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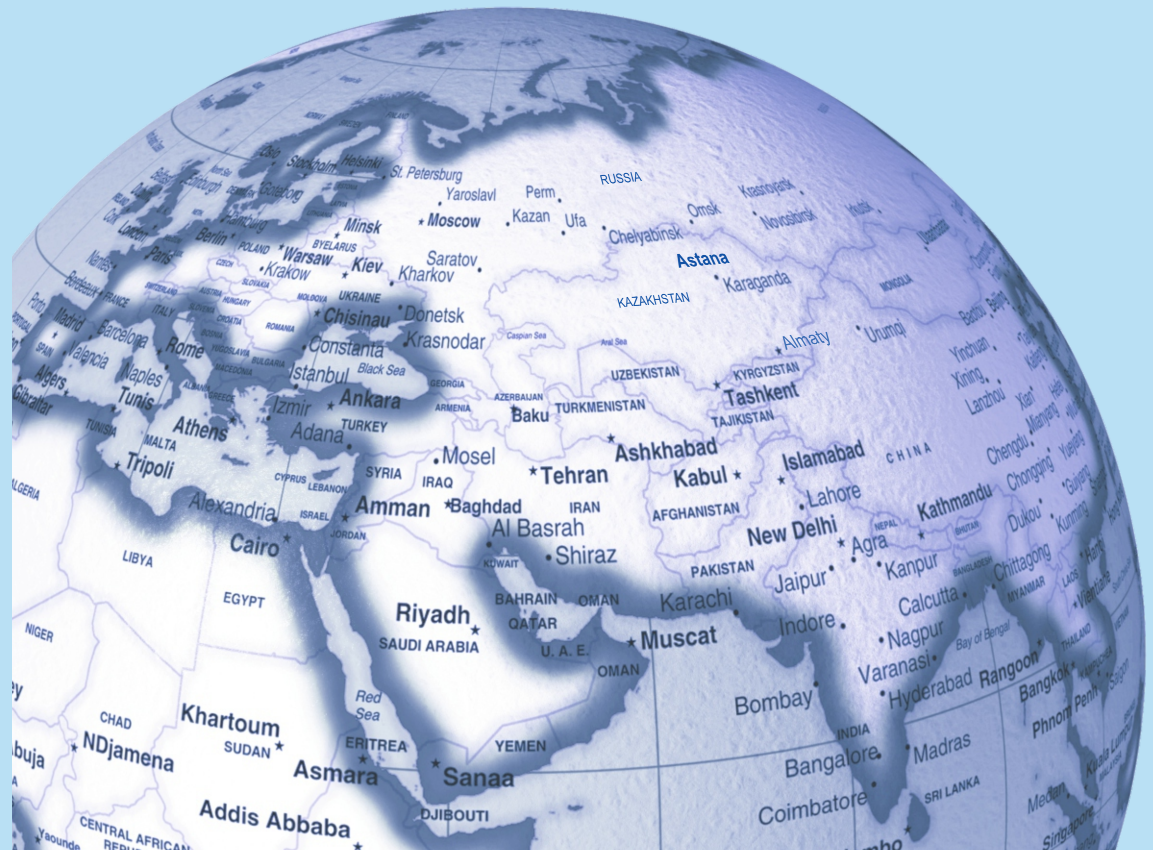

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ONE-DIMENSIONAL INTEGRAL RELLICH TYPE INEQUALITIES

T. Ozawa, P. Roychowdhury, D. Suragan

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Abstract. The motive of this note is twofold. Inspired by the recent development of a new kind of Hardy inequality, here we discuss the corresponding Hardy–Rellich and Rellich inequality versions in the integral form. The obtained sharp Hardy–Rellich type inequality improves the previously known result. Meanwhile, the established sharp Rellich type integral inequality seems new.

