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TWO-WEIGHT HARDY INEQUALITY ON TOPOLOGICAL MEASURE SPACES

K.T. Mynbaev, E.N. Lomakina

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Key words: Hardy operator, measure space, topological space, multidimensional Hardy inequality, approximation numbers.

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Abstract We consider a Hardy type integral operator T associated with a family of open subsets $\Omega(t)$ of an open set Ω in a Hausdorff topological space X. In the inequality

$$\left(\int_\Omega |Tf(x)|^q u(x) d\mu(x)\right)^{1/q} \le C \left(\int_\Omega |f(x)|^p v(x) d\nu(x)\right)^{1/p},$$

the measures μ, ν are σ-additive Borel measures; the weights u, v are positive and finite almost everywhere, $1 < p < \infty$, $0 < q < \infty$, and $C > 0$ is independent of f, u, v, μ, ν. We find necessary and sufficient conditions for the boundedness and compactness of the operator T and obtain two-sided estimates for its approximation numbers. All results are proved using domain partitions, thus providing a roadmap for generalizing many one-dimensional results to a Hausdorff topological space.

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1 Introduction

A one-dimensional Hardy inequality

$$\left[\int_0^\infty u(x) \left(\int_0^x f(y)\, dy\right)^q d\mu(x)\right]^{1/q} \le C \left(\int_0^\infty f^p(y)\, v(y)\, d\nu(y)\right)^{1/p}$$

has been studied in detail and complete characterizations of its validity for all non-negative functions f have been obtained in terms of pairs of weights u, v and measures μ, ν for all pairs of exponents p, q, see [11], [12], [13], [14], [20] for the history and extensive references. By a characterization we mean obtaining a functional $\Phi(u, v, \mu, \nu)$ such that for all weights and measures the inequality $c_1 C \le \Phi(u, v, \mu, \nu) \le c_2 C$ is true, where C is the best constant in the above inequality and c_1, $c_2 > 0$ can depend on p, q but are not allowed to depend on u, v, μ, ν. Those characterizations are very different for the cases $p \le q$ and $q < p$.

In the one-dimensional case most researchers have used tools of one-dimensional calculus, such as integration by parts [33]. The lack of such tools has been the main obstacle on the way to multidimensional results. Some general results for $p \le q$ and for Banach function spaces have been established in [7]. Obtaining full characterizations has been facilitated by the possibility to reduce the multidimensional case to the one-dimensional, by using spherical coordinates [31], [4], the polar decomposition [26], [27] or assuming that the weights are products of functions of one variable [35],

[36]. The result by Sawyer [29] does not allow reduction to dimension one but is limited to a quadrant on the plane R^2.

In a recent paper Sinnamon [32] suggested a very general method that covers totally ordered sets of domains on a measure space. The method relies on a non-increasing rearrangement involving the weights and measures and reduces the multidimensional case to the one-dimensional. Apart from generality, the method allows Sinnamon to improve the constants c_1, c_2. The analysis of ordered cores is of independent interest.

For applications it is desirable to have everything to be expressed in terms of original weights and measures, the most important examples being the Hardy-Steklov type operator [9] and the Hardy inequality on cones of monotone functions [30], [34]. In Sinnamon's method one additional step is required to derive the criteria in terms of original weights and measures from his one-dimensional formulations. K. Mynbaev [21] has obtained results in terms of original weights and measures under the assumptions on the domains that are close to the ones imposed by Sinnamon (see [21, Remark 1] for a more detailed comparison with Sinnamon's paper).

Here we develop a different approach to the norm estimation, compactness conditions and bounds for approximation numbers using domain partitions. The boundedness criteria obtained below can be derived from both [32] and [21]. Nevertheless, we give full proofs of boundedness, compactness and estimates of approximation numbers to show that domain partitions combined with the conditions on the operator T imposed here allow one to extend many of the existing one-dimensional results to the current setup in a Hausdorff space. Possible extensions include results that employ the Oinarov condition [22], [15]. Since Sinnamon's approach covers also discrete Hardy inequalities, it would be interesting to see if the results of [16] can deduced following Sinnamon.

We consider integration over expanding subsets $\Omega(t)$ of an arbitrary open set Ω in a Hausdorff topological space X with σ-additive Borel measures μ, ν. As in [32] and [21], neither $\Omega(t)$ nor their complements $\Omega \backslash \Omega(t)$ need to be connected and there are no requirements on the shape of $\Omega(t)$ when X is a linear space. In the classical case one can notice that the subdomain $\Omega(t) = (0, t)$ of $\Omega = (0, \infty)$ has $\omega(t) = t$ as the boundary in the relative topology and that $\Omega(t) = \{s \in \Omega : \omega(s) < \omega(t)\}$. Our conditions on the family $\{\Omega(t)\}$ are based on this observation.

The existing results on integral Hardy inequalities for R^n or measure metric spaces (in which $\Omega(t)$ are balls, see [4], [2], [26], [27], [28]) follow from ours, as well as from [32] and [21]. Product weights are not included as well as Sawyer's result [29] (his rectangles do not satisfy condition (2.1) below). In papers [26], [27], [28] a metric is required to generate balls and a polar decomposition to use the one-dimensional techniques, while we avoid these requirements. There is a number of situations (homogeneous groups, hyperbolic spaces, Cartan-Hadamard manifolds, and connected Lie groups) when the polar decomposition is available, see also [1] for a study of polarizable metric measure spaces. All such situations are covered by our statements. The authors of the article [28] employ the results from [21].

Unlike [32] and [21], our approach is elementary and does not require any advanced measure theory beyond σ-additivity. Note that binary partitions were used to prove sufficiency for the Hardy operator in the one-dimensional case in [3]. Unlike [3], we avoid their auxiliary functions Φ and Φ_1 and apply discretization both for the upper and lower bounds in terms of the same functional of the weights.

Note that we provide two different proofs of the sufficiency of the compactness conditions: in Sections 3 and 4. Both employ an explicit finite-rank approximation to the Hardy operator.

The study of the approximation numbers (a-numbers) of the Hardy operator in the Lebesgue spaces on the half-line for parameters satisfying $1 < p \leq q < \infty$ started with the papers by D.E. Edmunds, W.D. Evans, D.J. Harris [5], [6]. They found implicit and asymptotic bounds for a-numbers of the operator $T : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$. Next D.E. Edmunds, V.D. Stepanov [8] obtained the bounds for singular numbers of the Hardy type operator with a polynomial kernel acting in the spaces $L^2(\mathbb{R}^+)$. Those results were extended by E.N. Lomakina, V.D. Stepanov [19] to the case $1 < p, q < \infty$; besides, two-sided bounds for the Schatten–von-Neumann norm were proved. However, in the case $1 < q < p < \infty$ the upper bound for the a-numbers $a_N(T) \le N^{1/q-1/p}\varepsilon$ was not informative because of its dependence on N. In this paper for $1 < q < p < \infty$ we derive an upper bound that does not depend on N. We do not consider the case $0 < q < 1 < p < \infty$ studied by E.N. Lomakina [17], nor do we attempt to study the Hardy operator acting from the Lebesgue spaces to the Lorentz spaces in the spirit of [18].

2 Hardy operator boundedness

We write $A \asymp B$ to mean that $c_1 A \le B \le c_2 A$ with constants c_1, c_2 that do not depend on weights and measures.

Assumption 1. *Let Ω be an open subset of a Hausdorff topological space X with σ-additive measures μ, ν. The measures are defined on a σ-algebra $\mathfrak{M}$ that contains the Borel-measurable sets. The weights u, v are assumed to be positive and finite almost everywhere.*

Assumption 2. *a) $\{\Omega(t) : t \ge 0\}$ is a one-parametric family of open subsets of Ω which satisfy monotonicity*

$$\text{for } t_1 < t_2,\ \Omega(t_1) \text{ is a proper subset of } \Omega(t_2). \tag{2.1}$$

b) $\Omega(t)$ start at the empty set and eventually cover almost all Ω:

$$\Omega(0) = \bigcap_{t>0} \Omega(t) = \varnothing,\ \nu\left(\Omega \setminus \bigcup_{t>0} \Omega(t)\right) = 0.$$

c) Further, denote by $\omega(t) = \overline{\Omega(t)} \bigcap \overline{(\Omega\backslash\Omega(t))}$ the boundary of $\Omega(t)$ in the relative topology. We require the boundaries to be disjoint and cover almost all Ω:

$$\omega(t_1) \bigcap \omega(t_2) = \varnothing,\ t_1 \ne t_2,\ \nu(\Omega \setminus \bigcup_{t>0} \omega(t)) = 0. \tag{2.2}$$

d) Passing to a different parametrization, if necessary, we can assume that

$$\nu\left(\Omega \setminus \bigcup_{t \le N} \omega(t)\right) > 0 \text{ for any } N < \infty. \tag{2.3}$$

e) Finally, we assume that boundaries are thin in the sense that

$$\nu(\omega(t)) = 0 \text{ for all } t > 0. \tag{2.4}$$

This Assumption has simple implications.

1) (2.2) implies that for ν-almost each $y \in \Omega$ there exists a unique $\tau(y) > 0$ such that $y \in \omega(\tau(y))$, which allows us to define

$$Tf(y) = \int_{\Omega(\tau(y))} f d\nu,\ y \in \Omega, \tag{2.5}$$

for any non-negative $\mathfrak{M}$-measurable f. On the set $\Omega_0 \subset \Omega$ of those y for which $\tau(y)$ is not defined we can put $\tau(\Omega_0) = \varnothing$. (A more general definition of a Hardy-type operator is given in [7]. That definition is more difficult to work with what we call slices.)

2) (2.3) and the fact that $\omega(t) \ne \varnothing,\ t > 0$, lead to the equality $\tau(\Omega) = (0, \infty)$.

3) Because of (2.4) $\int_{\Omega(t)} f d\nu = \int_{\overline{\Omega(t)}} f d\nu$ and up to a set of ν-measure zero

$$\{x \in \Omega : \tau(x) > \tau(y)\} = \Omega \backslash \Omega(\tau(y)). \tag{2.6}$$

For $0 \le a < b \le \infty$ we denote $\Omega([a,b]) = \Omega(b) \setminus \Omega(a)$.

Since $\tau(y_1) = \tau(y_2)$ for any $y_1, y_2 \in \omega(t)$, the value $Tf(y)$ is the same for all $y \in \omega(t)$ and we can define $Sf(t) = Tf(y)$ if $y \in \omega(t)$. For a non-negative f, the function Sf is non-decreasing and its jumps are zero due to (2.4). Thus,

$$\text{for each } f \ge 0,\ Sf \text{ is continuous where it is finite, including } t = 0. \tag{2.7}$$

Let $L^p_{vd\nu}(\Omega)$ denote the space with the norm $\|f\|_{L^p_{vd\nu}(\Omega)} = \left(\int_\Omega |f|^p v d\nu\right)^{1/p}$ where v is a weight function and let $\|T\| = \|T\|_{L^p_{vd\nu}(\Omega) \to L^q_{ud\mu}(\Omega)}$ be the norm of a linear operator T acting from $L^p_{vd\nu}(\Omega)$ to $L^q_{ud\mu}(\Omega)$, hence

$$\left[\int_\Omega \left|\int_{\Omega(\tau(x))} f d\nu\right|^q u(x) d\mu(x)\right]^{1/q} \le \|T\| \left(\int_\Omega |f|^p \, v d\nu\right)^{1/p}.$$

Denote

$$\Psi(t) = \left(\int_{\Omega \backslash \Omega(t)} u d\mu\right)^{1/q} \left(\int_{\Omega(t)} v^{-p'/p} d\nu\right)^{1/p'}.$$

Theorem 2.1. *If $1 < p \le q < \infty$, then (2.5) is bounded if and only if $A < \infty$, where $A = \sup_{t>0} \Psi(t)$. Moreover, $A \le \|T\| \le 4A$.*

Proof. Lower bound. Let an operator $T : L^p_{vd\nu}(\Omega) \to L^q_{ud\mu}(\Omega)$ be bounded, then there exists a constant $C > 0$ such that $\|Tf\|_{L^q_{ud\mu}(\Omega)} \le C\|f\|_{L^p_{vd\nu}(\Omega)}$.

