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ON BOUNDEDNESS OF THE HARDY OPERATOR IN MORREY-TYPE SPACES

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Abstract. In this paper we study the boundedness of the Hardy operator H_α in local and global Morrey-type spaces $LM_{p\theta,w(\cdot)}$, $GM_{p\theta,w(\cdot)}$ respectively, characterized by numerical parameters p,θ and a functional parameter w. We reduce this problem to the problem of a continuous embedding of one local Morrey-type space to another one. This allows obtaining, for all admissible values of the numerical parameters $\alpha, p_1, p_2, \theta_1, \theta_2$, sufficient conditions on the functional parameters w_1 and w_2 ensuring the boundedness of H_α from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$ and from $GM_{p_1\theta_1,w_1(\cdot)}$ to $GM_{p_2\theta_2,w_2(\cdot)}$. Moreover, for a certain range of the numerical parameters and under certain a priori assumptions on w_1 and w_2 these sufficient conditions coincide with the necessary ones.

1 Introduction

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r)$ denote the open ball centered at x of radius r and $|B(x,r)|$ denote its Lebesgue measure.

We consider, for $-\infty < \alpha < \infty$, the Hardy operator $H_\alpha \equiv H_{n,\alpha}$ defined for $f \in L_1^{loc}(\mathbb{R}^n)$ by

$$(H_\alpha f)(x) = \frac{1}{|B(0,|x|)|^{1-\frac{\alpha}{n}}} \int\limits_{B(0,|x|)} f(y)dy, \qquad x \in \mathbb{R}^n. \tag{1.1}$$

This operator is to a certain extent related to the fractional maximal operator M_α defined for $0 \le \alpha < n$ for $f \in L_1^{loc}(\mathbb{R}^n)$ by

$$(M_\alpha f)(x) = \sup_{r>0} \frac{1}{|B(x,r)|^{1-\frac{\alpha}{n}}} \int\limits_{B(x,r)} |f(y)|dy, \qquad x \in \mathbb{R}^n, \tag{1.2}$$

and to the Riesz potential I_α defined for $0 < \alpha < n$ for $f \in L_1^{loc}(\mathbb{R}^n)$ by

$$I_\alpha f(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)dy}{|x-y|^{n-\alpha}}, \qquad x \in \mathbb{R}^n.$$

One can easily verify that

$$(M_\alpha f)(x) = \sup_{z\in\mathbb{R}^n} (H_\alpha(|f(\cdot + x)|))(z), \qquad x \in \mathbb{R}^n\,, \tag{1.3}$$

and

$$(H_\alpha(|f|))(x) = \frac{2^{n-\alpha}}{|B(x,2|x|)|^{1-\frac{\alpha}{n}}} \int\limits_{B(x,2|x|)} |f(y)|\chi_{B(0,|x|)}(y)\,dy$$

$$\le 2^{n-\alpha}(M_\alpha(f\chi_{B(0,|x|)}))(x) \le 2^{n-\alpha}(M_\alpha(f))(x), \quad x \in \mathbb{R}^n\,. \tag{1.4}$$

However the latter estimate is rather rough. It may easily happen that $(M_\alpha f)(x) = +\infty$ for all $x \in \mathbb{R}^n$ whilst $(H_\alpha(|f|)(x) < +\infty$ for all $x \in \mathbb{R}^n$. (For example, this happens if $f(x) = 0$ for $|x| \le 1$ and $f(x) = |x|^\beta$ for $|x| > 1$ where $\beta > -\alpha$.) The reason for that is that, for a fixed $x \in \mathbb{R}^n$, the definition of $(M_\alpha f)(x)$ takes into account the values of $f(y)$ for all $y \in \mathbb{R}^n$ while the definition of $(H_\alpha f)(x)$ takes into account the values of $f(y)$ only for $y \in B(0,|x|)$.

Recall also that, for $0 < \alpha < n$,

$$M_\alpha f(x) \le v_n^{\frac{\alpha}{n}-1} I_\alpha(|f|)(x)\,, \tag{1.5}$$

where v_n is the volume of the unit ball in $\mathbb{R}^n$.