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1 Introduction

In the celebrated paper, [9], Godfrey H. Hardy first stated the famous inequality. The result reads as follows. For any $1 < p < \infty$ and f be a p-integrable function on $(0, \infty)$, then the function $r \longmapsto \frac{1}{r}\int_0^r f(t)\,\mathrm{d}t$ is p-integrable over $(0, \infty)$ and there holds

$$\int_0^\infty \left|\frac{1}{r}\int_0^r f(t)\,\mathrm{d}t\right|^p \mathrm{d}r \le \left(\frac{p}{p-1}\right)^p \int_0^\infty |f(r)|^p\,\mathrm{d}r. \tag{1.1}$$

The constant on the right-hand side of (1.1) is sharp. The development of the famous Hardy inequality (1.1) during the period 1906–1928 has its own history and we refer to [12] (also, see the preface of [22]). Recent progress by Frank–Laptev–Weidl [10] presents a novel one-dimensional inequality with the same sharp constant, which improves the classical Hardy inequality (1.1).

This new version looks as follows. For any $1 < p < \infty$ and for any $f \in L^p(0, \infty)$, which vanishes at zero, there holds

$$\int_0^\infty \sup_{0<s<\infty}\left|\min\left\{\frac{1}{r}, \frac{1}{s}\right\}\int_0^s f(t)\,\mathrm{d}t\right|^p \mathrm{d}r \le \left(\frac{p}{p-1}\right)^p \int_0^\infty |f(r)|^p\,\mathrm{d}r. \tag{1.2}$$

Certainly, (1.2) gives an improvement of (1.1). Recently, the multidimensional version in the supercritical case and the discrete version of (1.2) have been established in [20] and [19], respectively. In the same spirit, one may ask about the possible structure of Hardy–Rellich and Rellich type inequalities. In this short note, we obtain the possible form of these two types of inequalities.

Let us recall the one-dimensional Hardy–Rellich inequality. For $f \in C^1[0, \infty)$ with $f(0) = 0$, there holds

$$\int_0^\infty \frac{|f(r)|^2}{r^2}\,\mathrm{d}r \le 4\int_0^\infty |f'(r)|^2\,\mathrm{d}r. \tag{1.3}$$

Starting from it, there have been several articles in which the authors studied many improvements in inequality (1.3). Here we mention only a few of them [3, 6, 7, 11, 13, 16, 17, 24, 23] and references therein. Now let us write (1.3) in the integral form. Note that it can be derived from the weighted one-dimensional classical Hardy inequality. This reads as follows. Let $f \in C^1(0,\infty)$, then there holds

$$\int_0^\infty \frac{|\int_0^r f'(t)\,\mathrm{d}t|^2}{r^2}\,\mathrm{d}r \le 4\int_0^\infty |f'(r)|^2\,\mathrm{d}r. \tag{1.4}$$

Here the constant 4 is sharp. We give an improved version of this inequality in Theorem 2.1.

Let us briefly mention another important function inequality the so-called Rellich inequality which was first introduced in [18]. It is worth recalling the one-dimensional Rellich inequality. The classical one-dimensional Rellich inequality states that for $f \in C^2[0,\infty)$ with $f(0) = 0$ and $f'(0) = 0$, there holds

$$\int_0^\infty \frac{|f(r)|^2}{r^4}\,\mathrm{d}r \le C\int_0^\infty |f''(r)|^2\,\mathrm{d}r, \tag{1.5}$$

where $C > 0$ is independent of f. Over the past few decades, there has been a constant effort to improve (1.5). Here are some closely related papers [8, 14, 1, 15, 4, 21, 5]. In this short contribution, we also obtain another type of Rellich inequality (see Theorem 2.2 with $p = 2$). To the best of our knowledge, the most recent progress in this direction was made in [4]. However, a one-dimensional study is still missing. As far as we know, a sharp constant in this inequality was not found. Thus, trying to fill this gap is another motivation for the present paper. Taking inspiration from there we obtain the following version of Rellich inequality. For any $f \in L^2(0,\infty)$ there holds

$$\begin{aligned}
&\int_0^\infty \frac{1}{r^4}\Big(\int_0^r\int_0^\tau |f(t)|\,\mathrm{d}t\,\mathrm{d}\tau\Big)^2\,\mathrm{d}r\\
&\le \int_0^\infty \frac{1}{r^4}\Big(\int_0^r \sup_{0<s<\infty}\min\Big\{1,\frac{\tau}{s}\Big\}\int_0^s |f(t)|\,\mathrm{d}t\,\mathrm{d}\tau\Big)^2\,\mathrm{d}r\\
&\le \frac{16}{9}\int_0^\infty |f(r)|^2\,\mathrm{d}r.
\end{aligned} \tag{1.6}$$

Moreover, we will show that the constant 16/9 is a sharp constant. Therefore, (1.6) can be compared with (1.5). Note that we have mentioned only the $L^2(0,\infty)$ case but we will discuss the result for the general $L^p(0,\infty)$ case.