Put $f_y(z) = v^{-p'/p}(z)\chi_{\Omega(\tau(y))}(z),\ y \in \Omega$. Then

$$Tf_y(x) = \int_{\Omega(\tau(x)) \cap \Omega(\tau(y))} v^{-p'/p} d\nu = \int_{\Omega(\tau(y))} v^{-p'/p} d\nu, \quad \text{for} \quad \tau(x) > \tau(y)$$

and

$$C\|f_y\|_{L^p_{vd\nu}(\Omega)} \ge \|Tf_y\|_{L^q_{ud\mu}(\Omega)}.$$

Therefore, by applying (2.6) and $\tau(\Omega) = (0,\infty)$ we see that

$$\begin{aligned} C &\ge \sup_{y \in \Omega} \frac{\left(\int_\Omega (Tf_y)^q u d\mu\right)^{1/q}}{\left(\int_\Omega f_y^p v d\nu\right)^{1/p}} \ge \sup_{y \in \Omega} \frac{\left(\int_{\{x:\tau(x)>\tau(y)\}} u d\mu\right)^{1/q} \int_{\Omega(\tau(y))} v^{-p'/p} d\nu}{\left(\int_{\Omega(\tau(y))} v^{-p'/p} d\nu\right)^{1/p}} \\ &= \sup_{y \in \Omega} \left(\int_{\Omega \backslash \Omega(\tau(y))} u d\mu\right)^{1/q} \left(\int_{\Omega(\tau(y))} v^{-p'/p} d\nu\right)^{1/p'} = A \end{aligned}$$

and $\|T\|_{L^p_{vd\nu}(\Omega) \to L^q_{ud\mu}(\Omega)} = \inf C \ge A$.

Upper bound. Without loss of generality we suppose that $0 < \sup_{y \in \Omega} Tf(y) < \infty$. Put $t_0 = \infty$, $\Omega(\infty) = \Omega$. By (2.7) $Sf(t) \to 0$, $t \to 0$, so the definition $t_1 = \sup\{t > 0 : 2Sf(t) \le Sf(t_0)\}$ is correct. By the continuity of Sf, we have $2Sf(t_1) = Sf(t_0)$ and $t_1 < t_0$. By induction, if t_k has been defined, we put $t_{k+1} = \sup\{t > 0 : 2Sf(t) \le Sf(t_k)\}$. Then

$$2Sf(t_{k+1}) = Sf(t_k),\ t_{k+1} < t_k. \tag{2.8}$$

This can be called a sliding property because it allows us to pass from $Sf(t_k)$ to $Sf(t_{k+j})$. Defining slices

$$s_{k+1} = \Omega(t_k) \backslash \Omega(t_{k+1}),\ k \geq 0,$$

we have

$$\begin{aligned} Sf(t_{k+1}) &= 2Sf(t_{k+1}) - Sf(t_{k+1}) \\ &= Sf(t_k) - Sf(t_{k+1}) = \int_{s_{k+1}} f d\nu,\ k \geq 0. \end{aligned} \tag{2.9}$$

Let $y \in s_{k+1}$ or, equivalently, $t_{k+1} \leq \tau(y) < t_k$. Using (2.8), (2.9) and Hölder's inequality we have

$$\begin{aligned} Tf(y) &= \int_{\Omega(\tau(y))} f d\nu \leq Sf(t_k) = 2Sf(t_{k+1}) = 4Sf(t_{k+2}) \\ &= 4\int_{s_{k+2}} f d\nu \leq 4\left(\int_{s_{k+2}} f^p v d\nu\right)^{1/p} \left(\int_{s_{k+2}} v^{-p'/p} d\nu\right)^{1/p'}. \end{aligned} \tag{2.10}$$

Denote

$$\alpha_k = \left(\int_{s_{k+1}} u d\mu\right)^{1/q} \left(\int_{s_{k+2}} v^{-p'/p} d\nu\right)^{1/p'}.$$

We can use (2.10) and the inequality $p \leq q$ to estimate

$$\begin{aligned} \left(\int_\Omega (Tf)^q u d\mu\right)^{1/q} &= \left(\sum_{k\geq 0} \int_{s_{k+1}} (Tf)^q u d\mu\right)^{1/q} \\ &\leq 4\left[\sum_{k\geq 0} \int_{s_{k+1}} u d\mu \left(\int_{s_{k+2}} f^p v d\nu\right)^{q/p} \left(\int_{s_{k+2}} v^{-p'/p} d\nu\right)^{q/p'}\right]^{1/q} \\ &= 4\left[\sum_{k\geq 0} \alpha_k^q \left(\int_{s_{k+2}} f^p v d\nu\right)^{q/p}\right]^{1/q} \leq 4 \sup_k \alpha_k \left(\sum_{k\geq 0} \int_{s_{k+2}} f^p v d\nu\right)^{1/p} \\ &\leq 4A \|f\|_{L^p_{vd\nu}(\Omega)}. \end{aligned} \tag{2.11}$$

The last transition uses the following inclusions

$$s_{k+1} = \Omega(t_k) \backslash \Omega(t_{k+1}) \subset \Omega \backslash \Omega(t_{k+1}) \quad \text{and} \quad s_{k+2} = \Omega(t_{k+1}) \backslash \Omega(t_{k+2}) \subset \Omega(t_{k+1}).$$

If $\sup\limits_{y\in\Omega} Tf(y) = \infty$ we can choose $t < \infty$ such that $\int_{\Omega(t)} f d\nu < \infty$, put $f_t(x) = \chi_{\Omega(t)}(x) f(x)$, and do all calculations leading to (2.11) with f_t in place of f. Since the constant in (2.11) does not depend on t, then we can let $t \to \infty$ thus completing the proof. □

Let u_0, v_0 be non-negative integrable functions such that $u_0 \leq u$, $v_0 \leq v^{1-p'}$. We can assume that $0 < \int_\Omega v_0 d\nu < \infty$ and by analogy with (2.8) define the points $t_0 = \infty > t_1 > ...$, where $t_1 = \sup\{t > 0 : 2Sv_0(t) \leq Sv_0(t_0)\}, ..., t_{k+1} = \sup\{t > 0 : 2Sv_0(t) \leq Sv_0(t_k)\}$ such that $\Omega(\infty) = \Omega$ and

$$\int_{\Omega(t_k)} v_0 d\nu = 2^{-k} \int_\Omega v_0 d\nu,\ k \geq 0. \tag{2.12}$$

This implies the following equality (as before, $s_{k+1} = \Omega(t_k)\backslash\Omega(t_{k+1})$):

$$\int_{s_{k+1}} v_0 d\nu = 2\int_{s_{k+2}} v_0 d\nu.$$

The partition $\{t_k\}$ generates non-negative numbers

$$\begin{aligned} V_k &= \int_{s_{k+1}} v_0 d\nu,\ U_k = \int_{s_{k+1}} u_0 d\mu, \\ x_k &= \int_{\Omega(t_k)} v_0 d\nu = \sum_{j\ge k} V_j, \quad y_k = \int_{\Omega\backslash\Omega(t_{k+1})} u_0 d\mu = \sum_{j\le k} U_j,\ \ k\ge 0. \end{aligned}$$

Here $\{x_k\}$ is non-increasing and $\{y_k\}$ is non-decreasing. We need the identities

$$\frac{r}{q} - 1 = \frac{r}{p}, \quad \frac{r}{p'q} - 1 = \frac{r}{pq'}, \quad \frac{r}{p'} - 1 = \frac{r}{q'}. \tag{2.13}$$

The next lemma provides a replacement for the one-dimensional techniques mentioned in the Introduction.

Lemma 2.1. *Let $a \ge 1$.*
a) We have

$$\sum_{j\ge k}\left(\sum_{i\ge j+1} V_i\right)^{a-1} V_{j+1} \ge \frac{1}{a}\left(\sum_{i\ge k+1} V_i\right)^{a}$$

for any non-negative numbers V_i such that the left side is finite.
b) Moreover,

$$\left(\sum_{i\ge k} V_i\right)^{a-1} V_{k-1} \le \frac{1}{a}\left[\left(\sum_{i\ge k-1} V_i\right)^{a} - \left(\sum_{i\ge k} V_i\right)^{a}\right],\ \ k\ge 1.$$

For this inequality to be true for $k=0$ we formally put $V_{-1}=0$ so that

$$x_{-1} = \sum_{i\ge -1} V_i = \sum_{i\ge 0} V_i = x_0$$

and the inequality holds trivially.
c) For the partition $\{t_k\}$ one has

$$y_{k+1}^{r/q} - y_k^{r/q} \le \frac{r}{q}\left(\int_{\Omega\backslash\Omega(t_{k+2})} u_0 d\mu\right)^{r/p}\int_{s_{k+2}} u_0 d\mu.$$

Proof. Let $g(x) = x^a$.
a) By the mean value theorem with some $\theta \in (x_{j+2}, x_{j+1})$

$$\begin{aligned} \left(\sum_{i\ge j+1} V_i\right)^{a-1} V_{j+1} &= \frac{1}{a} g'(x_{j+1})(x_{j+1} - x_{j+2}) \\ &\ge \frac{1}{a} g'(\theta)(x_{j+1} - x_{j+2}) = \frac{1}{a}\left(g(x_{j+1}) - g(x_{j+2})\right). \end{aligned}$$

It follows that

$$\begin{aligned}\sum_{j\ge k}\left(\sum_{i\ge j+1}V_i\right)^{a-1}V_{j+1} &\ge \frac{1}{a}\sum_{j\ge k}(g(x_{j+1})-g(x_{j+2}))\\ &= \frac{1}{a}g(x_{k+1})=\frac{1}{a}\left(\sum_{i\ge k+1}V_i\right)^{a}.\end{aligned}$$

b) Similarly, with some $\theta\in(x_k,x_{k-1})$

$$\begin{aligned}\left(\sum_{i\ge k}V_i\right)^{a-1}V_{k-1} &= \frac{1}{a}g'(x_k)(x_{k-1}-x_k)\\ &\le \frac{1}{a}g'(\theta)(x_{k-1}-x_k)=\frac{1}{a}(g(x_{k-1})-g(x_k)).\end{aligned}$$

c) With $a=r/q$ and $\theta\in(y_k,y_{k+1})$ by the first identity (2.13)

$$\begin{aligned}&\left(\int_{\Omega\setminus\Omega(t_{k+2})}u_0d\mu\right)^{r/q}-\left(\int_{\Omega\setminus\Omega(t_{k+1})}u_0d\mu\right)^{r/q}\\ =\;& g(y_{k+1})-g(y_k)=g'(\theta)(y_{k+1}-y_k)\le g'(y_{k+1})\int_{s_{k+2}}u_0d\mu\\ =\;& \frac{r}{q}\left(\int_{\Omega\setminus\Omega(t_{k+2})}u_0d\mu\right)^{r/p}\int_{s_{k+2}}u_0d\mu.\end{aligned}$$

□

Let $0<q<p$, $1<p<\infty$ and put $1/r=1/q-1/p$,

$$\Phi(y)=\left(\int_{\Omega\setminus\Omega(\tau(y))}ud\mu\right)^{1/p}\left(\int_{\Omega(\tau(y))}v^{-p'/p}d\nu\right)^{1/p'}.$$

For $\Omega=(0,\infty)$ [20], [33] have shown that $c_1\|\Phi\|_{L^r_{ud\mu}(\Omega)}\le\|T\|\le c_2\|\Phi\|_{L^r_{ud\mu}(\Omega)}$ with constants c_1,c_2 that depend on p,q and do not depend on the weights and measures.