We shall study the boundedness of the operator H_α in the Morrey spaces $M_{p\theta}^\lambda$, where $0 < p,\theta \le \infty$, $0 \le \lambda \le \frac{n}{p}$, the spaces of all functions $f \in L_p^{loc}(\mathbb{R}^n)$, for which

$$\|f\|_{M_{p\theta}^\lambda} = \sup_{x\in\mathbb{R}^n} \left(\int\limits_0^\infty \left(\frac{\|f\|_{L_p(B(x,r))}}{r^\lambda} \right)^\theta \frac{dr}{r} \right)^{\frac{1}{\theta}} < \infty \tag{1.6}$$

if $\theta < \infty$ and

$$\|f\|_{M_{p\infty}^\lambda} = \sup_{x\in\mathbb{R}^n} \sup_{r>0} \frac{\|f\|_{L_p(B(x,r))}}{r^\lambda} < \infty \tag{1.7}$$

if $\theta = \infty$. For $\theta = \infty$, the spaces $M_{p\infty}^\lambda \equiv M_p^\lambda$ were introduced in [17].

It will always be assumed that $0 < \lambda < \frac{n}{p}$ if $\theta < \infty$ and that $0 \le \lambda \le \frac{n}{p}$ if $\theta = \infty$. Otherwise, the space $M_{p\theta}^\lambda$ is trivial, i. e. consists only of functions equivalent to 0 on $\mathbb{R}^n$. Also,

$$M_p^0 \equiv M_{p\infty}^0 = L_p(\mathbb{R}^n), \qquad M_p^{\frac{n}{p}} \equiv M_{p\infty}^{\frac{n}{p}} = L_\infty(\mathbb{R}^n).$$

Compared with [17], we write r^λ in the definition of $M_{p\theta}^\lambda$ rather than $r^{\frac{\lambda}{p}}$ as in [17] for $\theta = \infty$, because with this notation the parameter λ is closely related to the order of smoothness. (See [12, 3].)

We shall also consider the local variant of the spaces $M_{p\theta}^\lambda$, namely the spaces $LM_{p\theta}^\lambda$, where $0 < p,\theta \le \infty$, $\lambda \ge 0$, of all functions $f \in L_p^{loc}(\mathbb{R}^n)$, for which

$$\|f\|_{LM_{p\theta}^\lambda} = \left(\int\limits_0^\infty \left(\frac{\|f\|_{L_p(B(0,r))}}{r^\lambda} \right)^\theta \frac{dr}{r} \right)^{\frac{1}{\theta}} < \infty \tag{1.8}$$

if $\theta < \infty$ and

$$\|f\|_{LM^{\lambda}_{p\infty}} = \sup_{r>0} \frac{\|f\|_{L_p(B(0,r))}}{r^{\lambda}} < \infty \tag{1.9}$$

if $\theta = \infty$. It will always be assumed that $\lambda > 0$ if $\theta < \infty$ and that $\lambda \geq 0$ if $\theta = \infty$. Otherwise, the space $LM^{\lambda}_{p\theta}$ is trivial. Also $LM^{0}_{p\infty} = L_p(\mathbb{R}^n)$.

We shall also call the spaces $M^{\lambda}_{p\theta}$ global Morrey spaces and denote them by $GM^{\lambda}_{p\theta}$. Note that

$$\|f\|_{M^{\lambda}_{p\theta}} \equiv \|f\|_{GM^{\lambda}_{p\theta}} = \sup_{x\in\mathbb{R}^n} \|f(\cdot + x)\|_{LM^{\lambda}_{p\theta}}\,.$$

Example 1. Let $-\infty < \beta < \infty$ and χ_{Ω} denote the characteristic function of a set $\Omega \subset \mathbb{R}^n$. Then $|x|^{\beta}\chi_{B(0,1)}(x) \in LM^{\lambda}_{p\theta}$ (or $GM^{\lambda}_{p\theta}$) if and only if $\beta > \lambda - \frac{n}{p}$ if $\theta < \infty$; $\beta \geq \lambda - \frac{n}{p}$ if $\theta = \infty$ and $\lambda > 0$; $\beta > -\frac{n}{p}$ if $\theta = \infty$, $\lambda = 0$ and $p < \infty$; and $\beta \geq 0$ if $p = \theta = \infty$ and $\lambda = 0$.

