1 + \lambda_7,$ $\lambda_2 + \lambda_8, 2\lambda_1, \lambda_3, \lambda_6, \lambda_1 + \lambda_8, \lambda_2$	$\lambda_3 + \lambda_8, \lambda_1 + \lambda_2, \lambda_5,$ $\lambda_1 + \lambda_7, \lambda_2 + \lambda_8, 2\lambda_1$
4	$\lambda_4 + \lambda_8, \lambda_2 + \lambda_3, \lambda_1 + \lambda_5,$ $\lambda_2 + \lambda_6, \lambda_3 + \lambda_7, \lambda_1 + \lambda_2 + \lambda_8,$ $\lambda_5 + \lambda_8, \lambda_1 + \lambda_3, 2\lambda_2, \lambda_4, \lambda_1 + \lambda_6,$ $\lambda_2 + \lambda_7, \lambda_3 + \lambda_8, \lambda_1 + \lambda_2, \lambda_5$	$\lambda_4 + \lambda_8, \lambda_2 + \lambda_3,$ $\lambda_1 + \lambda_5, \lambda_2 + \lambda_6,$ $\lambda_3 + \lambda_7, \lambda_1 + \lambda_2 + \lambda_8,$ $\lambda_5 + \lambda_8, \lambda_1 + \lambda_3, 2\lambda_2$
5	$\lambda_5 + \lambda_8, \lambda_4, \lambda_1 + \lambda_6, \lambda_2 + \lambda_7,$ $\lambda_3 + \lambda_8, \lambda_1 + \lambda_2, \lambda_6 + \lambda_8, \lambda_5,$ $\lambda_1 + \lambda_7, \lambda_2 + \lambda_8, \lambda_3, \lambda_6$	$\lambda_5 + \lambda_8, \lambda_4, \lambda_1 + \lambda_6, \lambda_2 + \lambda_7,$ $\lambda_3 + \lambda_8, \lambda_6 + \lambda_8,$ $\lambda_1 + \lambda_2,$
6	$\lambda_6 + \lambda_8, \lambda_5, \lambda_1 + \lambda_7,$ $\lambda_2 + \lambda_8, \lambda_3, \lambda_7 + \lambda_8,$ $\lambda_6, \lambda_1 + \lambda_8,$ λ_2, λ_7	$\lambda_6 + \lambda_8, \lambda_5, \lambda_1 + \lambda_7$ $\lambda_2 + \lambda_8, \lambda_3,$ $\lambda_7 + \lambda_8,$
7	$\lambda_7 + \lambda_8, \lambda_6, \lambda_1 + \lambda_8,$ $\lambda_2, 2\lambda_8, \lambda_7$ λ_1, λ_8	$\lambda_7 + \lambda_8, \lambda_6, \lambda_1 + \lambda_8,$ $\lambda_2, 2\lambda_8$
8	$2\lambda_8, \lambda_7, \lambda_1$ $\lambda_8, 0$	$2\lambda_8, \lambda_7, \lambda_1$

Table 4. *Weights of the composition factors for* $\mathbf{g} = F_4$.

i	$H^0(\lambda_i) \otimes H^0(\widetilde{\alpha})$	$H^3(G_1, H^0(p\lambda_i - \alpha_i))^{(-1)}$
1	$2\lambda_1, \lambda_2, 2\lambda_4, \lambda_1, 0$	$2\lambda_1, \lambda_2, 2\lambda_4, \lambda_1 + \lambda_4, \lambda_3$
2	$\lambda_1 + \lambda_2, 2\lambda_3, \lambda_1 + 2\lambda_4,$ $\lambda_3 + \lambda_4, 2\lambda_1, \lambda_2, 2\lambda_4, \lambda_1,$	$\lambda_1 + \lambda_2, 2\lambda_3, \lambda_1 + 2\lambda_4, \lambda_1 + \lambda_3,$ $\lambda_3 + \lambda_4, 2\lambda_1, \lambda_2 + \lambda_4$
3	$\lambda_1 + \lambda_3, \lambda_3 + \lambda_4, 2\lambda_4, \lambda_3, \lambda_4,$ $\lambda_1 + \lambda_4,$	$\lambda_1 + \lambda_3, \lambda_3 + \lambda_4, \lambda_1 + 2\lambda_4, \lambda_2,$ $\lambda_1 + \lambda_4, 2\lambda_4$
4	$\lambda_1 + \lambda_4, \lambda_3, \lambda_4$	$\lambda_1 + \lambda_4, \lambda_3, \lambda_1$

Table 5. *Weights of the composition factors for* $\mathbf{g} = G_2$.

i	$H^0(\lambda_i) \otimes H^0(\widetilde{\alpha})$	$H^3(G_1, H^0(p\lambda_i - \alpha_i))^{(-1)}$
1	$\lambda_1 + \lambda_2, 2\lambda_1, \lambda_1$	$\lambda_1 + \lambda_2, 2\lambda_1, \lambda_2$
2	$2\lambda_2, 3\lambda_1, 2\lambda_1,$ $\lambda_2, 0$	$2\lambda_2, 3\lambda_1,$ $\lambda_1 + \lambda_2, 2\lambda_1$

Comparing the composition factors of $H^0(\lambda_i) \otimes H^0(\widetilde{\alpha})$ with the composition factors of $H^3(G_1, H^0(\mu_)))^{(-1)}$, listed in the tables 1-5, we obtain the statements of Proposition 3.1. This completes the proof of Proposition 3.1. □

Finally, we can finish the proof of Theorem 2.1.

By (3.3), $H^1(G_1, H^0(p\nu + w_2 \cdot 0) = 0$ for all $w_2 \in \{w \in W \mid l(w) = 2\}$. Then, from the exactness of (3.2) it follows that

$$H^2(G_1, H^0(p\nu + w \cdot 0)) \cong H^2(\mathbf{g}, H^0(p\nu + w \cdot 0)) \ \text{for all} \ \ w_2 \in \{w \in W \mid l(w) = 2\}.$$

Thus, combining the statements of Lemma 3.2 and Proposition 3.1 we get the statement of Theorem 2.1. This completes the proof of Theorem 2.1. □

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