Theorem 2.2. *If $1<p<\infty$, $0<q<p$ and $\frac{1}{q}-\frac{1}{p}=\frac{1}{r}$, then (2.5) is bounded if and only if $B<\infty$, where $B=\left(\int_\Omega\Phi^r ud\mu\right)^{1/r}$. Moreover,*

$$\frac{q\,(p'/r)^{1/p'}\,2^{1-2r/p'q}}{\left(\left(1+\frac{r}{q}\right)2^{r+r/p'}\right)^{1/p}}B\le\|T\|\le 2^{2+1/q}B.$$

Proof. Upper bound. As in the proof of Theorem 2.1, it suffices to consider the case $0<\sup_{y\in\Omega}Tf(y)<\infty$. Begin with applying Hölder's inequality with exponents p/q and r/q in (2.11):

$$\|Tf\|_{L^q_{ud\mu}(\Omega)}\le 4\left[\sum_{k\ge0}\alpha_k^q\left(\int_{s_{k+2}}f^pvd\nu\right)^{q/p}\right]^{1/p}\le 4\left(\sum_{k\ge0}\alpha_k^r\right)^{1/r}\left(\sum_{k\ge0}\int_{s_{k+2}}f^pvd\nu\right)^{1/p}.$$

We want to bound

$$\begin{aligned}\alpha_k^r &= \left(\int_{s_{k+1}} u d\mu\right)^{r/q}\left(\int_{s_{k+2}} v^{-p'/p} d\nu\right)^{r/p'}\\ &= \int_{s_{k+1}} u d\mu \left(\int_{s_{k+1}} u d\mu\right)^{r/p}\left(\int_{s_{k+2}} v^{-p'/p} d\nu\right)^{r/p'}\end{aligned}$$

by an integral. Select $t'_k \in (t_{k+1}, t_k)$ so that

$$\int_{s'_{k+1}} u d\mu = \int_{s''_{k+1}} u d\mu = \frac{1}{2}\int_{s_{k+1}} u d\mu, \text{ where } s'_{k+1} = \Omega(t_k)\backslash\Omega(t'_k),\ s''_{k+1} = \Omega(t'_k)\backslash\Omega(t_{k+1}).$$

First, we replace integrals and explicitly write out the domains of integration:

$$\begin{aligned}\alpha_k^r &\equiv \int_{s_{k+1}} u d\mu \left(\int_{s_{k+1}} u d\mu\right)^{r/p}\left(\int_{s_{k+2}} v^{-p'/p} d\nu\right)^{r/p'}\\ &= 2^{r/q}\int_{s''_{k+1}} u d\mu \left(\int_{s'_{k+1}} u d\mu\right)^{r/p}\left(\int_{s_{k+2}} v^{-p'/p} d\nu\right)^{r/p'}\\ &= 2^{r/q}\int_{t_{k+1}\le\tau(y)<t'_k} u(y)d\mu(y)\left(\int_{t'_k\le\tau(z)<t_k} u(z)d\mu(z)\right)^{r/p}\\ &\quad\times\left(\int_{t_{k+2}\le\tau(z)<t_{k+1}} v^{-p'/p}(z)d\nu(z)\right)^{r/p'}.\end{aligned}$$

Next, we increase the domains of integration in the last two integrals:

$$\begin{aligned}\alpha_k^r &\le 2^{r/q}\int_{t_{k+1}\le\tau(y)<t'_k} u(y)d\mu(y)\left(\int_{\tau(y)\le\tau(z)<\infty} u(z)d\mu(z)\right)^{r/p}\\ &\quad\times\left(\int_{\tau(z)<\tau(y)} v^{-p'/p}(z)d\nu(z)\right)^{r/p'}.\end{aligned}$$

Finally, we increase the domain of integration in the outer integral:

$$\begin{aligned}\alpha_k^r &\le 2^{r/q}\int_{s_{k+1}} u(y)d\mu(y)\left(\int_{\Omega\backslash\Omega(\tau(y))} u d\mu)\right)^{r/p}\left(\int_{\Omega(\tau(y))} v^{-p'/p} d\nu\right)^{r/p'}\\ &= 2^{r/q}\int_{s_{k+1}} \Phi^r u d\mu.\end{aligned}$$

Thus,

$$\|Tf\|_{L^q_{ud\mu}(\Omega)} \le 2^{2+1/q}\|f\|_{L^p_{vd\nu}(\Omega)}\left(\sum_{k\ge 0}\int_{s_{k+1}} \Phi^r u d\mu\right)^{1/r} = 2^{2+1/q}\|f\|_{L^p_{vd\nu}(\Omega)}\|\Phi\|_{L^r_{ud\mu}(\Omega)}.$$

Lower bound. Inspired by [33] we define

$$f(t) = \left(\int_{\Omega\backslash\Omega(\tau(t))} u_0 d\mu\right)^{r/(pq)}\left(\int_{\Omega(\tau(t))} v_0 d\nu\right)^{r/(p'q)-1} v_0(t).$$

Then

$$\int_{\Omega(\tau(x))} f d\nu \geq \left(\int_{\Omega\setminus\Omega(\tau(x))} u_0 d\mu\right)^{r/(pq)} \left[\int_{\Omega(\tau(x))} \left(\int_{\Omega(\tau(t))} v_0 d\nu\right)^{r/(p'q)-1} v_0(t) d\nu(t)\right]. \tag{2.14}$$

Let $k = \min\{j : t_j \leq \tau(x)\}$, for which $t_k \leq \tau(x) < t_{k-1}$, and consider the integral in the square brackets:

$$\begin{aligned} I &\equiv \int_{\Omega(\tau(x))} \left(\int_{\Omega(\tau(t))} v_0 d\nu\right)^{r/(p'q)-1} v_0(t) d\nu(t) \\ &\geq \int_{\Omega(t_k)} \left(\int_{\Omega(\tau(t))} v_0 d\nu\right)^{r/(p'q)-1} v_0(t) d\nu(t) \\ &= \sum_{j\geq k} \int_{s_{j+1}} \left(\int_{\Omega(\tau(t))} v_0 d\nu\right)^{r/(p'q)-1} v_0(t) d\nu(t) \\ &\geq \sum_{j\geq k} \int_{s_{j+1}} \left(\int_{\Omega(t_{j+1})} v_0 d\nu\right)^{r/(p'q)-1} v_0(t) d\nu(t) = \sum_{j\geq k} \left(\sum_{i\geq j+1} V_i\right)^{r/(p'q)-1} V_j. \end{aligned}$$

By the sliding property and Lemma 2.1 a) with $a = \frac{r}{p'q}$

$$I \geq 2\sum_{j\geq k} \left(\sum_{i\geq j+1} V_i\right)^{r/(p'q)-1} V_{j+1} \geq \frac{2p'q}{r} \left(\sum_{i\geq k+1} V_i\right)^{r/(p'q)}.$$

(2.14), (2.12) and this bound give

$$\begin{aligned} \int_{\Omega(\tau(x))} f d\nu &\geq \frac{2p'q}{r} \left(\int_{\Omega\setminus\Omega(\tau(x))} u_0 d\mu\right)^{r/(pq)} \left(\int_{\Omega(t_{k+1})} v_0 d\nu\right)^{r/(p'q)} \\ &= \frac{2p'q}{r} 4^{-r/(p'q)} \left(\int_{\Omega\setminus\Omega(\tau(x))} u_0 d\mu\right)^{r/(pq)} \left(\int_{\Omega(t_{k-1})} v_0 d\nu\right)^{r/(p'q)} \\ &\geq c_1 \left(\int_{\Omega\setminus\Omega(\tau(x))} u_0 d\mu\right)^{r/(pq)} \left(\int_{\Omega(\tau(x))} v_0 d\nu\right)^{r/(p'q)} = c_1 \Phi_0^{r/q}(x), \end{aligned}$$

where $c_1 = \frac{p'q}{r} 2^{1-2r/(p'q)}$ and we have denoted

$$\Phi_0(x) = \left(\int_{\Omega\setminus\Omega(\tau(x))} u_0 d\mu\right)^{1/p} \left(\int_{\Omega(\tau(x))} v_0 d\nu\right)^{1/p'}.$$

Assuming that $\|T\| < \infty$ we have

$$\begin{aligned} c_1^q \int_\Omega \Phi_0^r u_0 d\mu &\leq \int_\Omega \left(\int_{\Omega(\tau(x))} f d\nu\right)^q u(x) d\mu(x) \leq \|T\|^q \left(\int_\Omega f^p v d\nu\right)^{q/p} \\ &\leq \|T\|^q \left[\int_\Omega \left(\int_{\Omega\setminus\Omega(\tau(x))} u_0 d\mu\right)^{r/q} \left(\int_{\Omega(\tau(x))} v_0 d\nu\right)^{r/q'} v_0(x) d\nu(x)\right]^{q/p}. \end{aligned} \tag{2.15}$$

We have applied the inequality $v_0^p v \le v_0^{p+1/(1-p')} = v_0^{p-p/p'} = v_0$. Further, we need to bound

$$\begin{aligned} I &\equiv \int_\Omega \left(\int_{\Omega\setminus\Omega(\tau(x))} u_0 d\mu\right)^{r/q} \left(\int_{\Omega(\tau(x))} v_0 d\nu\right)^{r/q'} v_0(x) d\nu(x) \\ &= \sum_{k\ge 0} \int_{s_{k+1}} \left(\int_{\Omega\setminus\Omega(\tau(x))} u_0 d\mu\right)^{r/q} \left(\int_{\Omega(\tau(x))} v_0 d\nu\right)^{r/q'} v_0(x) d\nu(x) \\ &\le \sum_{k\ge 0} \left(\int_{\Omega\setminus\Omega(t_{k+1})} u_0 d\mu\right)^{r/q} \left(\int_{\Omega(t_k)} v_0 d\nu\right)^{r/q'} \int_{s_{k+1}} v_0 d\nu. \end{aligned} \tag{2.16}$$