Example 2. Let, for $\varepsilon > 0$, $(\tau_{\varepsilon} f)(x) = f(\varepsilon x)$. Then

$$\|\tau_{\varepsilon} f\|_{LM^{\lambda}_{p\theta}} = \varepsilon^{\lambda - \frac{n}{p}}\, \|f\|_{LM^{\lambda}_{p\theta}}\,, \quad \|\tau_{\varepsilon} f\|_{GM^{\lambda}_{p\theta}} = \varepsilon^{\lambda - \frac{n}{p}}\, \|f\|_{GM^{\lambda}_{p\theta}}\,. \tag{1.10}$$

In the terminology, used in particular in [2], p. 32, $\lambda - \frac{n}{p}$ is the differential dimension of $LM^{\lambda}_{p\theta}$ and of $GM^{\lambda}_{p\theta}$. Equality (1.10) also holds if the space $LM^{\lambda}_{p\theta}$ or $GM^{\lambda}_{p\theta}$ is replaced by the homogeneous Nikols'ski-Besov space $\dot{B}^{\lambda}_{p\theta}$ of functions with fractional order of smoothness λ (see the definition, for example, in [2]), hence the differential dimensions of the spaces $LM^{\lambda}_{p\theta}$, $GM^{\lambda}_{p\theta}$, and $\dot{B}^{\lambda}_{p\theta}$ coincide.

Also

$$H_{\alpha}(\tau_{\varepsilon} f) = \varepsilon^{-\alpha}\tau_{\varepsilon}(H_{\alpha} f)\,, \quad M_{\alpha}(\tau_{\varepsilon} f) = \varepsilon^{-\alpha}\tau_{\varepsilon}(M_{\alpha} f)\,,$$
$$I_{\alpha}(\tau_{\varepsilon} f) = \varepsilon^{-\alpha}\tau_{\varepsilon}(I_{\alpha} f), \tag{1.11}$$

hence the order of homogeneity of all three operators H_{α}, M_{α}, and I_{α} is $-\alpha$.

By (1.10) and (1.11)

$$\|H_{\alpha}(\tau_{\varepsilon} f)\|_{LM^{\lambda}_{p\theta}} = \varepsilon^{\lambda - \frac{n}{p} - \alpha}\, \|H_{\alpha} f\|_{LM^{\lambda}_{p\theta}}\,,$$
$$\|H_{\alpha}(\tau_{\varepsilon} f)\|_{GM^{\lambda}_{p\theta}} = \varepsilon^{\lambda - \frac{n}{p} - \alpha}\, \|H_{\alpha} f\|_{GM^{\lambda}_{p\theta}}\,, \tag{1.12}$$

where the exponent of ε is the sum of the order of homogeneity of the operator H_{α} and of the differential dimension of the space $LM^{\lambda}_{p\theta}$ or $GM^{\lambda}_{p\theta}$. These equalities also hold if H_{α} is replaced by M_{α} or I_{α}.

In fact we shall consider more general spaces $LM_{p\theta,w(\cdot)}$ and $GM_{p\theta,w(\cdot)}$, the local and the global Morrey-type spaces, characterized by numerical parameters p, θ and a functional parameter w, defined in the next section. Note that if $w(r) = r^{-\lambda - \frac{1}{\theta}}$, then

$$LM_{p\theta, r^{-\lambda - \frac{1}{\theta}}} = LM^{\lambda}_{p\theta} \quad \text{and} \quad GM_{p\theta, r^{-\lambda - \frac{1}{\theta}}} = GM^{\lambda}_{p\theta}\,.$$

For this reason we may say that $LM^{\lambda}_{p\theta}$ and $GM^{\lambda}_{p\theta}$ are power type local Morrey spaces, power type global Morrey spaces respectively, while $LM_{p\theta,w(\cdot)}, GM_{p\theta,w(\cdot)}$ are general local Morrey-type spaces, general global Morrey-type spaces respectively.

In the case of local or global Morrey-type spaces we obtain sufficient conditions on the functions w_1 and w_2 ensuring that $H_\alpha : LM_{p_1\theta_1,w_1(\cdot)} \to LM_{p_2\theta_2,w_2(\cdot)}$ or $H_\alpha : GM_{p_1\theta_1,w_1(\cdot)} \to GM_{p_2\theta_2,w_2(\cdot)}$ (Theorem 3) and necessary conditions (Theorem 4), which, for a certain range of the numerical parameters $\alpha, p_1, p_2, \theta_1, \theta_2$ and under additional a priori assumptions on w_1 and w_2, coincide (Theorem 5). (Note that in [4-9] necessary and sufficient conditions on the functions w_1 and w_2 ensuring the boundedness of the operators M_α and I_α were obtained, for a certain range of the numerical parameters, only for the case of local Morrey-type spaces.)