2 Preliminaries and main results

Let us begin this section with basic facts about *a decreasing rearrangement*. For more details, we refer to [2, Section 2.1]. The decreasing rearrangement of f is the function f^* defined on $[0,\infty)$ by

$$f^*(x) = \inf\{\lambda : \mu_f(\lambda) \le x\}, \quad x \ge 0,$$

where $\mu_f(\lambda) = |\{x \in \mathbb{R} : |f(x)| > \lambda\}|, \quad \lambda \ge 0$. Here $|J|$ is the Lebesgue measure of the set $J \subset \mathbb{R}$. It is well known that f^* is a nonnegative and nonincreasing function. Irrespective of several properties of f^*, the useful property in our context is the equimeasurability property, i.e.

$$|\{|f| > \tau\}| = |\{f^* > \tau\}| \text{ for all } \tau \ge 0. \tag{2.1}$$

By using the *layer cake representation* and the above property, we have the following helpful identity:

$$\int_0^\infty |f(t)|^p\,\mathrm{d}t = \int_0^\infty |f^*(t)|^p\,\mathrm{d}t \quad \text{for all } p \ge 1. \tag{2.2}$$

Also, for any $s > 0$ there holds

$$\int_0^s |f(t)|\,\mathrm{d}t \le \int_0^s f^*(t)\,\mathrm{d}t. \tag{2.3}$$

These relations will be valuable in the proofs.

Now, we are ready to state the following important observation.

Lemma 2.1. *For any r>0 and $f \in L^1(0,r)$, the following identity holds:*

$$\sup_{0<s<\infty} \min\left\{1, \frac{r}{s}\right\} \int_0^s f^*(t)\,\mathrm{d}t = \int_0^r f^*(t)\,\mathrm{d}t. \tag{2.4}$$

Proof. We wish to calculate the supremum by using the monotonicity of f^*. For any fixed $r > 0$, we consider the following two cases:

Case 1. Let $0 < s \le r$. Then we obtain

$$\min\left\{1, \frac{r}{s}\right\} \int_0^s f^*(t)\,\mathrm{d}t = \int_0^s f^*(t)\,\mathrm{d}t \le \int_0^r f^*(t)\,\mathrm{d}t.$$

Case 2. Let $r \le s < \infty$. Then we have by change of variable

$$\min\left\{1, \frac{r}{s}\right\} \int_0^s f^*(t)\,\mathrm{d}t = \frac{r}{s} \int_0^s f^*(t)\,\mathrm{d}t \le \frac{r}{s} \int_0^s f^*(tr/s)\,\mathrm{d}t = \int_0^r f^*(v)\,\mathrm{d}v.$$

In both cases, we get

$$\min\left\{1, \frac{r}{s}\right\} \int_0^s f^*(t)\,\mathrm{d}t \le \int_0^r f^*(t)\,\mathrm{d}t.$$

Hence, the supremum is attained at $s = r$ and we arrive at

$$\sup_{0<s<\infty} \min\left\{1, \frac{r}{s}\right\} \int_0^s f^*(t)\,\mathrm{d}t = \int_0^r f^*(t)\,\mathrm{d}t.$$

□

Now, we are ready to present an improvement of (1.4). That is, this gives a natural improvement of the Hardy–Rellich inequality in the integral form. Below we will describe the corresponding differential form which improves the original Hardy–Rellich inequality (1.3) in a simple form.

Theorem 2.1. *Let $f \in L^2(0,\infty)$, then there holds*

$$\int_0^\infty \sup_{0<s<\infty} \left| \min\left\{\frac{1}{r}, \frac{1}{s}\right\} \int_0^s f(t)\,\mathrm{d}t \right|^2 \mathrm{d}r \le 4 \int_0^\infty |f(r)|^2\,\mathrm{d}r. \tag{2.5}$$

Moreover, the constant 4 in the above inequality is sharp in the sense that no inequality of the form

$$\int_0^\infty \sup_{0<s<\infty} \left| \min\left\{\frac{1}{r}, \frac{1}{s}\right\} \int_0^s f(t)\,\mathrm{d}t \right|^2 \mathrm{d}r \le C \int_0^\infty |f(r)|^2\,\mathrm{d}r$$

holds, for $f \in L^2(0,\infty)$ such that $f \not\sim 0$ on $(0,\infty)$, when $C < 4$.