Using the third identity in (2.13) and Lemma 2.1 b) with $a = r/p'$ we can write

$$\begin{aligned} \left(\int_{\Omega(t_k)} v_0 d\nu\right)^{r/q'} \int_{s_{k+1}} v_0 d\nu &= \left(\sum_{j\ge k} V_j\right)^{r/p'-1} V_k = \frac{1}{2}\left(\sum_{j\ge k} V_j\right)^{r/p'-1} V_{k-1} \\ &\le \frac{p'}{2r}\left[\left(\sum_{i\ge k-1} V_i\right)^{r/p'} - \left(\sum_{i\ge k} V_i\right)^{r/p'}\right] \\ &= \frac{p'}{2r}\left(x_{k-1}^{r/p'} - x_k^{r/p'}\right). \end{aligned} \tag{2.17}$$

Next combine (2.16) and (2.17), denoting $c_2 = \dfrac{p'}{2r}$ and keeping in mind that $x_{-1} = x_0$:

$$\begin{aligned} I/c_2 &\le \sum_{k\ge 0} y_k^{r/q}\left(x_{k-1}^{r/p'} - x_k^{r/p'}\right) = \sum_{k\ge 1} y_k^{r/q}\left(x_{k-1}^{r/p'} - x_k^{r/p'}\right) + y_0^{r/q}\left({x_0}^{r/p'} - {x_0}^{r/p'}\right) \\ &= y_0^{r/q}{x_0}^{r/p'} + \sum_{k\ge 0}\left(y_{k+1}^{r/q} - y_k^{r/q}\right) x_k^{r/p'}. \end{aligned}$$

By Lemma 2.1 c)

$$I/c_2 \le y_0^{r/q}{x_0}^{r/p'} + \frac{r}{q}\sum_{k\ge 0}\left(\int_{\Omega\setminus\Omega(t_{k+2})} u_0 d\mu\right)^{r/p}\left(\int_{\Omega(t_k)} v_0 d\nu\right)^{r/p'} \int_{s_{k+2}} u_0 d\mu. \tag{2.18}$$

Let $t'_{k+1} \in (t_{k+2}, t_{k+1})$ satisfy the following equality

$$\int_{\Omega(t'_{k+1})\setminus\Omega(t_{k+2})} u_0 d\mu = \int_{\Omega(t_{k+1})\setminus\Omega(t'_{k+1})} u_0 d\mu.$$

Then

$$\begin{aligned} &\left(\int_{\Omega\setminus\Omega(t_{k+2})} u_0 d\mu\right)^{r/p} \int_{s_{k+2}} u_0 d\mu \\ &= 2\left(2\int_{\Omega(t_{k+1})\setminus\Omega(t'_{k+1})} u_0 d\mu + \int_{\Omega\setminus\Omega(t_{k+1})} u_0 d\mu\right)^{r/p} \int_{\Omega(t'_{k+1})\setminus\Omega(t_{k+2})} u_0 d\mu \\ &\le 2^{1+r/p}\left(\int_{\Omega\setminus\Omega(t'_{k+1})} u_0 d\mu\right)^{r/p} \int_{\Omega(t'_{k+1})\setminus\Omega(t_{k+2})} u_0 d\mu \\ &\le 2^{1+r/p}\int_{s_{k+2}}\left(\int_{\Omega\setminus\Omega(\tau(t))} u_0 d\mu\right)^{r/p} u_0(t) d\mu(t). \end{aligned}$$

Besides,

$$\int_{\Omega(t_k)} v_0 d\nu = 2^{-k} \int_\Omega v_0 d\nu = 4 \cdot 2^{-(k+2)} \int_\Omega v_0 d\nu = 4 \int_{\Omega(t_{k+2})} v_0 d\nu.$$

Hence, the big sum in (2.18) is bounded since

$$\begin{aligned}
&\sum_{k\geq 0} \left(\int_{\Omega\setminus\Omega(t_{k+2})} u_0 d\mu\right)^{r/p} \left(\int_{\Omega(t_k)} v_0 d\nu\right)^{r/p'} \int_{s_{k+2}} u_0 d\mu \\
\leq\ & 2^{1+r+r/p'} \sum_{k\geq 0} \left(\int_{\Omega(t_{k+2})} v_0 d\nu\right)^{r/p'} \int_{s_{k+2}} \left(\int_{\Omega\setminus\Omega(\tau(t))} u_0 d\mu\right)^{r/p} u_0(t) d\mu(t) \\
\leq\ & 2^{1+r+r/p'} \sum_{k\geq 0} \int_{s_{k+2}} \left(\int_{\Omega\setminus\Omega(\tau(t))} u_0 d\mu\right)^{r/p} \left(\int_{\Omega(\tau(t))} v_0 d\nu\right)^{r/p'} u_0(t) d\mu(t) \\
\leq\ & 2^{1+r+r/p'} \int_\Omega \Phi_0^r u_0 d\mu. \qquad (2.19)
\end{aligned}$$

The first term in (2.18) has to be treated separatel. Note that

$$y_0^{r/q} {x_0}^{r/p'} = \left(\int_{\Omega\setminus\Omega(t_1)} u_0 d\mu\right)^{r/p} \left(\int_\Omega v_0 d\nu\right)^{r/p'} \int_{\Omega\setminus\Omega(t_1)} u_0 d\mu.$$

Let $t' \in [t_1, \infty)$ satisfy $\int_{\Omega(t')\setminus\Omega(t_1)} u_0 d\mu = \int_{\Omega\setminus\Omega(t')} u_0 d\mu$. Then,

$$\begin{aligned}
y_0^{r/q} {x_0}^{r/p'} &= 2^{r/p+r/p'+1} \left(\int_{\Omega\setminus\Omega(t')} u_0 d\mu\right)^{r/p} \left(\int_{\Omega(t_1)} v_0 d\nu\right)^{r/p'} \int_{\Omega(t')\setminus\Omega(t_1)} u_0 d\mu \\
&\leq 2^{1+r} \int_{\Omega(t')\setminus\Omega(t_1)} \left(\int_{\Omega\setminus\Omega(\tau(t))} u_0 d\mu\right)^{r/p} \left(\int_{\Omega(\tau(t))} v_0 d\nu\right)^{r/p'} u_0(t) d\mu(t) \\
&\leq 2^{1+r} \int_\Omega \Phi_0^r u_0 d\mu. \qquad (2.20)
\end{aligned}$$

Equations (2.18), (2.19) and (2.20) are summarized to give

$$I \leq \frac{p'}{r}\left(1+\frac{r}{q}\right) 2^{r+r/p'} \int_\Omega \Phi_0^r u_0 d\mu.$$

Recalling also (2.15) we get

$$c_1 \left(\int_\Omega \Phi_0^r u_0 d\mu\right)^{1/q} \leq \|T\|\, I^{1/p} \leq \|T\| \left(\frac{p'}{r}\left(1+\frac{r}{q}\right) 2^{r+r/p'}\right)^{1/p} \left(\int_\Omega \Phi_0^r u_0 d\mu\right)^{1/p}.$$

It remains to divide both sides by $\left(\int_\Omega \Phi_0^r u_0 d\mu\right)^{1/p}$. Finally, choose u_0, v_0 monotonously approaching to $u, v^{1-p'}$, respectively, and apply Fatou's lemma, which holds on a general measure space (no topology is needed) [10]. □

With simple changes in the proofs, analogues of Theorems 2.1 and 2.2 hold for the adjoint operator $T^* : L^p_{vd\nu}(\Omega) \to L^q_{ud\mu}(\Omega)$,

$$T^* f\,(y) = \int_{\Omega\setminus\Omega(y)} f d\nu.$$

Denote

$$\begin{aligned}\Psi^*(t) &= \left(\int_{\Omega(t)} u d\mu\right)^{1/q}\left(\int_{\Omega\backslash\Omega(t)} v^{-p'/p} d\nu\right)^{1/p'}, \ t>0,\\ \Phi^*(y) &= \left(\int_{\Omega(\tau(y))} u d\mu\right)^{1/p}\left(\int_{\Omega\backslash\Omega(\tau(y))} v^{-p'/p} d\nu\right)^{1/p'}, \ y\in\Omega,\end{aligned}$$

and consider the inequality

$$\left[\int_\Omega \left|\int_{\Omega\backslash\Omega(\tau(x))} f d\nu\right|^q u(x) d\mu(x)\right]^{1/q} \le \|T^*\| \left(\int_\Omega |f|^p v d\nu\right)^{1/p}.$$

Theorem 2.3. *1) If $1<p\le q<\infty$, then T^* is bounded if and only if $A^*<\infty$ where $A^*=\sup\limits_{t>0}\Psi^*(t)$, and $A^*\le\|T^*\|\le 4A^*$.*

2) If $0<q<p$, $1<p<\infty$, then T^* is bounded if and only if $B^*<\infty$ where $B^*=\left(\int_\Omega (\Phi^*)^r u d\mu\right)^{1/r}$, $1/r=1/q-1/p$. Moreover,

$$\frac{q\,(p'/r)^{1/p'}\,2^{1-2r/p'q}}{\left(\left(1+\frac{r}{q}\right)2^{r+r/p'}\right)^{1/p}} B^* \le \|T^*\| \le 2^{2+1/q} B^*.$$

Remark 1. Instead of being parameterized by $t\in(0,\infty)$ the family $\{\Omega(t)\}$ can be parameterized by $t\in[a,b]$ with $-\infty\le a<b\le\infty$. The resulting bounds will be applied below.

3 Hardy operator compactness

The next subject is the compactness of T. The notation below allows one to trace similarity with [32]. Denote

$$\begin{aligned} a(x) &= \int_{\Omega\backslash\Omega(x)} u d\mu, \ b(x) = \int_{\Omega(x)} v^{-p'/p} d\nu, \ 0<x<\infty,\\ l_i &= \limsup_{x\to i} a(x)^{1/q}\, b(x)^{1/p'}, \ \text{for } i=0,\infty, \ l=\max\{l_0,l_\infty\}.\end{aligned}$$

Lemma 3.1. *Suppose that $a(x)<\infty$, $b(x)<\infty$ on $(0,\infty)$. If $l>\varepsilon>0$, then there exists a sequence $\{g_n\}$ such that $\|g_n\|_{L^p_{vd\nu}(\Omega)}=1$, $\|Tg_n-Tg_m\|_{L^q_{ud\mu}(\Omega)}>\varepsilon$.*

Proof. Let $f_x(z)=b(x)^{-1/p}\chi_{\Omega(x)}(z)\,v^{-p'/p}(z)$, $x\in\Omega$. Then

$$\|f_x\|_{p,vd\nu} = b(x)^{-1/p}\left(\int_{\Omega(x)} v^{1-p'} d\nu\right)^{1/p} = 1 \tag{3.1}$$

and

$$Tf_x(y) = b(x)^{-1/p}\int_{\Omega(\tau(y))} \chi_{\Omega(x)} v^{-p'/p} d\nu = b(x)^{-1/p}\int_{\Omega(\min\{\tau(y),x\})} v^{-p'/p} d\nu. \tag{3.2}$$

Case $i=0$. Suppose that $l_0>\varepsilon>0$. Choose $x_1>0$ for which

$$a(x_1)^{1/q}\, b(x_1)^{1/p'} > \varepsilon. \tag{3.3}$$

Since $b(x) \to 0$ as $x \to 0$, we can select $x_2 < x_1$ so that

$$a(x_2)^{1/q} b(x_2)^{1/p'} > \varepsilon \text{ and } a(x_1)^{1/q} \left(b(x_1)^{1/p'} - b(x_2)^{1/p'} \right) > \varepsilon.$$

Similarly, we can choose $x_1 > x_2 > \ldots$ recursively so that for each n

$$a(x_n)^{1/q} b(x_n)^{1/p'} > \varepsilon \text{ and } a(x_n)^{1/q} \left(b(x_n)^{1/p'} - b(x_{n+1})^{1/p'} \right) > \varepsilon. \tag{3.4}$$

If $m > n$ then $x_n > x_{n+1} \geq x_m$ and

$$b(x_n) > b(x_{n+1}) \geq b(x_m). \tag{3.5}$$

If $\tau(y) \geq x_n$, then by (3.2) and (3.5)

$$Tf_{x_n} - Tf_{x_m} \geq b(x_n)^{1/p'} - b(x_m)^{1/p'} \geq 0.$$

Hence, by (3.4)

$$\begin{aligned} \|Tf_{x_n} - Tf_{x_m}\|_{L^q_{ud\mu}(\Omega)} &\geq \left(b(x_n)^{1/p'} - b(x_m)^{1/p'} \right) \left(\int_{\{y:\tau(y)\geq x_n\}} ud\mu \right)^{1/q} \\ &\geq a(x_n)^{1/q} \left(b(x_n)^{1/p'} - b(x_{n+1})^{1/p'} \right) > \varepsilon. \end{aligned}$$

This and (3.1) show that the functions $g_n = f_{x_n}$ possess the required properties.

