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A NEW WEIGHTED FRIEDRICHS–TYPE INEQUALITY FOR A PERFORATED DOMAIN WITH A SHARP CONSTANT

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Abstract. We derive a new three-dimensional Hardy-type inequality for a cube for the class of functions from the Sobolev space H^1 having zero trace on small holes distributed periodically along the boundary. The proof is based on a careful analysis of the asymptotic expansion of the first eigenvalue of a related spectral problem and the best constant of the corresponding Friedrichs-type inequality.

1 Introduction

Integral inequalities of Friedrichs and Hardy types are very important for different applications. In particular, they are often used for deriving some estimates for operator norms, for proving some embedding theorems, for solving various problems for partial differential equations, homogenization theory, spectral theory etc. In this paper we prove and discuss some new integral inequalities of Hardy-type for a domain with microinhomogeneous structure in a neighborhood of the boundary.

Let $\Omega \subseteq \mathbb{R}^n$. A Hardy-type inequality is an integral inequality of the form

$$\left(\int\limits_{\Omega} |U(x)|^q V(x)\,dx\right)^{\frac{1}{q}} \le \mathcal{C}\left(\int\limits_{\Omega} |\nabla U(x)|^p W(x)\,dx\right)^{\frac{1}{p}}, \tag{1.1}$$

where $U \in C_0^\infty(\Omega)$, $V(x) \ge 0, W(x) \ge 0$, $1 \le p, q < \infty$, and the constant $\mathcal{C}$ does not depend on the function U. There are several results concerning Hardy-type inequalities (see e.g. the books [19], [25], [26] and [31] and the references given therein).

One main aim of this paper (c.f. also [23, Paper F]) is to derive the Hardy-type inequality

$$\int\limits_{\Omega} |U(x)|^2 \rho^{\alpha-2}(x)\,dx \le C\int\limits_{\Omega} |\nabla U(x)|^2 \rho^{\alpha}(x)\,dx, \tag{1.2}$$

where Ω is bounded and has nontrivial microstructure. More precisely, we assume that Ω is a cube with perforation along a part of the boundary and that the weight function

ρ decreases to zero as x approaches the part of the boundary which is associated with the perforation. It should be mentioned that results in this direction are completely new in the theory of Hardy-type inequalities. In particular, it gives us possibility to use ideas developed within the homogenization theory to obtain estimates for the best constant in different Hardy-type inequalities. The first step in this direction was recently done in [21], where the inequality (1.2) was proved under the assumption that the function U vanishes on small alternating pieces of a part of the boundary.

Note that some analogous results concerning Friedrichs-type inequality for perforated domains were studied earlier in a number of papers. Some examples of perforated domains with the Friedrich's constant of order ε were given in [12] and [13]. Here ε is a small parameter characterizing the perforation. One such example is the domain perforated by an aperiodic lattice of holes studied by M. Briane, A. Damlamian and P. Donato in [3] for the homogenization of the Laplace equation with the Neumann boundary condition. The authors used the new generalized definition of aperiodically perforated material introduced by M. Briane in [2]. Another example is the domain perforated by quasi-periodic holes considered by L. Mascarenhas and D. Polisevski in [27] and D. Chenas, L. Mascarenhas and L. Trabucho in [11].

However, in all these examples it was assumed that the considered function has zero trace both on boundaries of the small sets and on the boundary of the domain. The Dirichlet condition on boundary of the domain was replaced by the Neumann boundary condition in papers [6] and [22]. The main result of these publications was the validity of the Friedrichs inequality for perforated domains under the assumption that the diameters of small sets, the distances between them and the distance to the boundary are of order ε. Moreover, the convergence of the Friedrichs constant to the constant in the limit inequality was established in these papers. Estimates of the difference between these constants were derived later on in [10] for two-dimensional perforated domain. In the present paper we also derive the error estimate for the difference between the constants in the Friedrichs inequalities in the three-dimensional case. We use this result to prove Hardy-type inequality (1.2) for a perforated domain.

Also we note that domains perforated along the boundary were considered in [1], [7].

The paper is organized as follows: In Section 2 we give some necessary definitions and formulate the main results, which are proved in Section 4. The proofs of the main results in Section 2 are based on some auxiliary lemmas, which are proved and discussed in Section 3. Finally, we reserve Section 5 for some concluding remarks and results.

2 Statement of the problem and the main result

Let $\Omega \subset \mathbb{R}^3$ be the cube

$$\left\{0 < x_1 < 1,\ -\frac{1}{2} < x_2 < \frac{1}{2},\ -\frac{1}{2} < x_3 < \frac{1}{2}\right\}.$$

We denote by $\partial\Omega$ the boundary of Ω, and by

$$\Gamma := \left\{x_1 = 0,\ -\frac{1}{2} < x_2 < \frac{1}{2},\ -\frac{1}{2} < x_3 < \frac{1}{2}\right\}$$

and

$$\widetilde{\Gamma} := \left\{x_1 = 1,\ -\frac{1}{2} < x_2 < \frac{1}{2},\ -\frac{1}{2} < x_3 < \frac{1}{2}\right\}.$$

Assume that $0 < c < \frac{1}{2}$ is a positive number. Here and further on $\varepsilon > 0$ is a small parameter. Denote

$$B_\varepsilon^{ij} = \{x \in \Omega : (x_1 - \varepsilon)^2 + (x_2 - i\varepsilon)^2 + (x_3 - j\varepsilon)^2 < (c\varepsilon)^2\},$$

$i, j \in \mathbb{Z},\ B_\varepsilon = \bigcup\limits_{i,j} B_\varepsilon^{ij}$, $\Gamma_\varepsilon = \partial B_\varepsilon$. Finally, we define the domain $\Omega_\varepsilon := \Omega \setminus \overline{B_\varepsilon}$ (see Figure 2). Fix a parameter $0 < \theta < 1$. Define the set $\Omega^\theta := \{x \in \Omega :\ x_1 >\ \theta\}$. Consider the

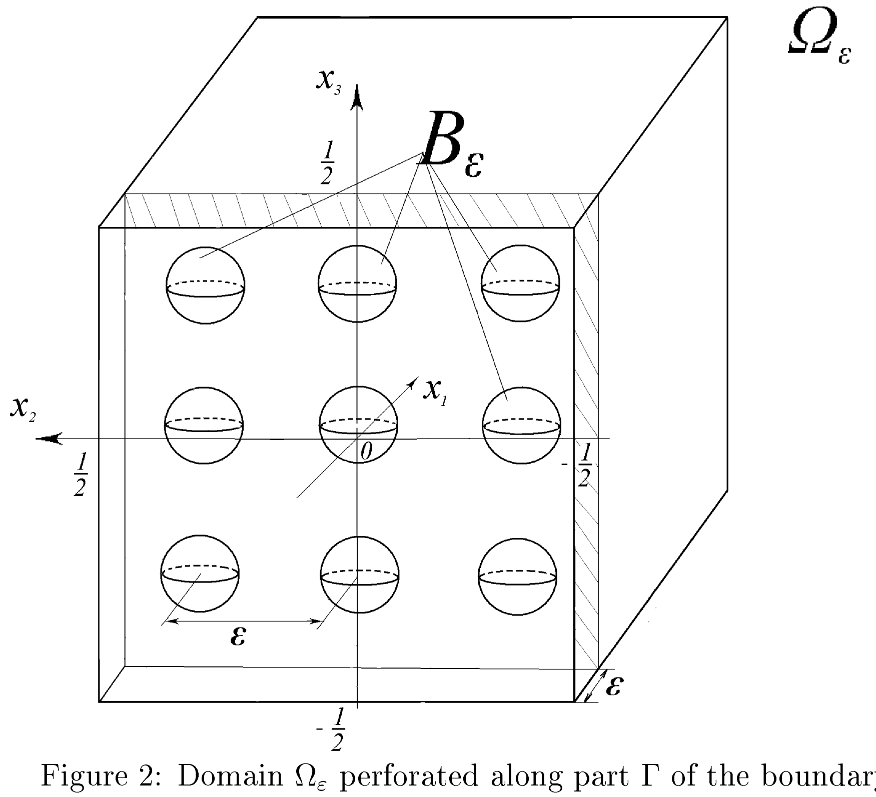

Figure 2: Domain Ω_ε perforated along part Γ of the boundary.

Sobolev-type spaces

$$H^1(\Omega_\varepsilon, \Gamma_\varepsilon) = \{U \in H^1(\Omega_\varepsilon) :\ U|_{\Gamma_\varepsilon} = 0\},$$

where $U|_{\Gamma_\varepsilon}$ is the trace of the function U on Γ_ε.