Now, we are going to discuss the second main result of this note. Before presenting the statement first let us recall the classical one-dimensional L^p-Rellich inequality (see, e.g. [1]). This reads as follows. Let $p > 1$, $f \in C^2[0,\infty)$ with $f(0) = 0$ and $f'(0) = 0$ there holds

$$\int_0^\infty \frac{|f(r)|^p}{r^{2p}}\,\mathrm{d}r \le \frac{p^{2p}}{(p-1)^p(2p-1)^p}\int_0^\infty |f''(r)|^p\,\mathrm{d}r. \tag{2.6}$$

Now, we are ready to demonstrate the one-dimensional Rellich-type inequality in the following integral form.

Theorem 2.2. *Let $f \in L^p(0,\infty)$, $p > 1$. Then we have*

$$\begin{aligned}
&\int_0^\infty \frac{1}{r^{2p}}\left(\int_0^r\int_0^\tau |f(t)|\,\mathrm{d}t\,\mathrm{d}\tau\right)^p \mathrm{d}r\\
&\le \int_0^\infty \frac{1}{r^{2p}}\left(\int_0^r \sup_{0<s<\infty}\min\left\{1,\frac{\tau}{s}\right\}\int_0^s |f(t)|\,\mathrm{d}t\,\mathrm{d}\tau\right)^p \mathrm{d}r\\
&\le \frac{p^{2p}}{(p-1)^p(2p-1)^p}\int_0^\infty |f(r)|^p\,\mathrm{d}r.
\end{aligned} \tag{2.7}$$

Moreover, the constant $\frac{p^{2p}}{(p-1)^p(2p-1)^p}$ in the above inequality turns out to be sharp in the sense that no inequality of the form

$$\int_0^\infty \frac{1}{r^{2p}}\left(\int_0^r\int_0^\tau |f(t)|\,\mathrm{d}t\,\mathrm{d}\tau\right)^p \mathrm{d}r \le C\int_0^\infty |f(r)|^p\,\mathrm{d}r.$$

for all $f \in L^p(0,\infty)$ such that $f \not\sim 0$ on $(0,\infty)$, when $C < \frac{p^{2p}}{(p-1)^p(2p-1)^p}$.

3 Proofs of Theorems 2.1 and 2.2

This section is concerned with the proofs of Theorems 2.1 and 2.2. Before going further let us recall the following lemma.

Lemma 3.1. [20, Lemma 3.1] *Let $1 < p < \infty$. Let w be any nonnegative measurable function on $(0,\infty)$. Assume h is a strictly positive non-decreasing function on $(0,\infty)$ such that $sh(r) \le rh(s)$ for any $r, s \in (0,\infty)$ with $r \le s$. Let $f \in L^1(0,r)$ for any $r > 0$. Then we have*

$$\int_0^\infty w(r)\sup_{0<s<\infty}\left|\min\left\{\frac{1}{h(r)},\frac{1}{h(s)}\right\}\int_0^s f(t)\,\mathrm{d}t\right|^p \mathrm{d}r \le \int_0^\infty w(r)\left|\frac{1}{h(r)}\int_0^r f^*(t)\,\mathrm{d}t\right|^p \mathrm{d}r.$$

Now, as a direct corollary of Lemma 3.1, we derive the proof of Theorem 2.1.