Case $i = \infty$. Suppose $l_\infty > \varepsilon$. Choose x_1 satisfying (3.3). Obviously, $a(x) \to 0$ as $x \to \infty$ and, therefore, $b(x) \to \infty$. Using (3.3) we can choose $z_1 > x_1$ such that

$$(a(x_1) - a(z_1))^{1/q} b(x_1)^{1/p'} > \varepsilon. \tag{3.6}$$

The inequality $l_\infty > \varepsilon$ and (3.6) imply that we can select $x_2 > z_1$ with $a(x_2)^{1/q} b(x_2)^{1/p'} > \varepsilon$ and

$$(a(x_1) - a(z_1))^{1/q} \left(b(x_1)^{1/p'} - b(z_1) b(x_2)^{-1/p} \right) > \varepsilon.$$

Continuing in this way, we obtain points $x_1 < z_1 < x_2 < z_2 < \ldots$ such that for each n

$$\begin{gathered} a(x_n)^{1/q} b(x_n)^{1/p'} > \varepsilon, \ (a(x_n) - a(z_n))^{1/q} b(x_n)^{1/p'} > \varepsilon, \\ (a(x_n) - a(z_n))^{1/q} \left(b(x_n)^{1/p'} - b(z_n) b(x_{n+1})^{-1/p} \right) > \varepsilon. \end{gathered} \tag{3.7}$$

If $m > n$, then $x_n < z_n < x_{n+1} \leq x_m$, leading to the inequlities

$$b(x_n) \leq b(z_n) \leq b(x_{n+1}) \leq b(x_m). \tag{3.8}$$

Let $x_n \leq \tau(y) \leq z_n$. Then by (3.2)

$$Tf_{x_n}(y) = b(x_n)^{1/p'}, \ Tf_{x_m}(y) = b(x_m)^{-1/p} b(\tau(y)).$$

Because of (3.7) and (3.8) this implies

$$\begin{aligned} Tf_{x_n}(y) - Tf_{x_m}(y) &= b(x_n)^{1/p'} - b(x_m)^{-1/p} b(\tau(y)) \\ &\geq b(x_n)^{1/p'} - b(x_m)^{-1/p} b(z_n) \\ &\geq b(x_n)^{1/p'} - b(x_{n+1})^{-1/p} b(z_n) > 0. \end{aligned}$$

Now we apply (3.7) to get

$$
\begin{aligned}
\|Tf_{x_n} - Tf_{x_m}\|_{L^q_{ud\mu}(\Omega)} &\geq \left(\int_{\{y: x_n \leq \tau(y) \leq z_n\}} (Tf_{x_n}(y) - Tf_{x_m}(y))^q \, u(y) \, d\mu(y)\right)^{1/q} \\
&\geq (a(x_n) - a(z_n))^{1/q} \left(b(x_n)^{1/p'} - b(z_n) \, b(x_{n+1})^{-1/p}\right) > \varepsilon.
\end{aligned}
$$

Denoting $g_n = f_{x_n}$ we see that this bound and (3.1) complete the proof of the lemma. □

Theorem 3.1. *a) If $1 < p \leq q < \infty$, then T is compact if and only if $l = 0$. b) If $1 < q < p$ and T is bounded, then T is compact.*

Proof. Approximation to T. The points $t_k = k2^{-n}$, $k = 0, ..., n2^n$, $t_{n2^n+1} = \infty$, lead to two partitions: one of $(0, \infty)$, consisting of the intervals

$$
\Delta_k = (t_k, t_{k+1}), \ k = 0, ..., n2^n - 1, \quad \Delta_{n2^n} = (t_{n2^n}, t_{n2^n+1}) = (n, \infty),
$$

and the another one of Ω, consisting of the sets

$$
\Omega_k = \Omega(t_{k+1}) \backslash \Omega(t_k), \ k = 0, ..., n2^n.
$$

Define $\kappa_n(t) = \sum_{k=0}^{n2^n} t_k \chi_{\Delta_k}(t)$, $t > 0$. Since $x \in \Omega_k$ is equivalent to $\tau(x) \in \Delta_k$ we have

$$
\begin{aligned}
\kappa_n(\tau(x)) &= t_k < \tau(x) < t_k + 2^{-n} \quad \text{for } x \in \Omega_k, \quad k = 0, ..., n2^n - 1, \\
\kappa_n(\tau(x)) &= n < \tau(x) \quad \text{for } x \in \Omega_{n2^n}.
\end{aligned}
$$

Put

$$
T_n f(y) = \int_{\Omega(\kappa_n(\tau(y)))} f d\nu = \int_{\Omega\left(\sum_{k=0}^{n2^n} t_k \chi_{\Omega_k}(y)\right)} f d\nu = \sum_{k=0}^{n2^n} \int_{\Omega(t_k)} f d\nu \chi_{\Omega_k}(y).
$$

Obviously T_n is a finite-rank operator.

For the difference $T - T_n$ we have the representation

$$
\begin{aligned}
Tf(y) - T_n f(y) &= \sum_{k=0}^{n2^n} \left(\int_{\Omega(\tau(y))} f d\nu - \int_{\Omega(t_k)} f d\nu\right) \chi_{\Omega_k}(y) \\
&= \sum_{k=0}^{n2^n} \int_{\Omega(\tau(y)) \backslash \Omega(t_k)} f d\nu \chi_{\Omega_k}(y). \qquad (3.9)
\end{aligned}
$$

Case $p \leq q$. *Sufficiency.* By Theorem 2.1

$$
\left[\int_{\Omega_k} \left|\int_{\Omega(\tau(y)) \backslash \Omega(t_k)} f d\nu\right|^q u(y) d\mu(y)\right]^{1/q} \leq c a_k \left(\int_{\Omega_k} |f|^p v d\nu\right)^{1/p} \qquad (3.10)
$$

where

$$
a_k = \sup_{t_k < t < t_{k+1}} \left(\int_{\Omega(t_{k+1}) \backslash \Omega(t)} u d\mu\right)^{1/q} \left(\int_{\Omega(t) \backslash \Omega(t_k)} v^{-p'/p} d\nu\right)^{1/p'}.
$$

Since $p \le q$ we see by (3.9)-(3.10) that

$$\left(\int_\Omega |Tf - T_n f|^q \, ud\mu\right)^{1/q} = \left[\sum_{k=0}^{n2^n} \int_{\Omega_k} |Tf - T_n f|^q \, ud\mu\right]^{1/q}$$
$$\le \quad c\left[\sum_{k=0}^{n2^n} a_k^q \left(\int_{\Omega_k} |f^p| \, vd\nu\right)^{q/p} ud\mu\right]^{1/q} \le c\sup_k a_k \|f\|_{L^p_{vd\nu}(\Omega)}. \tag{3.11}$$

Let us prove the compactness of T assuming that $l = 0$. For any $\varepsilon > 0$ we can choose $x_1 < x_2$ such that

$$a\,(x)^{1/q}\, b\,(x)^{1/p'} < \varepsilon, \quad x \in (0, x_1] \cup [x_2, \infty). \tag{3.12}$$

Since $v^{-p'/p}$ and u are positive almost everywhere, this implies that $a\,(x)$ and $b\,(x)$ are positive in the range in (3.12) and then $a\,(x) \le a\,(x_1) < \infty$, $b\,(x) \le b\,(x_2) < \infty$ for $x \in [x_1, x_2]$. This justifies the calculations that yielded (3.11).

We want to evaluate the sets

$$\widetilde{\Omega}_1 = \bigcup_{t_{k+1} < x_1} \Omega_k, \quad \widetilde{\Omega}_2 = \bigcup_{x_1 \le t_{k+1} \le 2x_2} \Omega_k, \quad \widetilde{\Omega}_3 = \bigcup_{t_{k+1} > 2x_2} \Omega_k.$$

Obviously, $\widetilde{\Omega}_1 \subseteq \Omega\,(x_1)$. Assuming that $2^{-n} \le x_2$ we see that $t_{k+1} > 2x_2$ implies that $t_k > x_2$ and $\widetilde{\Omega}_3 \subseteq \Omega \backslash \Omega\,(x_2)$. Further, provided that $2^{-n} \le x_1/2$ from $x_1 \le t_{k+1} \le 2x_2$ we have $t_k \ge x_1/2$ and $\widetilde{\Omega}_2 \subseteq \Omega\,(2x_2) \backslash \Omega\,(x_1/2)$. We have shown that

$$\widetilde{\Omega}_1 \subseteq \Omega\,(x_1), \quad \widetilde{\Omega}_2 \subseteq \Omega\,(2x_2) \backslash \Omega\,(x_1/2), \quad \widetilde{\Omega}_3 \subseteq \Omega \backslash \Omega\,(x_2). \tag{3.13}$$

Since $a_k \le \sup\limits_{t_k < t < t_{k+1}} \Psi\,(t)$, the inclusions in (3.13) and (3.12) give

$$\sup_{t_{k+1} < x_1} a_k < \varepsilon, \quad \sup_{t_{k+1} > 2x_2} a_k < \varepsilon. \tag{3.14}$$

For Δ_k with $x_1 \le t_{k+1} \le 2x_2$ we have

$$a_k \le \left(\int_{\Omega(t_k + 2^{-n}) \backslash \Omega(t_k)} ud\mu\right)^{1/q} \left(\int_{\Omega(t_k + 2^{-n}) \backslash \Omega(t_k)} v^{-p'/p} d\nu\right)^{1/p'} = \psi\left(t_k, 2^{-n}\right) \tag{3.15}$$

where ψ is defined as

$$\psi\,(x, \delta) \;=\; \left(\int_{\Omega(x+\delta) \backslash \Omega(x)} ud\mu\right)^{1/q} \left(\int_{\Omega(x+\delta) \backslash \Omega(x)} v^{-p'/p} d\nu\right)^{1/p'},$$
$$(x, \delta) \;\in\; \left[\frac{x_1}{2}, 2x_2\right] \times [0, \delta_0]$$

for some $\delta_0 > 0$. This function is continuous on a compact domain and has the property that $\lim\limits_{\delta \to 0} \psi\,(x, \delta) = 0$ for any $x \in [x_1/2, 2x_2]$. Hence, there exists $\delta_1 \in (0, \delta_0]$ such that

$$a_k \le \sup_{(x,\delta) \in [x_1/2, 2x_2] \times (0, \delta_1]} \psi\,(x, \delta) < \varepsilon.$$

If we choose n satisfying the inequalities $2^{-n} \le x_1/2 < x_2$ and $2^{-n} \le \delta_1$ then (3.11), (3.14), (3.15) give the desired bound from above: $\|T - T_n\| \le c\varepsilon$ and T is compact.