Remark 2.1. Without loss of generality we can assume that $U \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon)$ is extended to be $U \equiv 0$ in B_ε, and we denote by $H^1(\Omega, \Gamma_\varepsilon)$ the space of all such extensions of functions $U \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon)$.

Analogously, we define $H^1(\Omega,\Gamma)=\{U\in H^1(\Omega):\ U|_\Gamma=0\}$, $H^1(\Omega,\Gamma_\varepsilon\cup\widetilde{\Gamma})=\{U\in H^1(\Omega):\ U|_{\Gamma\varepsilon\cup\widetilde{\Gamma}}=0\}$ and $H^1(\Omega,\Gamma\cup\widetilde{\Gamma})=\{U\in H^1(\Omega):\ U|_{\Gamma\cup\widetilde{\Gamma}}=0\}$. Let $\rho(x):=\operatorname{dist}(x,\Gamma)$ for $x\in\Omega_\varepsilon$. Our new Hardy-type inequality has the following form.

Theorem 2.1. *Let $0\le\alpha<\alpha_0=\frac{2\theta}{\sqrt{K_0}}$, where $K_0>0$ is the best constant in the Friedrichs-type inequality*

$$\int\limits_{\Omega^\theta} U^2 dx \le K_0 \int\limits_{\Omega^\theta} |\nabla U|^2 dx \tag{2.1}$$

for functions from $H^1(\Omega,\Gamma\cup\widetilde{\Gamma})$. Then the Hardy-type inequality

$$\int\limits_{\Omega^\theta} U^2\rho^{\alpha-2}\,dx \le C(\theta,\alpha)\int\limits_{\Omega^\theta} |\nabla U|^2\rho^\alpha\,dx \tag{2.2}$$

holds for any function $U\in H^1(\Omega,\Gamma_\varepsilon\cup\widetilde{\Gamma})$, where $C(\theta,\alpha)=\frac{4K_0}{(2\theta-\sqrt{K_0}\alpha)^2}$.

Our next main result is the corresponding Friedrichs-type inequality which is of independent interest and is crucial for the proof of Theorem 2.1.

Theorem 2.2. *Let $0<\varepsilon<<1$. Then the Friedrichs-type inequality*

$$\int\limits_{\Omega} U^2\,dx \le K_\varepsilon \int\limits_{\Omega} |\nabla U|^2\,dx \tag{2.3}$$

holds for any function $U\in H^1(\Omega,\Gamma_\varepsilon)$, where for any sufficiently small $\mu>0$

$$K_\varepsilon=\frac{4}{\pi^2}+\varepsilon K+o\left(\varepsilon^{\frac{3}{2}-\mu}\right) \tag{2.4}$$

as $\varepsilon\to0^+$. Here K_ε is the best constant in Friedrichs-type inequality (2.3). The precise formula for the constant $K<0$ is given by (4.38) *later on.*

It is well known that the best constant in Friedrichs-type inequalities can be expressed via the first eigenvalue of the corresponding spectral problem. This is why we first study an auxiliary spectral problem and construct the asymptotic expansion for its first eigenvalue via the method of matching of asymptotic expansions. After that we derive the asymptotics (2.4) for the constant K_ε in Friedrichs-inequality (2.3) by using the relations between it and the first eigenvalue. More exactly, we consider the following spectral problem:

$$\begin{cases} -\Delta u_\varepsilon=\lambda_\varepsilon u_\varepsilon & \text{in } \Omega_\varepsilon,\\ u_\varepsilon=0 & \text{on } \Gamma_\varepsilon,\\ \frac{\partial u_\varepsilon}{\partial\nu}=0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

Here and in the sequel we denote by ν the outward unit vector.

The problem

$$\begin{cases} -\Delta u_0 = \lambda_0 u_0 & \text{in } \Omega, \\ u_0 = 0 & \text{on } \Gamma, \\ \frac{\partial u_0}{\partial \nu} = 0 & \text{on } \partial\Omega \setminus \Gamma \end{cases} \tag{2.6}$$

is the limit problem for (1.1). This fact can be established analogously as in [9] for the two-dimensional case. In particular, the convergence of any eigenvalue λ_ε of the problem (1.1) to the corresponding eigenvalue λ_0 of the problem (1.3) as $\varepsilon \to 0^+$ was proved. Moreover, the convergence of the corresponding eigenfunctions in the norm of Sobolev space H^1 was derived. The next result gives a more exact description of the asymptotics of all eigenvalues of the problem (1.1) and is crucial for the proof of Theorem 2.2 (and, thus, of Theorem 2.1) and also of independent interest. Due to the geometry of our domain it is not difficult to derive that all eigenvalues of the problem (1.3) (and, hence, of the problem (1.1)) are positive and simple. In particular, the next result is valid for the first eigenvalue of (1.1).

Theorem 2.3. *The following asymptotics holds for the first eigenvalue of* (1.1)

$$\lambda_\varepsilon = \lambda_0 + \varepsilon\lambda_1 + o(\varepsilon^{\frac{3}{2}-\mu}), \tag{2.7}$$

where $0 < \mu$ *is an arbitrary small real number,*

$$\lambda_1 = -C(B) \int\limits_\Gamma \left(\frac{\partial u_0}{\partial \nu} \right)^2 ds < 0, \tag{2.8}$$

(λ_0, u_0) *is the corresponding eigenelement of* (1.3) *and* $C(B)$ *is a strictly positive constant (the precise formula for* $C(B)$ *is given by* (3.8) *in Section* 3*). Here* $\lambda_0 = \frac{\pi^2}{4}$.

Remark 2.2. The corresponding two-dimensional result was proved in [9].

3 Some auxiliary results

Define the sets

$$\Pi = \left\{ \xi_1 > 0, \, -\frac{1}{2} < \xi_2 < \frac{1}{2}, \, -\frac{1}{2} < \xi_3 < \frac{1}{2} \right\},$$

$$\gamma = \left\{ \xi_1 = 0, \, -\frac{1}{2} < \xi_2 < \frac{1}{2}, \, -\frac{1}{2} < \xi_3 < \frac{1}{2} \right\},$$

$$B := \{ (\xi_1 - 1)^2 + \xi_2^2 + \xi_3^2 < c^2, \; 0 < c < 1 \}$$

(see Figure 3). The following three auxiliary Lemmas are necessary for our proofs of the main results.

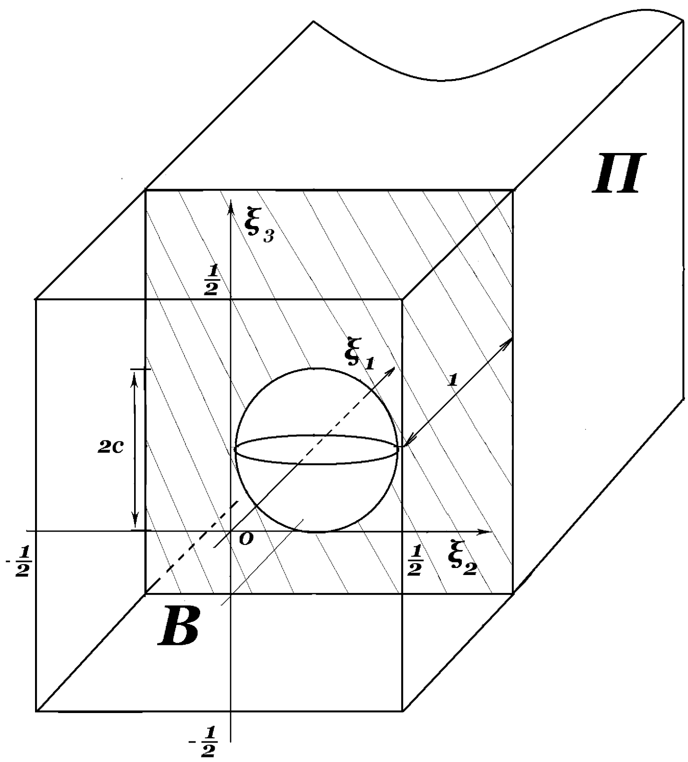

Figure 3: Cell of periodicity.