Proof of Theorem 2.1. Let us consider $w(r) = 1$ and $h(r) = r$ to be functions on $(0,\infty)$ and substitute these in Lemma 3.1 with $p = 2$, then we have

$$\int_0^\infty \sup_{0<s<\infty}\left|\min\left\{\frac{1}{r},\frac{1}{s}\right\}\int_0^s f(t)\,\mathrm{d}t\right|^2 \mathrm{d}r \le \int_0^\infty \frac{1}{r^2}\left|\int_0^r f^*(t)\,\mathrm{d}t\right|^2 \mathrm{d}r.$$

By using the Hardy–Rellich inequality in form (1.4) for the function f^*, we obtain

$$\begin{aligned}
\int_0^\infty \sup_{0<s<\infty}\left|\min\left\{\frac{1}{r},\frac{1}{s}\right\}\int_0^s f(t)\,\mathrm{d}t\right|^2 \mathrm{d}r &\le 4\int_0^\infty |f^*(r)|^2\,\mathrm{d}r\\
&= 4\int_0^\infty |f(r)|^2\,\mathrm{d}r.
\end{aligned}$$

In the last step, we have used norm preserving property (2.2). The sharpness follows from the optimality of the constant in (1.4). This completes the proof.

Proof of Theorem 2.2. The first inequality follows from the property of the supremum. Now taking the integral of (2.4) from 0 to r we have

$$\int_0^r \sup_{0<s<\infty} \min\left\{1, \frac{\tau}{s}\right\} \int_0^s f^*(t)\,\mathrm{d}t\,\mathrm{d}\tau = \int_0^r \int_0^\tau f^*(t)\,\mathrm{d}t\,\mathrm{d}\tau. \tag{3.1}$$

Then

$$\begin{aligned}
&\int_0^\infty \frac{1}{r^{2p}}\left(\int_0^r \sup_{0<s<\infty} \min\left\{1, \frac{\tau}{s}\right\} \int_0^s |f(t)|\,\mathrm{d}t\,\mathrm{d}\tau\right)^p \mathrm{d}r \\
&\overset{(2.3)}{\leq} \int_0^\infty \frac{1}{r^{2p}}\left(\int_0^r \sup_{0<s<\infty} \min\left\{1, \frac{\tau}{s}\right\} \int_0^s f^*(t)\,\mathrm{d}t\,\mathrm{d}\tau\right)^p \mathrm{d}r \\
&\overset{(3.1)}{=} \int_0^\infty \frac{1}{r^{2p}}\left(\int_0^r \int_0^\tau f^*(t)\,\mathrm{d}t\,\mathrm{d}\tau\right)^p \mathrm{d}r \\
&\overset{(2.6)}{\leq} \frac{p^{2p}}{(p-1)^p(2p-1)^p} \int_0^\infty |f^*(r)|^p\,\mathrm{d}r \\
&\overset{(2.2)}{=} \frac{p^{2p}}{(p-1)^p(2p-1)^p} \int_0^\infty |f(r)|^p\,\mathrm{d}r.
\end{aligned}$$

Optimality. We set

$$C_p := \sup_{f\in L^p(0,\infty)\setminus\{0\}} \frac{\int_0^\infty \frac{1}{r^{2p}}\left(\int_0^r \int_0^\tau |f(t)|\,\mathrm{d}t\,\mathrm{d}\tau\right)^p \mathrm{d}r}{\int_0^\infty |f(r)|^p\,\mathrm{d}r}. \tag{3.2}$$

The validity of (2.7) immediately implies

$$C_p \leq \frac{p^{2p}}{(p-1)^p(2p-1)^p}.$$

So, it remains to show the reverse inequality and this will be done by giving a proper minimizing sequence. We divide the proof into some steps.

Step 1. Let us start with a cut-off function $\chi : [0,\infty) \to \mathbb{R}$ with the following properties:

1. $\chi(r) \in [0,1]$ for all $r \in [0,\infty)$ and χ is smooth;
2. χ satisfies the following
$$\chi(r) = \begin{cases} 1, & 0 \leq r \leq 1, \\ 0, & 2 \leq r < \infty \end{cases}$$
3. χ is decreasing function, i.e. $\chi'(r) \leq 0$ for all $r \in (0,\infty)$.

Now for a small $\epsilon > 0$, let us define the minimizing functions $\{f_\epsilon\}$ as follows:

$$f_\epsilon(r) := r^{\frac{\epsilon-1}{p}}\chi(r).$$

Step 2. In this step we will estimate the right-hand side of (2.7). The denominator of (3.2) gives

$$\begin{aligned}
\int_0^\infty |f_\epsilon(r)|^p\,\mathrm{d}r &= \int_0^\infty r^{\epsilon-1}\chi^p(r)\,\mathrm{d}r \\
&= \int_0^1 r^{\epsilon-1}\,\mathrm{d}r + \int_1^2 r^{\epsilon-1}\chi^p(r)\,\mathrm{d}r \\
&= \frac{1}{\epsilon} + O(1).
\end{aligned} \tag{3.3}$$

Therefore, for a fixed positive ϵ, we have $f_\epsilon \in L^p(0,\infty)$.