Necessity. Suppose that T is compact, which implies that $\|T\| < \infty$ and $A < \infty$ by Theorem 2.1. As above, it follows that a and b are finite on $(0, \infty)$ and we can use Lemma 3.1. Suppose $l > 0$. Taking $\varepsilon = l/2$ in Lemma 3.1 we obtain a sequence $\{g_n\}$ such that $\|g_n\|_{L^p_{vd\nu}(\Omega)} = 1$, $\|Tg_n - Tg_m\|_{L^q_{ud\mu}(\Omega)} > \varepsilon$. This shows that T cannot be compact and that the condition $l = 0$ is necessary.

Case $q < p$. Let $\|T\| < \infty$. By Theorem 2.2, instead of (3.11) we have

$$\left(\int_\Omega |Tf - T_n f|^q \, ud\mu\right)^{1/q} \le c \left(\sum_{k=0}^{n2^n} b_k^r\right)^{1/r} \|f\|_{L^p_{vd\nu}(\Omega)}, \tag{3.16}$$

where

$$b_k = \left[\int_{\Omega_k} \left(\int_{\Omega(t_{k+1})\backslash\Omega(\tau(y))} ud\mu\right)^{r/p} \left(\int_{\Omega(\tau(y))\backslash\Omega(t_k)} v^{-p'/p} d\nu\right)^{r/p'} u(y)\, d\mu(y)\right]^{1/r}.$$

Also $B < \infty$. Therefore, we can select $x_1 < x_2$ so that $\left(\int_{\widetilde{\Omega}} \Phi^r ud\mu\right)^{1/r} < \varepsilon$ for both $\widetilde{\Omega} = \Omega(x_1)$ and $\widetilde{\Omega} = \Omega\backslash\Omega(x_2)$. This implies that

$$\sum_{t_{k+1}<x_1} b_k^r \le \sum_{t_{k+1}<x_1} \int_{\Omega_k} \Phi^r ud\mu \le \int_{\Omega(x_1)} \Phi^r ud\mu < \varepsilon. \tag{3.17}$$

Assuming that $2^{-n} \le x_2$ we can use (3.13) to obtain

$$\sum_{\{k: t_{k+1} > 2x_2\}} b_k^r \le \int_{\Omega\backslash\Omega(x_2)} \Phi^r ud\mu < \varepsilon. \tag{3.18}$$

Again using (3.13) with $2^{-n} \le x_1/2$ we get

$$\begin{aligned}\sum_{\{k: x_1 \le t_{k+1} \le 2x_2\}} b_k^r &\le \sum_{\{k: x_1 \le t_{k+1} \le 2x_2\}} \int_{\Omega_k} ud\mu \; \psi\left(t_k, 2^{-n}\right)^r \\ &\le \int_{\Omega(2x_2)\backslash\Omega(x_1/2)} ud\mu \sup_{x_1/2 \le x \le 2x_2} \psi\left(x, 2^{-n}\right)^r.\end{aligned} \tag{3.19}$$

The function Φ is integrable and the choice of x_1 can be subject to one more condition: $\Phi(x_1/2) < \infty$. As $b(x_1/2) > 0$, by the inequality

$$\int_{\Omega(2x_2)\backslash\Omega(x_1/2)} ud\mu \le \int_{\Omega\backslash\Omega(x_1/2)} ud\mu = \frac{\Phi(x_1/2)^p}{b(x_1/2)^{p/p'}} < \infty$$

we see that the right-hand side in (3.19) tends to zero as $n \to \infty$. Bounds (3.16)-(3.19) imply that T can be approximated arbitrarily well with finite-rank operators and thus is compact. □

4 Bounds for approximation numbers

Our next task is to obtain bounds for the approximation numbers (*a-numbers*) of operator (2.5). Let X, Y be two Banach spaces. For a bounded linear operator $T : X \to Y$ its *n-th a-number*, $n \in N$, is defined by

$$a_n(T) = \inf\{\|T - P\| : P : X \to Y \text{ is a bounded linear operator and } \operatorname{rank} P < n\}.$$

For $[a,b]\subseteq[0,\infty)$ we initially consider the problem of how well the operator $\chi_{[a,b]}T$ is approximated by averages. To this end, successively define

$$\begin{aligned}\mu_u\left(\Omega\left[a,b\right]\right) &= \int_{\Omega[a,b]} ud\mu,\ \bar{T}_{[a,b]}f=\frac{1}{\mu_u\left(\Omega\left[a,b\right]\right)}\int_{\Omega[a,b]}(Tf)ud\mu, \\ T_{[a,b]}f\left(x\right) &= \chi_{\Omega[a,b]}\left(x\right)\left(Tf\left(x\right)-\bar{T}_{[a,b]}f\right).\end{aligned}\tag{4.1}$$

Theorem 4.1. *Choose the point c so that*

$$\mu_u\left(\Omega\left[a,c\right]\right)=\mu_u\left(\Omega\left[c,b\right]\right)=\frac{1}{2}\mu_u\left(\Omega\left[a,b\right]\right).$$

a) Let $1<p\le q<\infty$,

$$\begin{aligned}A^*\left[a,c\right] &= \sup_{a<\tau(x)<c}\left(\int_{\Omega[a,\tau(x)]}ud\mu\right)^{1/q}\left(\int_{\Omega[\tau(x),c]}v^{-p'/p}d\nu\right)^{1/p'}, \\ A\left[c,b\right] &= \sup_{c<\tau(x)<b}\left(\int_{\Omega[\tau(x),b]}ud\mu\right)^{1/q}\left(\int_{\Omega[c,\tau(x)]}v^{-p'/p}d\nu\right)^{1/p'}\end{aligned}$$

and $\mathbf{A}[a,b]=\max\{A^*[a,c],A[c,b]\}$. Then

$$\left(1-2^{-1/q}\right)\mathbf{A}[a,b]\le\left\|T_{[a,b]}\right\|\le 8\mathbf{A}[a,b].$$

b) Let $1<q<p<\infty$, $1/r=1/q-1/p$,

$$\begin{aligned}B^*\left[a,c\right] &= \left[\int_{\Omega[a,c]}\left(\int_{\Omega[a,\tau(x)]}ud\mu\right)^{r/p}\left(\int_{\Omega[\tau(x),c]}v^{-p'/p}d\nu\right)^{r/p'}u\left(x\right)d\mu\left(x\right)\right]^{1/r}, \\ B\left[c,b\right] &= \left[\int_{\Omega[c,b]}\left(\int_{\Omega[\tau(x),b]}ud\mu\right)^{r/p}\left(\int_{\Omega[c,\tau(x)]}v^{-p'/p}d\nu\right)^{r/p'}u\left(x\right)d\mu\left(x\right)\right]^{1/r}\end{aligned}$$

and $\mathbf{B}\left[a,b\right]=\max\left\{B^*\left[a,c\right],B\left[c,b\right]\right\}$. Then

$$\frac{q\left(p'/r\right)^{1/p'}2^{1-2r/p'q}\left(1-2^{-1/q}\right)}{\left(\left(1+\frac{r}{q}\right)2^{r+r/p'}\right)^{1/p}}\mathbf{B}\left[a,b\right]\le\left\|T_{[a,b]}\right\|\le 2^{4+1/q}\mathbf{B}\left[a,b\right].$$

Proof. Define

$$F_{[a,b]}f\left(x\right)=\begin{cases}-\displaystyle\int_{\Omega[\tau(x),c]}fd\nu, & a<\tau\left(x\right)<c,\\ \displaystyle\int_{\Omega[c,\tau(x)]}fd\nu, & c<\tau\left(x\right)<b,\end{cases}$$

$$\bar{F}_{[a,b]}f=\frac{1}{\mu_u\left(\Omega\left[a,b\right]\right)}\int_{\Omega[a,b]}(F_{[a,b]}f)ud\mu.$$

With this notation, we have the following identity

$$Tf\left(x\right)-\bar{T}_{[a,b]}f=F_{[a,b]}f(x)-\bar{F}_{[a,b]}f,\ a<\tau\left(x\right)<b.\tag{4.2}$$

To prove it, we start with adding and subtracting terms in

$$\begin{aligned}
&\int_{\Omega[a,b]} (Tf) u d\mu \\
= \ & \left[\int_{\Omega[a,c]} \left(\int_{\Omega[0,\tau(x)]} f d\nu \right) u(x)\, d\mu(x) + \int_{\Omega[c,b]} \left(\int_{\Omega[0,\tau(x)]} f d\nu \right) u(x)\, d\mu(x) \right] \\
& - \left[\int_{\Omega[a,c]} \left(\int_{\Omega[0,c]} f d\nu \right) u(x)\, d\mu(x) + \int_{\Omega[c,b]} \left(\int_{\Omega[0,c]} f d\nu \right) u(x)\, d\mu(x) \right] \\
& + \int_{\Omega[a,b]} \left(\int_{\Omega[0,c]} f d\nu \right) u(x)\, d\mu(x)
\end{aligned}$$

(joining similar terms in the square brackets and then using the definition of F)

$$\begin{aligned}
= \ & \left[\int_{\Omega[a,c]} \left(- \int_{\Omega[\tau(x),c]} f d\nu \right) u(x)\, d\mu(x) + \int_{\Omega[c,b]} \left(\int_{\Omega[c,\tau(x)]} f d\nu \right) u(x)\, d\mu(x) \right] \\
& + \mu_u (\Omega[a,b]) \int_{\Omega[0,c]} f d\nu \\
= \ & \int_{\Omega[a,b]} (F_{[a,b]} f) u d\mu + \mu_u (\Omega[a,b]) \begin{cases} \displaystyle\int_{\Omega[0,\tau(x)]} f d\nu + \int_{\Omega[\tau(x),c]} f d\nu, & a < \tau(x) < c \\ \displaystyle\int_{\Omega[0,\tau(x)]} f d\nu - \int_{\Omega[c,\tau(x)]} f d\nu, & c < \tau(x) < b \end{cases} \\
= \ & \int_{\Omega[a,b]} (F_{[a,b]} f) u d\mu + \mu_u (\Omega[a,b]) \left[Tf(x) - F_{[a,b]} f(x) \right].
\end{aligned}$$

Rearranging this gives (4.2).