Lemma 3.1. *The following boundary-value problem*

$$\begin{cases} \Delta X_1 = 0 & in \quad \Pi\backslash\overline{B}, \\ X_1 = 0 & on \quad \partial B, \\ \frac{\partial X_1}{\partial \xi_1} = 0 & on \quad \gamma, \\ \frac{\partial X_1}{\partial \xi_2} = 0 & as \quad \xi_2 = \pm\frac{1}{2}, \\ \frac{\partial X_1}{\partial \xi_3} = 0 & as \quad \xi_3 = \pm\frac{1}{2}, \\ X_1 \sim \xi_1 & as \quad \xi_1 \to +\infty \end{cases} \tag{3.1}$$

has a unique even solution with respect to the variables ξ_2 *and* ξ_3. *Moreover, this solution has the asymptotics*

$$X_1(\xi) = \xi_1 + C(B) + O(e^{-2\pi\xi_1}) \quad as \quad \xi_1 \to +\infty, \tag{3.2}$$

where $C(B)$ *is a strictly positive constant (the precise formula for* $C(B)$ *is given later on in the proof of Lemma* 3.1 *(see* (3.8)*)).*

Lemma 3.2. *Let* X_1 *be the solution of* (3.1). *The boundary-value problem*

$$\begin{cases} \Delta X_2 = \frac{\partial X_1}{\partial \xi_2} & in\ \Pi\backslash\overline{B}, \\ X_2 = 0 & on\ \partial B, \\ \frac{\partial X_2}{\partial \xi_1} = 0 & on\ \gamma, \\ X_2 = 0 & as\ \xi_2 = \pm\frac{1}{2}, \\ X_2 = 0 & as\ \xi_3 = \pm\frac{1}{2} \end{cases} \tag{3.3}$$

has a unique solution which is odd with respect to ξ_2 *and even with respect to* ξ_3 *and satisfies the following asymptotics*

$$X_2(\xi) = O(e^{-\xi_1}) \text{ as } \xi_1 \to +\infty. \tag{3.4}$$

Lemma 3.3. *Let* X_1 *be the solution of* (3.1). *The boundary-value problem*

$$\begin{cases} \Delta_\xi X_3 = \frac{\partial X_1}{\partial \xi_3} & \text{in } \Pi\backslash\overline{B}, \\ X_3 = 0 & \text{on } \partial B, \\ \frac{\partial X_3}{\partial \xi_1} = 0 & \text{on } \gamma, \\ X_3 = 0 & \text{as } \xi_2 = \pm\frac{1}{2}, \\ X_3 = 0 & \text{as } \xi_3 = \pm\frac{1}{2} \end{cases} \tag{3.5}$$

has a unique solution which is even with respect to ξ_2, *odd with respect to* ξ_3 *and has the following asymptotics*

$$X_3(\xi) = O(e^{-\xi_1}) \text{ as } \xi_1 \to +\infty. \tag{3.6}$$

Due to 1-periodicity, with respect to ξ_2 and ξ_3, of the right-hand side of the equation in problem (3.3) and the boundary conditions we can extend X_2 and X_3 1-periodically. We will use the same notation for the extended functions.

In the remaining part of this section we describe how these lemmas can be proved.

Proof of Lemma 3.1. First we note that the proof of Lemma 3.1 is based on the following Lemma which can be proved exactly in the same way as Proposition 1.2 from [29] was proved. We omit the details.

Lemma 3.4. *Assume that* $e^{\delta_0\xi_1}F \in L_2(\Pi \setminus B)$, $e^{\delta_0\xi_1}H \in L_2(\partial\Pi)$, $G \in H^{\frac{1}{2}}(\partial B)$ *and* $\delta_0 > 0$. *Then there exists a unique weak solution of the following boundary-value problem:*

$$\begin{cases} -\Delta Z = F & \text{in } \Pi \setminus \overline{B} \\ Z = G & \text{on } \partial B, \\ \dfrac{\partial Z}{\partial \nu} = H & \text{on } \partial \Pi. \end{cases}$$

This solution is given by the formula

$$Z(\xi) = C + \widetilde{Z}(\xi),$$

where C *is a constant,* $e^{\delta\xi_1}\widetilde{Z} \in H^1(\Pi\backslash\overline{B})$ *and* δ *is an arbitrary number satisfying the conditions* $\delta \le \delta_0$ *and* $\delta < \pi$.

Consider now the boundary-value problem

$$\begin{cases} \Delta Y = 0 & \text{in } \Pi\backslash\overline{B}, \\ Y = -\xi_1 & \text{on } \partial B, \\ \frac{\partial Y}{\partial \xi_1} = -1 & \text{on } \gamma, \\ \frac{\partial Y}{\partial \xi_2} = 0 & \text{as } \xi_2 = \pm\frac{1}{2}, \\ \frac{\partial Y}{\partial \xi_3} = 0 & \text{as } \xi_3 = \pm\frac{1}{2}. \end{cases} \tag{3.7}$$

Due to Lemma 3.4 there exists a unique weak solution of this boundary-value problem of the form

$$Y(\xi) = C(B) + \widetilde{Z}(\xi),$$

where $C(B)$ is a constant and the function $\widetilde{Z}(\xi)$ satisfies the conditions of Lemma 3.4. The function Y is even with respect to ξ_2 and ξ_3 due to the symmetry of B. Denote by $\Pi^R = \Pi \cap \{\xi_1 > R\}$, $\gamma_R = \{\xi \in \Pi, \xi_1 = R\}$, $y_R = Y\big|_{\gamma_R}$. Obviously, the function Y is also a unique classical bounded solution of the following boundary-value problem

$$\begin{cases} \Delta Y = 0 & \text{in } \Pi^R, \\ \frac{\partial Y}{\partial \xi_1} = y_R & \text{on } \gamma_R, \\ \frac{\partial Y}{\partial \xi_2} = 0 & \text{as } \xi_2 = \pm\frac{1}{2}, \\ \frac{\partial Y}{\partial \xi_3} = 0 & \text{as } \xi_3 = \pm\frac{1}{2} \end{cases}$$

when R is a sufficiently large number. Hence, taking into account that Y is an even function with respect to ξ_2 and ξ_3 we conclude that the asymptotics has the following structure:

$$Y(\xi) = C(B) + O(e^{-2\pi\xi_1}) \quad \text{as} \quad \xi_1 \to +\infty,$$

and, consequently, (3.2) holds.

It only remains to prove that

$$C(B) = \int\limits_{\Pi\setminus\overline{B}} |\nabla Y|^2 d\xi + |B|. \tag{3.8}$$

Denote by $\Pi_R = \Pi \cap \{\xi_1 < R\}$. Multiplying the equation of the problem (3.7) by X_1, integrating over $\Pi_R \setminus \overline{B}$ and taking into account the properties of the function X_1 we obtain that

$$\begin{aligned} 0 = \int\limits_{\Pi_R\setminus\overline{B}} X_1 \Delta Y \, d\xi = \int\limits_{\gamma_R} X_1 \frac{\partial Y}{\partial \xi_1} \, d\xi_2 \, d\xi_3 - \\ - \int\limits_{\gamma} X_1 \frac{\partial Y}{\partial \xi_1} \, d\xi_2 \, d\xi_3 - \int\limits_{\Pi_R\setminus\overline{B}} \nabla Y \nabla X_1 \, d\xi. \end{aligned} \tag{3.9}$$

By first using integration by parts to rewrite the right-hand side of (3.9) and thereafter passing to the limit as $R \to \infty$, we find that

$$\begin{aligned} 0 &= -\int\limits_{\gamma} \frac{\partial Y}{\partial \xi_1} X_1 \; d\xi_2 \; d\xi_3 + \int\limits_{\partial B} Y \frac{\partial X_1}{\partial \nu} \; ds_\xi - C(B) \\ &= \int\limits_{\gamma} Y \; d\xi_2 \; d\xi_3 + \int\limits_{\partial B} Y \frac{\partial Y}{\partial \nu} \; ds_\xi - \int\limits_{\partial B} \xi_1 \frac{\partial \xi_1}{\partial \nu} \; ds_\xi - C(B). \end{aligned} \tag{3.10}$$

Analogously, multiplying (3.7) by Y, integrating the obtained formula by parts over $\Pi_R \setminus \overline{B}$ and passing to the limit as $R \to +\infty$, we have that

$$0 = -\int\limits_{\Pi\setminus\overline{B}} |\nabla Y|^2 \; d\xi + \int\limits_{\partial B} \frac{\partial Y}{\partial \nu} Y \; ds_\xi + \int\limits_{\gamma} Y \; d\xi_2 \; d\xi_3. \tag{3.11}$$

The estimates (3.10) and (3.11) lead to

$$C(B) = \int\limits_{\Pi\setminus\overline{B}} |\nabla Y|^2 \, d\xi - \int\limits_{\partial B} \xi_1 \frac{\partial \xi_1}{\partial \nu} \, ds_\xi. \tag{3.12}$$