Step 3. In this part we will evaluate the numerator of (3.2). Using the integration by parts, we have

$$\begin{aligned}
&\int_0^\infty \frac{1}{r^{2p}}\Big(\int_0^r\int_0^\tau |f_\epsilon(t)|\,\mathrm{d}t\mathrm{d}\tau\Big)^p \mathrm{d}r\\
&=\int_0^\infty \frac{1}{r^{2p}}\Big(\int_0^r\int_0^\tau t^{\frac{\epsilon-1}{p}}\chi(t)\,\mathrm{d}t\mathrm{d}\tau\Big)^p \mathrm{d}r\\
&=\Big(\frac{p}{\epsilon-1+p}\Big)^p\int_0^\infty \frac{1}{r^{2p}}\Big[\int_0^r \chi(\tau)\tau^{\frac{\epsilon-1+p}{p}}\,\mathrm{d}\tau-\int_0^r\int_0^\tau t^{\frac{\epsilon-1+p}{p}}\chi'(t)\,\mathrm{d}t\,\mathrm{d}\tau\Big]^p \mathrm{d}r\\
&\geq\Big(\frac{p}{\epsilon-1+p}\Big)^p\int_0^\infty \frac{1}{r^{2p}}\Big[\int_0^r \chi(\tau)\tau^{\frac{\epsilon-1+p}{p}}\,\mathrm{d}\tau\Big]^p \mathrm{d}r\\
&=\Big(\frac{p}{\epsilon-1+p}\Big)^p\Big(\frac{p}{\epsilon-1+2p}\Big)^p\int_0^\infty \frac{1}{r^{2p}}\Big[\chi(r)r^{\frac{\epsilon-1+2p}{p}}-\int_0^r \tau^{\frac{\epsilon-1+2p}{p}}\chi'(\tau)\,\mathrm{d}\tau\Big]^p \mathrm{d}r\\
&\geq\Big(\frac{p}{\epsilon-1+p}\Big)^p\Big(\frac{p}{\epsilon-1+2p}\Big)^p\int_0^\infty r^{\epsilon-1}\chi^p(r)\,\mathrm{d}r\\
&=\Big(\frac{p}{\epsilon-1+p}\Big)^p\Big(\frac{p}{\epsilon-1+2p}\Big)^p\Big[\int_0^1 r^{\epsilon-1}\,\mathrm{d}r+\int_1^2 r^{\epsilon-1}\chi^p(r)\,\mathrm{d}r\Big]\\
&=\frac{1}{\epsilon}\Big(\frac{p}{\epsilon-1+p}\Big)^p\Big(\frac{p}{\epsilon-1+2p}\Big)^p+O(1).
\end{aligned}\tag{3.4}$$

In between, exploiting $\chi'\leq 0$, we used an obvious inequality $(a+b)^p\geq a^p$ twice, for nonnegative real numbers a and b.

Step 4. Finally, by using (3.3) and (3.4) we estimate the ratio

$$\begin{aligned}
&\frac{\int_0^\infty \frac{1}{r^{2p}}\big(\int_0^r\int_0^\tau |f(t)|\,\mathrm{d}t\,\mathrm{d}\tau\big)^p\mathrm{d}r}{\int_0^\infty |f(r)|^p\,\mathrm{d}r}\\
&\geq\frac{\frac{1}{\epsilon}\big(\frac{p}{\epsilon-1+p}\big)^p\big(\frac{p}{\epsilon-1+2p}\big)^p+O(1)}{\frac{1}{\epsilon}+O(1)}\to\frac{p^{2p}}{(p-1)^p(2p-1)^p}\quad\text{for }\epsilon\to 0.
\end{aligned}$$

Hence $\{f_\epsilon\}$ is a required minimizing sequence and, in turn, we have

$$C_p=\frac{p^{2p}}{(p-1)^p(2p-1)^p}.$$

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