Upper bound. a) Equation (4.2) implies

$$\begin{aligned}
& \left(\int_{\Omega[a,b]} \left| T_{[a,b]} f \right|^q u d\mu \right)^{1/q} \\
\leq \ & \left\| F_{[a,b]} f \right\|_{L^q_{ud\mu}(a,b)} + \left| \bar{F}_{[a,b]} f \right| \left(\int_{\Omega[a,b]} u d\mu \right)^{1/q} \\
\leq \ & \left\| F_{[a,b]} f \right\|_{L^q_{ud\mu}(a,b)} + \frac{1}{\mu_u (\Omega[a,b])} \int_{\Omega[a,b]} \left| F_{[a,b]} f \right| u d\mu \left(\int_{\Omega[a,b]} u d\mu \right)^{1/q} \\
& \text{(applying Hölder's inequality)} \\
\leq \ & 2 \left\| F_{[a,b]} f \right\|_{L^q_{ud\mu}(a,b)}.
\end{aligned}$$

Next we apply the definition of $F_{[a,b]}$ and Theorems 2.1, 2.3:

$$\begin{aligned}
& \left\| T_{[a,b]} f \right\|_{L^q_{ud\mu}(a,b)} \\
\leq \ & 2 \left(\int_{\Omega[a,c]} \left| \int_{\Omega[\tau(x),c]} f d\nu \right|^q u(x)\, d\mu(x) + \int_{\Omega[c,b]} \left| \int_{\Omega[c,\tau(x)]} f d\nu \right|^q u(x)\, d\mu(x) \right)^{1/q} \\
\leq \ & 2 \left[\left((4A^*[a,c])^p \int_{\Omega[a,c]} |f|^p v d\nu \right)^{q/p} + \left((4A[c,b])^p \int_{\Omega[c,b]} |f|^p v d\nu \right)^{q/p} \right]^{1/q} \\
\leq \ & 8\, \mathbf{A}[a,b] \left(\int_{\Omega[a,b]} |f|^p v d\nu \right)^{1/p}.
\end{aligned}$$

This proves the upper bound.

The upper bound in the case b) is proved similarly, using Theorems 2.2, 2.3.

Lower bound. a) For any $f \geq 0$ supported in $[a,c]$ we have

$$\begin{aligned} \left| \int_{\Omega[a,b]} \left(F_{[a,b]} f\right) u d\mu \right| &\leq \left(\int_{\Omega[a,b]} \left|F_{[a,b]} f\right|^q u d\mu \right)^{1/q} \left(\int_{\Omega[a,b]} u d\mu \right)^{1/q'} \\ &= \left(\int_{\Omega[a,c]} \left|F_{[a,b]} f\right|^q u d\mu \right)^{1/q} \left(2\mu_u\left(\Omega\left[a,c\right]\right)\right)^{1/q'}. \end{aligned}$$

Therefore, by (4.2)

$$\begin{aligned} &\left\|T_{[a,b]}\right\| \|f\|_{L^p_{vd\nu}(a,c)} \geq \left\|T_{[a,b]} f\right\|_{L^q_{ud\mu}(a,b)} \geq \left\|Tf - \bar{T}_{[a,b]} f\right\|_{L^q_{ud\mu}(a,c)} \\ = &\left\|F_{[a,b]} f - \bar{F}_{[a,b]} f\right\|_{L^q_{ud\mu}(a,c)} \geq \left\|F_{[a,b]} f\right\|_{L^q_{ud\mu}(a,c)} - \left|\bar{F}_{[a,b]} f\right| \left(\int_{\Omega[a,c]} u d\mu \right)^{1/q} \\ = &\left\|F_{[a,b]} f\right\|_{L^q_{ud\mu}(a,c)} - \frac{1}{\mu_u\left(\Omega\left[a,b\right]\right)} \left| \int_{\Omega[a,b]} (F_{[a,b]} f) u d\mu \right| \left(\int_{\Omega[a,c]} u d\mu \right)^{1/q} \\ \geq &\left\|F_{[a,b]} f\right\|_{L^q_{ud\mu}(a,c)} \left(1 - \frac{1}{\mu_u\left(\Omega\left[a,b\right]\right)} 2^{1/q'} \mu_u\left(\Omega\left[a,c\right]\right)\right) \\ = &\left\|F_{[a,b]} f\right\|_{L^q_{ud\mu}(a,c)} \left(1 - 2^{-1/q}\right). \end{aligned}$$

The conclusion is that

$$\begin{aligned} \left\|T_{[a,b]}\right\| &\geq \frac{\left\|F_{[a,b]} f\right\|_{L^q_{ud\mu}(a,c)}}{\|f\|_{L^p_{vd\nu}(a,c)}} \left(1 - 2^{-1/q}\right) \\ &= \frac{\left[\int_{\Omega[a,c]} \left| \int_{\Omega[\tau(x),c]} f d\nu \right|^q u(x)\, d\mu(x) \right]^{1/q}}{\|f\|_{L^p_{vd\nu}(a,c)}} \left(1 - 2^{-1/q}\right) \end{aligned}$$

for any non-negative f with supp $\subseteq [a,c]$ and by Theorem 2.3

$$\left\|T_{[a,b]}\right\| \geq \left(1 - 2^{-1/q}\right) A^*\left[a,c\right].$$

Selecting f supported in $[c,b]$ similarly yields

$$\left\|T_{[a,b]}\right\| \geq \left(1 - 2^{-1/q}\right) A\left[c,b\right].$$

b) The lower bound in this case is obtained similarly using Theorems 2.2, 2.3. □

Obviously, for any $0 < x < \infty$ we have

$$\mathbf{A}\left[a,b\right] \to 0 \text{ if } a,b \to x; \ \ \mathbf{A}\left[a,b\right] > 0 \text{ if } a < b.$$

Everywhere below we assume that T is a compact operator.

Lemma 4.1. *Let $1 < p \leq q < \infty$ and $0 < \varepsilon < \max \Psi$. There exist points $0 = t_0 < t_1 < ... < t_N < t_{N+1} = \infty$ such that with the notation $\Delta_k = [t_k, t_{k+1})$, $k = 0, ..., N$ one has*

$$\sup_{t \in \Delta_0} \Psi(t) = \varepsilon, \ \max_{k=1,\dots,N-2} \mathbf{A}(\Delta_k) = \varepsilon, \ \mathbf{A}(\Delta_{N-1}) \leq \varepsilon, \ \sup_{t \in \Delta_N} \Psi(t) = \varepsilon. \tag{4.3}$$

Proof. Let $t_0 = 0$. Since Ψ is continuous and $\Psi(t) \to 0$ when $t \to 0$ or $t \to \infty$, we can define $t' = \min\{t > 0 : \Psi(t) \geq \varepsilon\}$ and $t'' = \max\{t > 0 : \Psi(t) \geq \varepsilon\}$. Then

$$\max_{t \leq t'} \Psi(t) = \varepsilon, \ \max_{t \geq t''} \Psi(t) = \varepsilon,$$
$$c' = \int_{\Omega[t',\infty]} u d\mu < \infty, \ c'' = \int_{\Omega(t'')} v^{-p'/p} d\nu < \infty. \tag{4.4}$$

and we put $t_1 = t'$. On the n-th step, if $\sup_{t \in [t_n, t'']} \mathbf{A}[t_n, t] \leq \varepsilon$, we set $t_{n+1} = t''$, $t_{n+2} = \infty$. If, on the other hand, $\sup_{t \in [t_n, t'']} \mathbf{A}[t_n, t] > \varepsilon$, then we put $t_{n+1} = \min\{t > t_n : \mathbf{A}[t_n, t] \geq \varepsilon\}$ so that by continuity $\mathbf{A}[t_n, t_{n+1}] = \varepsilon$. Thus, we have disjoint segments $[t_n, t_{n+1}] \subseteq [t', \infty)$.

We want to show that this process stops in a finite number of steps. Suppose it does not and $t_n \to t \leq t''$. Then $\mathbf{A}[t_n, t_{n+1}] = \varepsilon$ for $n = 1, 2, \ldots$ Obviously,

$$\mathbf{A}^*[t_n, t_{n+1}] \leq (c'')^{1/p'} \left(\int_{\Omega[t_n, \tau(x_n)]} u d\mu \right)^{1/q},$$
$$\mathbf{A}[t_n, t_{n+1}] \leq (c')^{1/p'} \left(\int_{\Omega[\tau(x_n), t_{n+1}]} u d\mu \right)^{1/q}.$$

Hence,

$$\varepsilon^q = \mathbf{A}[t_n, t_{n+1}]^q \leq \max\left\{ (c')^{1/p'}, (c'')^{1/p'} \right\}^q \int_{\Omega[t_n, t_{n+1}]} u d\mu \text{ for all } n.$$

This contradicts the fact that in (4.4) $c' < \infty$. □

With $\Omega_k = \Omega(\Delta_k)$ put for $k = 1, \ldots, N-1$

$$T_k f(x) = \int_{\Omega(\tau(x)) \setminus \Omega(t_k)} f d\nu, \ \mu_u(\Omega_k) = \int_{\Omega_k} u d\mu, \ \bar{T}_k f = \frac{1}{\mu_u(\Omega_k)} \int_{\Omega_k} (Tf) u d\mu,$$

$$P_k f(x) = \chi_{\Omega_k}(x) \left\{ Tf(x) - \left[T_k f(x) - \bar{T}_k f \right] \right\} = \chi_{\Omega_k}(x) \left\{ \int_{\Omega(t_k)} f d\nu + \bar{T}_k f \right\}, \tag{4.5}$$

$$P_0 f = 0, \ P_N f(x) = \chi_{\Omega_N}(x) \int_{\Omega(t_N)} f d\nu.$$

Each of P_k is one-dimensional, so $P = \sum_{k=1}^{N}$ has $\operatorname{rank} P \leq N$.

We use the approach developed in [5].

Theorem 4.2. *Let $1 < p \leq q < \infty$ and suppose the covering $\{\Omega_k : k = 0, \ldots, N\}$ satisfies (4.3). Then*

$$\frac{\left(2^{1/q} - 1\right)}{\left(2^{1/q+1}\right)} \varepsilon (N-2)^{1/q - 1/p} \leq a_{N-1}(T), \ a_{N+1}(T) \leq 8\varepsilon. \tag{4.6}$$

Proof. *Upper bound.* By Theorem 2.1

$$\left(\int_{\Omega_0} |Tf|^q u d\mu \right)^{1/q} \leq 4A[0, t_1] \left(\int_{\Omega_0} |f|^p v d\nu \right)^{1/p} \leq 4 \sup_{t \in \Delta_0} \Psi(t) \left(\int_{\Omega_0} |f|^p v d\nu \right)^{1/p},$$

$$\begin{aligned}\left(\int_{\Omega_N}|Tf-P_Nf|^q\,ud\mu\right)^{1/q} &\leq 4A\left[t_N,t_{N+1}\right]\left(\int_{\Omega_N}|f|^p\,vd\nu\right)^{1/p}\\ &\leq 4\sup_{t\in\Delta_N}\Psi\left(t\right)\left(\int_{\Omega_N}|f|^p\,vd\nu\right)^{1/p}.\end{aligned}$$

By Theorem 4.1

$$\begin{aligned}\left(\int_{\Omega_k}|Tf-P_kf|^q\,ud\mu\right)^{1/q} &= \left(\int_{\Omega_k}\left|T_kf-\bar{T}_kf\right|^q\,ud\mu\right)^{1/q}\\ &\leq 8\mathbf{A}\left(\Omega_k\right)\left(\int_{\Omega_k}|f|^p\,vd\nu\right)^{1/p},\ k=1,...,N-1.\end{aligned}$$

Summing these bounds and remembering (4.3) we get

$$\left(\int_{\Omega}|Tf-Pf|^q\,ud\mu\right)^{1/q}\leq 8\varepsilon\left(\int_{\Omega}|f|^p\,vd\nu\right)^{1/p}$$

which implies the upper bound in (4.6).