By integrating by parts the left-hand side of the formula

$$\int\limits_B \xi_1 \Delta \xi_1 d\xi = 0,$$

we get that

$$\int\limits_{\partial B} \xi_1 \frac{\partial \xi_1}{\partial \nu_B} \, ds_\xi = |B|, \tag{3.13}$$

where ν_B is an outward normal vector to B. The formula (3.8) follows from (3.12) and (3.13). □

Proof of Lemma 3.2. The proof of this Lemma is based on the following Lemma from [29]:

Lemma 3.5. *Assume that* $e^{\delta_0 \xi_1} F \in L_2(\Pi \setminus B)$ *and* $\delta_0 > 0$. *Then there exists a unique solution of the following boundary-value problem:*

$$\begin{cases} -\Delta Z = F & \text{in } \Pi \setminus \overline{B} \\ Z = 0 & \text{on } \partial B \cup \partial\Pi \setminus \overline{\gamma}, \\ \dfrac{\partial Z}{\partial \nu} = 0 & \text{on } \gamma, \end{cases}$$

where $e^{\delta \xi_1} Z \in H^1(\Pi \setminus \overline{B})$ *and* δ *is an arbitrary number satisfying the conditions* $\delta \leq \delta_0$ *and* $\delta < \pi$.

By applying this Lemma with $Z = X_2$ and $F = \frac{\partial X_1}{\partial \xi_2}$, we conclude that X_2 has the asymptotics (3.4). The solution X_2 is odd with respect to ξ_2 and even with respect to ξ_3 due to the equation of the boundary-value problem (3.3) and the properties of its right-hand side. □

Proof of Lemma 3.3. This Lemma can be proved analogously to Lemma 3.2, so we omit the details. □

4 Proofs of the main results

Proof of Theorem 2.3.

Proof. The proof is based on several steps, which sometimes are stated as Lemmas of independent interest. Our aim is to construct the first two terms of the asymptotic expansion for simple eigenvalues of the spectral problem (1.1).

The behavior of u_ε in a boundary layer close to Γ strongly differs from the behavior outside the boundary layer. We will use the method of matching of inner and outer expansions of u_ε. The inner expansion is valid in the boundary layer and the outer expansion is valid outside the boundary layer (for more information concerning the method of matching expansions see e.g. [17]).

Without loss of generality we may assume that the function u_0 is normalized in $L_2(\Omega)$.

It is natural to construct the asymptotic expansion for λ_ε in the form

$$\lambda_\varepsilon \approx \widehat{\lambda}_\varepsilon = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2, \tag{4.1}$$

while we use the formula

$$u_\varepsilon(x) \approx \widehat{u}_\varepsilon = u_0(x) + \varepsilon u_1(x) + \varepsilon^2 u_2(x) \tag{4.2}$$

for the asymptotics of u_ε.

We have that $u_0 \in C^\infty(\overline{\Omega})$, see [9]. If we substitute the expansions (4.1) and (4.2) into the spectral problems (1.1) and collect terms of the same order of ε, then, by taking into account (1.3), we obtain the expansion

$$u_0(x) = \alpha_0^1(x_2, x_3)x_1 + O(x_1^3) \tag{4.3}$$

as $x_1 \to 0$, where

$$\alpha_0^1 = \left.\frac{\partial u_0}{\partial x_1}\right|_{x_1=0} \in C^\infty\left[-\frac{1}{2}, \frac{1}{2}\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right] \tag{4.4}$$

and

$$\frac{\partial^{2j+1}\alpha_0^1}{\partial x_2^{2j+1}}\left(\pm\frac{1}{2}, x_3\right) = 0, \qquad \frac{\partial^{2j+1}\alpha_0^1}{\partial x_3^{2j+1}}\left(x_2, \pm\frac{1}{2}\right) = 0, \tag{4.5}$$

for $j = 0, 1, 2, \ldots$.

We choose the functions u_1 and u_2 satisfying the boundary-value problems

$$\begin{cases} -\Delta u_1 & = \lambda_0 u_1 + \lambda_1 u_0 \quad \text{in } \Omega, \\ \dfrac{\partial u_1}{\partial \nu} & = 0 \quad \text{on } \partial\Omega \setminus \Gamma, \\ u_1 & = \alpha_1^0 \quad \text{on } \Gamma, \end{cases} \tag{4.6}$$

$$\begin{cases} -\Delta u_2 & = \lambda_0 u_2 + \lambda_1 u_1 + \lambda_2 u_0 \quad \text{in } \Omega, \\ \dfrac{\partial u_2}{\partial \nu} & = 0 \quad \text{on } \partial\Omega \setminus \Gamma, \\ u_2 & = \alpha_2^0 \quad \text{on } \Gamma, \end{cases} \tag{4.7}$$

where $\alpha_1^0(x_2, x_3), \alpha_2^0(x_2, x_3)$ are unknown functions, which will be defined later on.

Remark 4.1. The equations of the boundary-value problems (4.6) and (4.7) together with the boundary conditions (except condition on Γ) are just the result of substituting the expansions (4.1) and (4.2) into (1.1) and collecting terms of the same order of ε.

The validity of the following Lemma can be established by using the same technique as in the proof of the analogous result in [4]. We omit the details.

Lemma 4.1. *Assume that* $\alpha_1^0, \alpha_2^0 \in C^\infty\left[-\frac{1}{2}, \frac{1}{2}\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right]$ *and that odd derivatives of the functions* α_1^0 *and* α_2^0 *with respect to* x_2, x_3 *vanish as* $x_2 = \pm\frac{1}{2}$, $x_3 = \pm\frac{1}{2}$. *Then there exist constants* λ_1, λ_2 *and functions* $u_1(x), u_2(x) \in C^\infty(\overline{\Omega})$, *which are the solutions of problems* (4.6) *and* (4.7), *respectively. Moreover,* λ_1 *satisfies that*

$$\lambda_1 = -\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} \alpha_1^0(x_2, x_3)\alpha_0^1(x_2, x_3)\, dx_2\, dx_3. \tag{4.8}$$

By using a Taylor expansion we obtain that

$$\begin{aligned} u_1(x) &= \alpha_1^0(x_2, x_3) + \alpha_1^1(x_2, x_3)x_1 + O(x_1^2), \\ u_2(x) &= \alpha_2^0(x_2, x_3) + O(x_1) \end{aligned} \tag{4.9}$$

as $x_1 \to 0$, where $\alpha_1^1 \in C^\infty\left[-\frac{1}{2}, \frac{1}{2}\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right]$ and

$$\frac{\partial^{2j+1}\alpha_1^1}{\partial x_2^{2j+1}}\left(\pm\frac{1}{2}, x_3\right) = 0, \quad \frac{\partial^{2j+1}\alpha_1^1}{\partial x_2^{2j+1}}\left(x_2, \pm\frac{1}{2}\right) = 0, \tag{4.10}$$

for $j = 0, 1, 2, \ldots$ due to (4.6).

Taking into account Remark 4.1 and Lemma 4.1 we conclude that the following Lemma holds:

Lemma 4.2. *Assume that* $\alpha_1^0, \alpha_2^0 \in C^\infty\left[-\frac{1}{2}, \frac{1}{2}\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right]$ *and odd derivatives of the functions* α_1^0 *and* α_2^0 *with respect to* x_2 *and* x_3 *vanish at the points* $\left(\pm\frac{1}{2}, x_3\right)$ *and* $\left(x_2, \pm\frac{1}{2}\right)$. *Then* $\widehat{u}_\varepsilon \in C^\infty(\overline{\Omega})$ *and the formulas*

$$\begin{cases} -\Delta\widehat{u}_\varepsilon &= \widehat{\lambda}_\varepsilon\widehat{u}_\varepsilon + O(\varepsilon^3) \quad \text{in } \Omega, \\ \dfrac{\partial\widehat{u}_\varepsilon}{\partial\nu} &= 0 \quad \text{on } \partial\Omega \setminus \Gamma \end{cases}$$

are valid.

We construct another interpolation for the function u_ε in a small neighborhood of Γ (*inner expansion*) since the function $\widehat{u}_\varepsilon(x)$ does not satisfy the boundary conditions of the problem (1.1) on Γ and on Γ_ε.

The formulas (4.2), (4.3) and (4.9) lead to the following:

$$\begin{aligned} \widehat{u}_\varepsilon(x) =& \alpha_0^1(x_2, x_3)x_1 + \varepsilon(\alpha_1^0(x_2, x_3) + \alpha_1^1(x_2, x_3)x_1) + \varepsilon^2\alpha_2^0(x_2, x_3) \\ &+ O(x_1^3 + \varepsilon x_1^2 + \varepsilon^2 x_1) \quad \text{as } x_1 \to 0. \end{aligned}$$

Put $\xi_1 = \frac{x_1}{\varepsilon}$. Then we conclude that

$$\begin{aligned} \widehat{u}_\varepsilon(x) =& \varepsilon V_1(\xi_1; x_2, x_3) + \varepsilon^2 V_2(\xi_1; x_2, x_3) \\ &+ O(x_1^3 + \varepsilon x_1^2 + \varepsilon^2 x_1) \quad \text{as } x_1 \to 0, \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} V_1(\xi_1; x_2, x_3) &= \alpha_0^1(x_2, x_3)\xi_1 + \alpha_1^0(x_2, x_3), \\ V_2(\xi_1; x_2, x_3) &= \alpha_1^1(x_2, x_3)\xi_1 + \alpha_2^0(x_2, x_3). \end{aligned} \tag{4.12}$$

According to the method of matching of asymptotic expansions we conclude that the internal expansion have to be of the following structure in a neighborhood of Γ:

$$u_\varepsilon(x) \approx \widehat{v}_\varepsilon(x) = \varepsilon v_1\left(\xi; x_2, x_3\right) + \varepsilon^2 v_2\left(\xi; x_2, x_3\right), \tag{4.13}$$

where $\xi = \frac{x}{\varepsilon}$ and

$$v_q(\xi; x_2, x_3) \sim V_q(\xi_1; x_2, x_3) \quad \text{as} \quad \xi_1 \to +\infty,\ q = 1, 2. \tag{4.14}$$

Here x_2, x_3 are so called "slow" variables while ξ is the "fast" variable.

The equation of the problem (1.1) with respect to the variables $(\xi; x_2, x_3)$ has the following form:

$$-\varepsilon^{-2}\Delta_\xi u_\varepsilon - 2\varepsilon^{-1}\frac{\partial^2 u_\varepsilon}{\partial x_2 \partial \xi_2} - 2\varepsilon^{-1}\frac{\partial^2 u_\varepsilon}{\partial x_3 \partial \xi_3} - \frac{\partial^2 u_\varepsilon}{\partial x_2^2} - \frac{\partial^2 u_\varepsilon}{\partial x_3^2} = \lambda_\varepsilon u_\varepsilon. \tag{4.15}$$

The boundary conditions on the lateral surface of the cell of periodicity Π except γ are

$$\frac{\partial u_\varepsilon}{\partial \nu} = \pm\varepsilon^{-1}\frac{\partial u_\varepsilon}{\partial \xi_2} \pm \frac{\partial u_\varepsilon}{\partial x_2} = 0, \tag{4.16}$$

$$\frac{\partial u_\varepsilon}{\partial \nu} = \pm\varepsilon^{-1}\frac{\partial u_\varepsilon}{\partial \xi_3} \pm \frac{\partial u_\varepsilon}{\partial x_3} = 0, \tag{4.17}$$

and on γ it yields that

$$\frac{\partial u_\varepsilon}{\partial \nu} = -\varepsilon^{-1}\frac{\partial u_\varepsilon}{\partial \xi_1} - \frac{\partial u_\varepsilon}{\partial x_1} = 0. \tag{4.18}$$

Next, we construct the internal expansion for (4.13) as 1-periodic function with respect to ξ_2 and ξ_3. In order to do this, we rewrite the equation and boundary conditions in ξ variables (see (4.15) – (4.18)), substitute (4.13) and (4.1) in (1.1) and, finally, equate terms at ε^q corresponding to the same q. Then, by taking into account (4.14), (4.12) and Remark 4.1, we get the following boundary-value problem for v_1:

$$\begin{cases} \Delta_\xi v_1 = 0 \quad \text{in } \Pi\backslash\overline{B}, \\ v_1 = 0 \quad \text{on } \partial B, \\ \frac{\partial v_1}{\partial \xi_1} = 0 \quad \text{on } \gamma, \\ \frac{\partial v_1}{\partial \xi_2}(\xi; \pm\frac{1}{2}, x_3) = 0 \quad \text{as } \xi_2 = \pm\frac{1}{2}, \\ \frac{\partial v_1}{\partial \xi_3}(\xi; x_2, \pm\frac{1}{2}) = 0 \quad \text{as } \xi_3 = \pm\frac{1}{2}, \\ v_1 \sim V_1 \quad \text{as } \xi_1 \to +\infty \end{cases} \tag{4.19}$$

and for v_2:

$$\begin{cases} -\Delta_\xi v_2 = 2\frac{\partial^2 v_1}{\partial x_2 \partial \xi_2} + 2\frac{\partial^2 v_1}{\partial x_3 \partial \xi_3} \quad \text{in } \Pi\backslash\overline{B}, \\ v_2 = 0 \quad \text{on } \partial B, \\ \frac{\partial v_2}{\partial \xi_1} = 0 \quad \text{on } \gamma, \\ \frac{\partial v_2}{\partial \xi_2}(\xi; \pm\frac{1}{2}, x_3) = -\frac{\partial v_1}{\partial x_2}(\xi; \pm\frac{1}{2}, x_3) \quad \text{as } \xi_2 = \pm\frac{1}{2}, \\ \frac{\partial v_2}{\partial \xi_3}(\xi; x_2, \pm\frac{1}{2}) = -\frac{\partial v_1}{\partial x_3}(\xi; x_2, \pm\frac{1}{2}) \quad \text{as } \xi_3 = \pm\frac{1}{2}, \\ v_2 \sim V_2 \quad \text{as } \xi_1 \to +\infty. \end{cases} \tag{4.20}$$

Due to the boundary-value problems (4.19) and (4.20) we conclude that the following Lemma is valid:

Lemma 4.3. *Assume that the solutions to problems* (4.19) *and* (4.20) *exist and are* 1*-periodic functions with respect to* ξ_2 *and* ξ_3*. Then the functions* $\widehat{v}_\varepsilon$ *and* $\widehat{\lambda}_\varepsilon$*, which are given by* (4.13) *and* (4.1)*, respectively, satisfy to the following formulas for each sufficiently small* $h > 0$ *:*

$$\begin{aligned} -\Delta\widehat{v}_\varepsilon &= \widehat{\lambda}_\varepsilon\widehat{v}_\varepsilon + \widehat{F}^v_\varepsilon \quad \text{in } \Omega_\varepsilon \cap \{x_1 < h\}, \\ \widehat{v}_\varepsilon &= 0 \quad \text{on } \Gamma_\varepsilon, \\ \frac{\partial \widehat{v}_\varepsilon}{\partial \nu} &= 0 \quad \text{on } \partial\Omega, \\ \frac{\partial \widehat{v}_\varepsilon}{\partial x_j} &= \varepsilon^2 \frac{\partial v_2}{\partial x_j}, \quad j = 2, 3, \quad \text{on } (\partial\Omega \setminus \Gamma) \cap \{x_1 < h\}, \end{aligned} \tag{4.21}$$

where

$$\begin{aligned} \widehat{F}^v_\varepsilon = &- \varepsilon\left(\frac{\partial^2 v_1}{\partial x_2^2} + \frac{\partial^2 v_1}{\partial x_3^2} + \lambda_0 v_1 + 2\frac{\partial^2 v_2}{\partial x_2 \partial \xi_2} + 2\frac{\partial^2 v_2}{\partial x_3 \partial \xi_3}\right) - \\ &- \varepsilon^2\left(\lambda_1 v_1 + \frac{\partial^2 v_2}{\partial x_2^2} + \frac{\partial^2 v_2}{\partial x_3^2} + \lambda_0 v_2\right) - \varepsilon^3\left(\lambda_1 v_2 + \lambda_2 v_1\right) - \varepsilon^4 \lambda_2 v_2. \end{aligned} \tag{4.22}$$

Now we study the solvability of the boundary-value problem (4.19) and determine a formula for the function $\alpha_1^0(x_2, x_3)$.