Lower bound. By Theorem 4.1 we can choose functions f_k satisfying $\operatorname{supp} f_k\subseteq\Omega_k$ and

$$\left(\int_{\Omega_k}\left|T_kf_k-\bar{T}_kf_k\right|^q\,ud\mu\right)^{1/q}\geq\left(1-2^{-1/q}\right)\mathbf{A}\left(\Omega_k\right)\left(\int_{\Omega_k}|f|^p\,vd\nu\right)^{1/p},\ k=1,...,N-1. \tag{4.7}$$

Let $P:L^p_{vd\nu}\to L^q_{ud\mu}$ be an arbitrary bounded linear operator, $\operatorname{rank}P<N-1$. Then because of the linear independence of Pf_k, $k=1,...,N-1$, there are constants $\alpha_1,...,\alpha_{N-1}$ such that $P\left(\sum_{k=1}^{N-1}\alpha_kf_k\right)=0$. Denote $f=\sum_{k=1}^{N-1}\alpha_kf_k$. For $\tau\left(x\right)\in\Delta_k$, $k=1,...,N-1$, we have

$$Tf\left(x\right)=\int_{\Omega(t_k)}fd\nu+\alpha_k\int_{\Omega(\tau(x))\backslash\Omega(t_k)}f_kd\nu=\beta_k+\alpha_kT_kf_k,$$

where the value of the constant $\beta_k=\int_{\Omega(t_k)}fd\nu$ does not matter, as we will see shortly. We need a well-known property that in L_p spaces the average of a function is a good approximation to it in the sense that

$$\left(\int_{\Omega_k}\left|T_kf_k-\bar{T}_kf_k\right|^q\,ud\mu\right)^{1/q}\leq 2\inf_c\left(\int_{\Omega_k}|T_kf_k-c|^q\,ud\mu\right)^{1/q}. \tag{4.8}$$

Now using (4.8) and Theorem 4.1 we can proceed with the following estimate:

$$\begin{aligned}\int_{\Omega_k}|\beta_k+\alpha_kT_kf_k|^q\,ud\mu &\geq \left(\frac{1}{2}\right)^q\int_{\Omega_k}\left|\alpha_kT_kf_k-\alpha_k\bar{T}_kf_k\right|^q\,ud\mu\\ &\geq \left(\frac{1}{2}\left(1-2^{-1/q}\right)\mathbf{A}\left(\Omega_k\right)\right)^q|\alpha_k|^q\left(\int_{\Omega_k}|f_k|^p\,vd\nu\right)^{q/p}.\end{aligned}$$

Therefore, by (4.3) and discrete Hölder's inequality

$$\begin{aligned}\int_{\Omega}|Tf-Pf|^{q}\,ud\mu &\geq \left(\frac{\left(2^{1/q}-1\right)\varepsilon}{\left(2^{1/q+1}\right)}\right)^{q}\sum_{k=1}^{N-2}|\alpha_k|^{q}\left(\int_{\Omega_k}|f_k|^{p}\,vd\nu\right)^{q/p}\\ &= \left(\frac{\left(2^{1/q}-1\right)\varepsilon}{\left(2^{1/q+1}\right)}\right)^{q}\sum_{k=1}^{N-2}\left(\int_{\Omega_k}|\alpha_k f_k|^{p}\,vd\nu\right)^{q/p}\\ &\geq \left(\frac{\left(2^{1/q}-1\right)\varepsilon}{\left(2^{1/q+1}\right)}\right)^{q}(N-2)^{1-q/p}\left(\int_{\Omega}|f|^{p}\,vd\nu\right)^{q/p}.\end{aligned}$$

In the first line the term with $k=N-1$ was omitted because $\mathbf{A}\left(\Delta_{N-1}\right)$ may be less than ε. The last inequality proves the lower bound. □

Remark 2. Obviously, when $p=q$, (4.6) gives a same-order two-sided bound for a-numbers. Besides, the upper bound on a-numbers gives an upper bound for the Gelfand, Kolmogorov and entropy numbers because the a-numbers are the largest among s-numbers of linear operators [23].

To consider the case $1<q<p<\infty$ we assume that $\|T\|<\infty$ and therefore $B<\infty$ by Theorem 2.2. Denote

$$\begin{aligned}\Phi^{*}_{[a,b]}(x) &= \left(\int_{\Omega[a,\tau(x)]}ud\mu\right)^{1/p}\left(\int_{\Omega[\tau(x),b]}v^{-p'/p}d\nu\right)^{1/p'},\\ \Phi_{[a,b]}(x) &= \left(\int_{\Omega[\tau(x),b]}ud\mu\right)^{1/p}\left(\int_{\Omega[a,\tau(x)]}v^{-p'/p}d\nu\right)^{1/p'}\\ \mathbf{\Phi}[a,b] &= \left[\int_{\Omega[a,b]}\left(\Phi^{*}_{[a,c]}\chi_{[a,c]}+\Phi_{[c,b]}\chi_{[c,b]}\right)^{r}ud\mu\right]^{1/r}\end{aligned}$$

where $c=c(a,b)$ is the constant defined in Theorem 4.1.

Theorem 4.3. *Suppose that $1<q<p<\infty$, $1/r=1/q-1/p$, T is bounded and $0<\varepsilon<B$. Then*

$$a_{N+1}(T)\leq 32^{1/q}\varepsilon.$$

Proof. Upper bound. Let $0<\varepsilon<B$. Select t',t'' to satisfy

$$\left(\int_{\Omega(t')}\Phi^{r}ud\mu\right)^{1/r}=\varepsilon,\ \left(\int_{\Omega[t'',\infty]}\Phi^{r}ud\mu\right)^{1/r}=\varepsilon. \tag{4.9}$$

This implies, in particular, (4.4). Let $\{\Delta_k : k=1,...,N\}$ be a uniform (and finite) partition of $[t',t'']$ into segments Δ_k of length m. From the bound

$$\sum_{k=1}^{N}\mathbf{\Phi}\left(\Delta_k\right)^{r}\leq\max_{k}\sup_{\tau(x)\in\Delta_k}\left(\Phi^{*}_{\Delta_k}(x)+\Phi_{\Delta_k}(x)\right)\int_{\Omega[t',t'']}ud\mu$$

we see that m can be chosen so that

$$\left(\sum_{k=1}^{N}\mathbf{\Phi}\left(\Delta_k\right)^{r}\right)^{1/r}=\varepsilon. \tag{4.10}$$

With definitions (4.5) and putting $\Omega_k = \Omega(\Delta_k)$ we have for each k by Theorem 4.1

$$\left(\int_{\Omega_k} |Tf - P_k f|^q \, ud\mu\right)^{1/q} \le 2^{4+1/q} \mathbf{\Phi}(\Delta_k) \left(\int_{\Omega_k} |f|^p \, vd\nu\right)^{1/p}.$$

By Theorem 2.2 (4.9) implies

$$\begin{aligned}\left(\int_{\Omega(t')} |Tf|^q \, ud\mu\right)^{1/q} &\le 2^{2+1/q}\varepsilon \left(\int_{\Omega(t')} |f|^p \, vd\nu\right)^{1/p}, \\ \left(\int_{\Omega[t'',\infty]} |Tf|^q \, ud\mu\right)^{1/q} &\le 2^{2+1/q}\varepsilon \left(\int_{\Omega[t'',\infty]} |f|^p \, vd\nu\right)^{1/p}.\end{aligned}$$

We use definitions of $P_0, ..., P_N$ from Theorem 4.2. Put $P = \sum_{k=1}^{N} P_k$. The last three estimates and (4.10) give

$$\left(\int_{\Omega} |Tf - Pf|^q \, ud\mu\right)^{1/q} \le 2^{4+1/q}\varepsilon \left(\int_{\Omega} |f|^p \, vd\nu\right)^{1/p}.$$

Since rank$P \le N$ this proves that $a_{N+1}(T) \le 2^{4+1/q}\varepsilon$. □

Lower bound. Let t', t'' be chosen as in (4.9) and put $t_0 = 0$, $t_1 = t'$. On the n-th step, if $\sup_{t>t_n} \mathbf{\Phi}(t_n, t) \ge \varepsilon$ then we put $t_{n+1} = \min\{t > t_n : \mathbf{\Phi}(t_n, t) = \varepsilon\}$. If $\sup_{t>t_n} \mathbf{\Phi}(t_n, t) < \varepsilon$ we put $t_{n+1} = \infty$. This process stops in a finite number of steps. Suppose that it does not and that $t_n \to t \le \infty$. From (4.4) we conclude that

$$\begin{aligned}\max\left\{\Phi^*_{[a,b]}(x), \Phi_{[a,b]}(x)\right\} &\le \left(\int_{\Omega[t',\infty]} ud\mu\right)^{1/p} \left(\int_{\Omega(t'')} v^{-p'/p} d\nu\right)^{1/p'} = c \\ \text{for } t' &\le a < b \le t''.\end{aligned}$$

Hence, for each k, $\varepsilon^r = \mathbf{\Phi}(\Delta_k)^r \le (2c)^r \int_{\Omega(\Delta_k)} ud\mu$, $\sum_k \int_{\Omega(\Delta_k)} ud\mu = \infty$, which contradicts (4.4).

Denoting N the total number of segments, for an arbitrary bounded linear operator $P : L^p_{vd\nu} \to L^q_{ud\mu}$, rank$P < N - 1$, instead of (4.7) we have

$$\left(\int_{\Omega_k} \left|T_k f_k - \bar{T}_k f_k\right|^q ud\mu\right)^{1/q} \ge c\varepsilon \left(\int_{\Omega_k} |f|^p \, vd\nu\right)^{1/p}, \; k = 1, ..., N-1,$$

where c is defined in Theorem 4.1 b). Repeating the argument based on (4.8) we get

$$\begin{aligned}\int_{\Omega} |Tf - Pf|^q \, ud\mu &\ge \left(\frac{q\,(p'/r)^{1/p'}\, 2^{1-2r/p'q}\left(1 - 2^{-1/q}\right)}{2\left(\left(1 + \frac{r}{q}\right) 2^{r+r/p'}\right)^{1/p}}\varepsilon\right)^q \sum_{k=1}^{N-2} \left(\int_{\Omega_k} |\alpha_k f_k|^p \, vd\nu\right)^{q/p} \\ &\ge \left(\frac{q\,(p'/r)^{1/p'}\, 2^{1-2r/p'q}\left(1 - 2^{-1/q}\right)}{2\left(\left(1 + \frac{r}{q}\right) 2^{r+r/p'}\right)^{1/p}}\varepsilon\right)^q \left(\int_{\Omega} |f|^p \, vd\nu\right)^{q/p}\end{aligned}$$

where the last transition is by Jensen's inequality with the exponent $0 < q/p < 1$. Thus, $a_{N-1}(T) \ge c\varepsilon$ with the partition we have defined here and the constant c that depends only on p and q. We have proved the following statement.

Theorem 4.4. *Suppose that* $1 < q < p < \infty$, T *is bounded and* $0 < \varepsilon < B$. *Then*

$$a_{N-1}(T) \geq \frac{q\left(p'/r\right)^{1/p'} 2^{1-2r/p'q}\left(1-2^{-1/q}\right)}{2\left(\left(1+\frac{r}{q}\right)2^{r+r/p'}\right)^{1/p}}\varepsilon.$$

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