It should be noted that the function X_1, defined in (3.1), can be extended 1-periodically with respect to ξ_2 and ξ_3. We save the same notation for the extended function. Put

$$v_1(\xi; x_2, x_3) = \alpha_0^1(x_2, x_3) X_1(\xi). \tag{4.23}$$

Due to (3.2) this function has the following asymptotics

$$v_1(\xi; x_2, x_3) = \alpha_0^1(x_2, x_3)\xi_1 + \alpha_0^1(x_2, x_3) C(B) + O(e^{-2\pi\xi_1}) \tag{4.24}$$

as $\xi_1 \to +\infty$. Consequently, by using Lemma 3.1 and assuming that

$$\alpha_1^0(x_2, x_3) = \alpha_0^1(x_2, x_3) C(B), \tag{4.25}$$

we conclude that the function v_1 is a solution of (4.19). Moreover,

$$v_1(\xi; x_2, x_3) = V_1(\xi_1; x_2, x_3) + O(e^{-2\pi\xi_1}) \quad \text{as } \xi_1 \to +\infty, \tag{4.26}$$

$\alpha_1^0 \in C^\infty\left[-\frac{1}{2}, \frac{1}{2}\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right]$ and, according to (4.5), it yields that

$$\frac{\partial^{2j+1}\alpha_1^0}{\partial x_2^{2j+1}}\left(\pm\frac{1}{2}, x_3\right) = 0, \qquad \frac{\partial^{2j+1}\alpha_1^0}{\partial x_3^{2j+1}}\left(x_2, \pm\frac{1}{2}\right) = 0, \qquad j = 0, 1, 2, \ldots$$

Summarizing all results, we deduce that the condition of solvability for (4.19) lead us to get the precise formula for the function $\alpha_1^0(x_2, x_3)$ in the boundary-value problem (4.6), which satisfies the conditions of Lemma 4.1. On the other hand, the formula (2.8) follows directly from (4.8) and (4.25).

Note that, due to (4.5) and (4.23), we have that

$$\frac{\partial v_1}{\partial x_2}(\xi; x_2, x_3) = 0, \qquad \frac{\partial v_1}{\partial x_3}(\xi; x_2, x_3) = 0, \tag{4.27}$$

as $x_2 = \pm\frac{1}{2}$, $x_3 = \pm\frac{1}{2}$, respectively.

Now we begin to study the problem (4.20). Put

$$v_2(\xi; x_2, x_3) = \alpha_1^1(x_2, x_3)X_1(\xi) - 2\frac{\partial \alpha_0^1}{\partial x_2}(x_2, x_3)X_2(\xi) - 2\frac{\partial \alpha_0^1}{\partial x_3}(x_2, x_3)X_3(\xi).$$

This function is 1-periodic with respect to ξ_2 and ξ_3 and, in view of (3.2), (3.4) and (3.6), it has the asymptotics

$$v_2(\xi; x_2, x_3) = \alpha_1^1(x_2, x_3)\xi_1 + C(B)\alpha_1^1(x_2, x_3) + O(\xi_1 e^{-\xi_1}) \tag{4.28}$$

as $\xi_1 \to \infty$. Hence, taking into account Lemmas 3.1, 3.2, 3.3 and formulas (4.24), (4.5) and (4.27), we deduce that v_2 is a solution of (4.20) if

$$\alpha_2^0(x_2, x_3) = \alpha_1^1(x_2, x_3)C(B).$$

Moreover,

$$v_2(\xi; x_2, x_3) = V_2(\xi_1; x_2, x_3) + O(\xi_1 e^{-\xi_1}) \text{ as } \xi_1 \to \infty, \tag{4.29}$$

$\alpha_2^0 \in C^\infty\left[-\frac{1}{2}, \frac{1}{2}\right]$ and, due to (4.10), we have that

$$\frac{\partial^{2j+1}\alpha_2^0}{\partial x_2^{2j+1}}\left(\pm\frac{1}{2}, x_3\right) = 0, \qquad \frac{\partial^{2j+1}\alpha_2^0}{\partial x_3^{2j+1}}\left(x_2, \pm\frac{1}{2}\right) = 0, \qquad j = 0, 1, 2, \ldots$$

Hence, the solvability conditions for the boundary-value problem (4.20) determine the function $\alpha_2^0(x_2, x_3)$, which satisfies the conditions of Lemma 4.1.

Note that, according to (4.10) and the boundary conditions $X_2 = X_3 = 0$ on $\partial\Pi$, it yields that

$$\begin{aligned} \frac{\partial v_2}{\partial x_2}\left(\xi_1, \pm\frac{1}{2}, \xi_3; x_2, x_3\right) &= 0 \text{ as } x_2 = \pm\frac{1}{2}, \\ \frac{\partial v_2}{\partial x_3}\left(\xi_1, \xi_2, \pm\frac{1}{2}; x_2, x_3\right) &= 0 \text{ as } x_3 = \pm\frac{1}{2}. \end{aligned} \tag{4.30}$$

Consequently, taking into account (4.30) and (4.21), we obtain that

$$\frac{\partial \widehat{v}_\varepsilon}{\partial \nu}\left(\frac{x}{\varepsilon}; x_2, x_3\right) = 0 \text{ on } \Gamma \cup ((\partial\Omega \setminus \Gamma) \cap \{x_1 < h\}).$$

We have completed the construction of the asymptotical expansions. Now we have to prove that the constructed expansion interpolates the limit element. Lemma 4.3 together with the formulas (4.24) and (4.28) lead to the following result:

Lemma 4.4. *If* $0 < \beta < 1$, *then the estimate*

$$\|\widehat{F}_\varepsilon^v\|_{L_2(\Omega_\varepsilon \cap \{x_1 < 2\varepsilon^\beta\})} = O\left(\varepsilon^{\frac{3}{2}\beta}\right)$$

holds for the function $\widehat{F}_\varepsilon^v$ *given by* (4.22).

Proof. Taking into account (4.24) and (4.28), we have that

$$\|\widehat{F}_\varepsilon^v\|^2_{L_2(\Omega_\varepsilon\cap\{x_1<2\varepsilon^\beta\})} = \int\limits_{\Omega_\varepsilon\cap\{x_1<2\varepsilon^\beta\}} \Bigg[-\varepsilon\Bigg(\frac{\partial^2 v_1}{\partial x_2^2}+\frac{\partial^2 v_1}{\partial x_3^2}+\lambda_0 v_1+$$
$$+2\frac{\partial^2 v_2}{\partial x_2\partial\xi_2}+2\frac{\partial^2 v_2}{\partial x_3\partial\xi_3}\Bigg)-\varepsilon^2\Bigg(\lambda_1 v_1+\frac{\partial^2 v_2}{\partial x_2^2}+\frac{\partial^2 v_2}{\partial x_3^2}+\lambda_0 v_2\Bigg)-$$
$$-\varepsilon^3\Bigg(\lambda_1 v_2+\lambda_2 v_1\Bigg)-\varepsilon^4\lambda_2 v_2\Bigg]^2 dx = \int\limits_{\Omega_\varepsilon\cap\{x_1<2\varepsilon^\beta\}} \Bigg[-\varepsilon\Bigg(\frac{\partial^2 v_1}{\partial x_2^2}+\frac{\partial^2 v_1}{\partial x_3^2}+$$
$$+\lambda_0 v_1+2\frac{\partial^2 v_2}{\partial x_2\partial\xi_2}+2\frac{\partial^2 v_2}{\partial x_3\partial\xi_3}\Bigg)+O(\varepsilon^2)\Bigg]^2 dx = \int\limits_{\Omega_\varepsilon\cap\{x_1<2\varepsilon^\beta\}} \Bigg[-\varepsilon\Bigg((\xi_1+$$
$$+C(B))\Bigg(\frac{\partial^2\alpha_0^1}{\partial x_2^2}+\frac{\partial^2\alpha_0^1}{\partial x_3^2}\Bigg)+\lambda_0[\alpha_0^1(\xi_1+C(B))+O(\xi_1 e^{-\xi_1}))]\Bigg)+$$
$$+O(\varepsilon^2)\Bigg]^2 dx = \int\limits_{\Omega_\varepsilon\cap\{x_1<2\varepsilon^\beta\}} \Big[\varepsilon\xi_1(\lambda_0\alpha_0^1-1)+O(\varepsilon)+O(\varepsilon^2)\Big]^2 dx =$$
$$= \int\limits_{\Omega_\varepsilon\cap\{x_1<2\varepsilon^\beta\}} \big[x_1(\lambda_0\alpha_0^1-1)+O(\varepsilon)\big]^2\, dx.$$

Finally, we deduce that

$$\|\widehat{F}_\varepsilon^v\|_{L_2(\Omega_\varepsilon\cap\{x_1<2\varepsilon^\beta\})} = \Bigg(\int\limits_{\Omega_\varepsilon\cap\{x_1<2\varepsilon^\beta\}} O(x_1^2)\Bigg)^{\frac{1}{2}} = O(\varepsilon^{\frac{3}{2}\beta}).$$

□

On the other hand, the formulas (4.11), (4.26) and (4.29) give us the validity of the following Lemma:

Lemma 4.5. *Assume that* $0<\beta<1$. *Then the estimates*

$$\widehat{v}_\varepsilon-\widehat{u}_\varepsilon = O(\varepsilon^{3\beta}), \qquad \frac{\partial}{\partial x_1}(\widehat{v}_\varepsilon-\widehat{u}_\varepsilon) = O(\varepsilon^{2\beta})$$

hold as $\varepsilon^\beta<x_1<2\varepsilon^\beta$ $(\varepsilon^{\beta-1}<\xi_1<2\varepsilon^{\beta-1})$.

Proof. By applying (4.11), (4.26) and (4.29), we get that

$$\widehat{v}_\varepsilon-\widehat{u}_\varepsilon = \varepsilon V_1(\xi_1;x_2,x_3)+\varepsilon^2 V_2(\xi_1;x_2,x_3)-\varepsilon V_1(\xi_1;x_2,x_3)-$$
$$-\varepsilon^2 V_2(\xi_1;x_2,x_3)+O(\varepsilon e^{-2\pi\xi_1}+\varepsilon^2\xi_1 e^{-\pi\xi_1}+x_1^3+\varepsilon x_1^2+\varepsilon^2 x_1) =$$
$$= O(x_1^3) = O(\varepsilon^{3\beta}) \qquad \text{as } \varepsilon^\beta<x_1<2\varepsilon^\beta.$$

From this it follows that

$$\frac{\partial}{\partial x_1}(\widehat{v}_\varepsilon - \widehat{u}_\varepsilon) = O(x_1^2) = O(\varepsilon^{2\beta})$$

as $\varepsilon^\beta < x_1 < 2\varepsilon^\beta$. □

Let $\chi(t) \in C^\infty$ be a cutoff function, which equals to zero as $t < 1$ and equals to 1 as $t > 2$, $\chi_\beta(x_1) = \chi\left(\frac{x_1}{\varepsilon^\beta}\right)$.

Lemma 4.6. *Suppose that* $0 < \beta < 1$. *Then the function*

$$\widehat{\mathcal{U}}_\varepsilon(x) = \chi_\beta(x_1)\,\widehat{u}_\varepsilon(x) + (1 - \chi_\beta(x_1))\,\widehat{v}_\varepsilon(x).$$

is a solution of the following boundary-value problem:

$$\begin{cases} -\Delta\widehat{\mathcal{U}}_\varepsilon = \widehat{\lambda}_\varepsilon\widehat{\mathcal{U}}_\varepsilon + \widehat{f}_\varepsilon & \text{in } \Omega_\varepsilon, \\ \widehat{\mathcal{U}}_\varepsilon = 0 & \text{on } \Gamma_\varepsilon, \\ \frac{\partial\widehat{\mathcal{U}}_\varepsilon}{\partial\nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.31}$$

where

$$\|\widehat{f}_\varepsilon\|_{L_2(\Omega_\varepsilon)} = O(\varepsilon^{\frac{3}{2}\beta}), \tag{4.32}$$

and, moreover,

$$\lim_{\varepsilon\to 0}\|\widehat{\mathcal{U}}_\varepsilon\|_0 \geq 1. \tag{4.33}$$

Proof. The validity of (4.33) is obvious. The function $\widehat{\mathcal{U}}_\varepsilon$ satisfies the boundary conditions of problem (3.1) due to Lemmas 4.2, 4.3 and formula (4.30). By applying the operator $-(\triangle + \widehat{\lambda}_\varepsilon)$ to $\widehat{\mathcal{U}}_\varepsilon$ we get that

$$\widehat{f}_\varepsilon = I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 =& -\chi_\beta(\triangle\widehat{u}_\varepsilon + \widehat{\lambda}_\varepsilon\widehat{u}_\varepsilon), \\ I_2 =& -(1-\chi_\beta)(\triangle\widehat{v}_\varepsilon + \widehat{\lambda}_\varepsilon\widehat{v}_\varepsilon) = -(1-\chi_\beta)\widehat{F}_\varepsilon^v, \\ I_3 =& (\widehat{v}_\varepsilon - \widehat{u}_\varepsilon)\triangle\chi_\beta + 2\nabla\chi_\beta\nabla_x(\widehat{v}_\varepsilon - \widehat{u}_\varepsilon) \\ =& \varepsilon^{-2\beta}\chi''\left(\frac{x_1}{\varepsilon^\beta}\right)(\widehat{v}_\varepsilon - \widehat{u}_\varepsilon) + 2\varepsilon^{-\beta}\chi'\left(\frac{x_1}{\varepsilon^\beta}\right)\frac{\partial}{\partial x_1}(\widehat{v}_\varepsilon - \widehat{u}_\varepsilon). \end{aligned}$$

Using Lemma 4.2, we obtain that

$$\|I_1\|_{L_2(\Omega_\varepsilon)} = \left(\int\limits_{\Omega_\varepsilon} I_1^2(x)\,dx\right)^{\frac{1}{2}} = \big(O(\varepsilon^6)\big)^{\frac{1}{2}} = O(\varepsilon^3). \tag{4.34}$$

Lemma 4.4 together with the definition of function χ_β give the following asymptotics:

$$\|I_2\|_{L_2(\Omega_\varepsilon)} = \left(\int\limits_{\Omega_\varepsilon} I_2^2(x)\,dx\right)^{\frac{1}{2}} = \left(\int\limits_{\Omega_\varepsilon\cap\{x_1<2\varepsilon^\beta\}} I_2^2(x)\,dx\right)^{\frac{1}{2}} =$$

$$= \big(O(\varepsilon^{3\beta})\big)^{\frac{1}{2}} = O\left(\varepsilon^{\frac{3}{2}\beta}\right).$$

Taking into account the fact that the support of I_3 belongs to the set $\{x :\ \varepsilon^\beta < x_1 < 2\varepsilon^\beta\}$ and using Lemma 4.5, we have that

$$\|I_3\|_{L_2(\Omega_\varepsilon)} = O\left(\varepsilon^{\frac{3}{2}\beta}\right). \tag{4.35}$$

Finally, the asymptotics (4.34)–(4.35) lead us to (4.32). □

The estimate

$$\|\widehat{\mathcal{U}}_\varepsilon\|_{L_2(\Omega_\varepsilon)} \leq C\frac{\|\widehat{f}\|_{L_2(\Omega_\varepsilon)}}{\left|\lambda_\varepsilon - \widehat{\lambda}_\varepsilon\right|}$$

can be proved exactly in the same way as it was done in paper the [8] for the two-dimensional case. Here $\widehat{\mathcal{U}}_\varepsilon$ is a solution of the boundary-value problem (3.1) and the constant C does not depend on ε. This fact together with (4.32) and (4.33) give us the following formula:

$$|\lambda_\varepsilon - \widehat{\lambda}_\varepsilon| = O(\varepsilon^{\frac{3}{2}\beta}). \tag{4.36}$$

The formula (2.7) holds due to (4.1) and (4.36) since β is an arbitrary number in the interval $(0, 1)$. The proof of Theorem 2.3 is complete. □

Proof of Theorem 2.2.

Proof. Note that the validity of (2.3) and the convergence K_ε to K_0 as $\varepsilon \to 0$ are proved in paper [20] (see also [22] for an aperiodical case). Taking into account the variational definition of K_ε, we have that

$$\frac{1}{K_\varepsilon} = \inf_{U_\varepsilon \in H^1(\Omega,\Gamma_\varepsilon)\setminus\{0\}} \frac{\int\limits_{\Omega_\varepsilon} |\nabla U_\varepsilon|^2\, dx}{\int\limits_{\Omega_\varepsilon} U_\varepsilon^2\, dx} = \lambda_\varepsilon^1,$$

where λ_ε^1 is the first eigenvalue of spectral problem (1.1).
Finally, by using the asymptotics (2.7) for the first eigenvalue we get that

$$K_\varepsilon = \left(\frac{\pi^2}{4} + \varepsilon\lambda_1^1 + o(\varepsilon^{\frac{3}{2}-\mu})\right)^{-1}. \tag{4.37}$$

Denote by

$$K = \frac{\pi^2}{4}\lambda_1^1. \tag{4.38}$$

The formula (2.4) follows directly from (4.37) and (4.38). The proof of Theorem 2.2 is complete. □

Proof of Theorem 2.1.

Proof. First we consider the case $\alpha = 0$. By using the definition of the domain Ω^θ, the Friedrichs-inequality (2.1) and the respective asymptotics we find that

$$\int\limits_{\Omega^\theta} \rho^{-2} U_\varepsilon^2 \, dx \le \frac{K_0}{\theta^2} \int\limits_{\Omega^\theta} |\nabla U_\varepsilon|^2 \, dx. \tag{4.39}$$

The next step is to prove (2.2) for $\alpha > 0$. Choose $\sigma > 0$ and put $V_\varepsilon = U_\varepsilon \rho^\sigma$. It is not difficult to derive that

$$\begin{aligned} |\nabla V_\varepsilon|^2 &= \left(\frac{\partial U_\varepsilon}{\partial x_1} \rho^\sigma + \sigma \rho^{\sigma-1} \frac{\partial \rho}{\partial x_1} U_\varepsilon \right)^2 + \rho^{2\sigma} |\nabla_{x_2 x_3} U_\varepsilon|^2 \le \\ &\le \left(1 + \frac{1}{\varpi}\right) \rho^{2\sigma} |\nabla U_\varepsilon|^2 + (1+\varpi) \sigma^2 \rho^{2\sigma-2} U_\varepsilon^2 \end{aligned} \tag{4.40}$$

with arbitrary ϖ. By applying (4.39) to V_ε, we obtain that

$$\int\limits_{\Omega^\theta} \rho^{-2+2\sigma} U_\varepsilon^2 \, dx \le \frac{K_0}{\theta^2} \left(\left(1 + \frac{1}{\varpi}\right) \int\limits_{\Omega^\theta} \rho^{2\sigma} |\nabla U_\varepsilon|^2 \, dx + \sigma^2 (1+\varpi) \int\limits_{\Omega^\theta} \rho^{2(\sigma-1)} U_\varepsilon^2 \, dx \right).$$

If $1 - \frac{(1+\varpi)K_0}{\theta^2} \sigma^2 > 0$, then

$$\int\limits_{\Omega^\theta} \rho^{-2+2\sigma} U_\varepsilon^2 \, dx \le \frac{(1+\frac{1}{\varpi})K_0}{\theta^2 - (1+\varpi) K_0 \sigma^2} \int\limits_{\Omega^\theta} \rho^{2\sigma} |\nabla U_\varepsilon|^2 \, dx.$$

Finally, choosing $\alpha = 2\sigma$ and the constant $\varpi = \frac{2\theta}{\sqrt{K_0}\alpha} - 1$, we obtain (2.2). □

5 Concluding remarks and result

Remark 5.1. Note that in (2.2) we have different domains on the right and left hand sides. By using well-known theorems from the theory of Hardy type inequalities we can derive inequalities with the same domain on both sides but then we must replace U_ε by another function $U_\varepsilon - M_\varepsilon$. For example the following result holds:

Theorem 5.1. *Assume that* $\rho(x) = dist(x, \Gamma)$, $0 \le \alpha \ne 1$, $U_\varepsilon \in H^1(\Omega, \Gamma_\varepsilon)$. *Then there exists a function* $M_\varepsilon = M_\varepsilon(x_2, x_3)$,

$$\|M_\varepsilon\|_{L_2(\Gamma)} \le \mathcal{C}\sqrt{\varepsilon}, \tag{5.1}$$

such that the Hardy-type inequality

$$\int\limits_{\Omega} |U_\varepsilon - M_\varepsilon|^2 \rho^{\alpha-2} \, dx \le \frac{4}{(\alpha-1)^2} \int\limits_{\Omega} |\nabla U_\varepsilon|^2 \rho^\alpha \, dx \tag{5.2}$$

holds.

Proof. We assume at first that $U_\varepsilon \in C^\infty(\Omega, \Gamma_\varepsilon)$. Fix the variables x_2, x_3 and use first the following one-dimensional Hardy-type inequality:

$$\int_0^1 \frac{v^2(x_1)}{x_1^{2-\alpha}}\,dx_1 \le \frac{4}{(\alpha-1)^2}\int_0^1 x_1^\alpha (v')^2\,dx_1,$$

where $v \in AC[0,1]$ and $v(0) = 0$. By applying this inequality to the function $v(x_1) = U_\varepsilon(x_1, \cdot, \cdot) - U_\varepsilon(0, \cdot, \cdot)$, we obtain that

$$\int_0^1 (U_\varepsilon(x_1,x_2,x_3) - U_\varepsilon(0,x_2,x_3))^2 \rho^{\alpha-2}\,dx_1 \le \frac{4}{(\alpha-1)^2}\int_0^1 \rho^\alpha |\nabla U_\varepsilon|^2\,dx_1. \tag{5.3}$$

Denote by $M_\varepsilon(x_2, x_3) := U_\varepsilon(0, x_2, x_3)$. By integrating the inequality (5.3) with respect to x_2 and x_3, we deduce that

$$\int_\Omega (U_\varepsilon - M_\varepsilon)^2 \rho^{\alpha-2}\,dx \le \frac{4}{(\alpha-1)^2}\int_\Omega \rho^\alpha |\nabla U_\varepsilon|^2\,dx. \tag{5.4}$$

Finally, we approximate the functions $U_\varepsilon \in H^1(\Omega, \Gamma_\varepsilon)$ by smooth functions from $C^\infty(\Omega, \Gamma_\varepsilon)$ and conclude that (5.4) is valid also for $U_\varepsilon \in H^1(\Omega, \Gamma_\varepsilon)$. The next step is to derive the estimate (5.1).

There exists a sequence of functions $U_\varepsilon^k \in C^\infty(\Omega, \Gamma_\varepsilon)$ such that U_ε^k converges to U_ε in H^1 as $k \to \infty$. Denote by $M_\varepsilon^k := U_\varepsilon^k(0, x_2, x_3)$. Consequently, M_ε^k converges to M_ε as $k \to \infty$ in $H^{\frac{1}{2}}$. Choose a number K such that

$$\|M_\varepsilon - M_\varepsilon^k\|_{L_2(\Gamma)} \le C\sqrt{\varepsilon} \ \text{ for any } k > K. \tag{5.5}$$

Let us prove that there is a constant C such that

$$\|M_\varepsilon^k\|_{L_2(\Gamma)} \le C\sqrt{\varepsilon}. \tag{5.6}$$

Suppose that there exists a subsequence ε_n, $n \to \infty$, such that

$$\|M_{\varepsilon_n}^k\|^2_{L_2(\Gamma)} > n^2 \varepsilon_n.$$

Hence, $\max\limits_\Gamma |M_{\varepsilon_n}^k| > n\sqrt{\varepsilon_n}$. Consequently,

$$\int_{L_2(\Omega\cap\{x_1\le\varepsilon_n\})} \left|\nabla U_{\varepsilon_n}^k\right|^2 dx > \varepsilon_n \left(\frac{\max\limits_\Gamma |M_{\varepsilon_n}^k|}{\varepsilon_n}\right)^2 > n^2,$$

that is U_ε^k does not belong to H^1. This contradiction proves (5.6).

Taking into account (5.6) and (5.5) we deduce that

$$\|M_\varepsilon\|_{L_2(\Gamma)} \le \|M_\varepsilon - M_\varepsilon^k\|_{L_2(\Gamma)} + \|M_\varepsilon^k\|_{L_2(\Gamma)} \le 2C\sqrt{\varepsilon}.$$

The last estimate with $2C = \mathcal{C}$ proves (5.1). □

Remark 5.2. We have shown in our proof that the estimate (5.1) is the best possible in the sense that $C\varepsilon^{\frac{1}{2}}$ on the right-hand side can not be replaced by $C\varepsilon^q$ for any $q > \frac{1}{2}$. In fact, it even yields that $\|M_\varepsilon\|_{L_2(\Gamma)} > \quad C\varepsilon^q$ for $q > \frac{1}{2}$ and any C.

Remark 5.3. We have proved the inequalities (2.2) and (2.3) and have constructed the asymptotics for the best constants only in the case $p = q = 2$. However, by using the techniques in this paper an analogous result for arbitrary $p, q > 1$ can also be proved, but here it is not so easy to construct the asymptotics for the best constant, since we have to consider a nonlinear spectral problem for $-\Delta_p$ operator. This is an interesting question to study in the future.

Remark 5.4. An interesting extension of the results in this paper would be to consider domains with more general microstructure.

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