Given $\theta_1 \le \theta_2$ and a function w_2 satisfying certain regularity assumptions, we find, for the operator H_α with the target space $LM_{p_2\theta_2,w_2(\cdot)}$, the maximal domain space in the scale of spaces $\{LM_{p_1\theta_1,w_1(\cdot)}\}$ (Theorem 6). If the target space is the power type local Morrey space $LM^{\lambda_2}_{p_2\theta_2}$, then, under the appropriate assumptions on the numerical parameters, it appears that the maximal domain spaces in the scale of general Morrey-type spaces $\{LM_{p_1\theta_1,w_1(\cdot)}\}$ is the power type local Morrey space $LM^{\lambda_1}_{p_1\theta_1}$ with $\lambda_1 = \lambda_2 + n(\frac{1}{p_1} - \frac{1}{p_2}) - \alpha$ (Corollary 4).

2 Definitions and basic properties of Morrey-type spaces

Definition 1. Let $0 < p, \theta \le \infty$ and let w be a non-negative measurable function on $(0,\infty)$. We denote by $LM_{p\theta,w(\cdot)}$ and $GM_{p\theta,w(\cdot)}$, the local Morrey-type spaces, the global Morrey-type spaces respectively, the spaces of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ with finite quasi-norms

$$\|f\|_{LM_{p\theta,w(\cdot)}} \equiv \|f\|_{LM_{p\theta,w(\cdot)}(\mathbb{R}^n)} = \left\| w(r)\, \|f\|_{L_p(B(0,r))} \right\|_{L_\theta(0,\infty)}, \tag{2.1}$$

$$\|f\|_{GM_{p\theta,w(\cdot)}} = \sup_{x\in\mathbb{R}^n} \|f(x+\cdot)\|_{LM_{p\theta,w(\cdot)}} = \sup_{x\in\mathbb{R}^n} \left\| w(r)\, \|f\|_{L_p(B(x,r))} \right\|_{L_\theta(0,\infty)} \tag{2.2}$$

respectively.

Note that

$$\|f\|_{LM_{p\theta,r^{-\lambda-\frac{1}{\theta}}}} = \|f\|_{LM^\lambda_{p\theta}}, \quad \|f\|_{GM_{p\theta,r^{-\lambda-\frac{1}{\theta}}}} = \|f\|_{GM^\lambda_{p\theta}} \equiv \|f\|_{M^\lambda_{p\theta}}.$$

The boundedness of the maximal operator M, the fractional maximal operator M_α, the Riesz potential I_α and the singular integral operator in Morrey or Morrey-type spaces was studied in [1], [11], [16], [18], [12], [13], [11], [4-9]. In [4-9], for a certain range of the numerical parameters $\alpha, p_1, p_2, \theta_1, \theta_2$, necessary and sufficient conditions on the functions w_1 and w_2 were obtained ensuring the boundedness of the aforementioned operators from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$.

Definition 2. Let $0 < p, \theta \le \infty$. We denote by Ω_θ the set of all functions w which are non-negative and measurable on $(0,\infty)$, not equivalent to 0, and such that for some $t > 0$

$$\|w(r)\|_{L_\theta(t,\infty)} < \infty. \tag{2.3}$$

Moreover, we denote by $\Omega_{p\theta}$, the set of all functions w which are non-negative and measurable on $(0,\infty)$, not equivalent to 0, and such that for all $t>0$

$$\left\|w(r)r^{n/p}\right\|_{L_\theta(0,t)} < \infty\,, \qquad \|w(r)\|_{L_\theta(t,\infty)} < \infty\,. \tag{2.4}$$

Lemma 1. *Let $0 < p,\theta \le \infty$ and let w be a non-negative measurable function on $(0,\infty)$, which is not equivalent to 0.*

Then the space $LM_{p\theta,w}$ is nontrivial if and only if $w \in \Omega_\theta$, and the space $GM_{p\theta,w}$ is notrivial if and only if $w \in \Omega_{p\theta}$.

Moreover, if $w \in \Omega_\theta$ and $\tau = \inf\{s>0 : \|w\|_{L_\theta(s,\infty)} < \infty\}$, then the space $LM_{p\theta,w(\cdot)}$ contains all functions $f \in L_p(\mathbb{R}^n)$ such that $f=0$ on $B(0,t)$ for some $t>\tau$. If $w \in \Omega_{p\theta}$, then

$$L_p(\mathbb{R}^n)\cap L_\infty(\mathbb{R}^n) \subset GM_{p\theta,w(\cdot)}\,.$$

Proof. Let w be a non-negative measurable function on $(0,\infty)$ which is not equivalent to 0.

1. In [5] it was proved that if $w \notin \Omega_\theta$, then the spaces $LM_{p\theta,w(\cdot)}$ and $GM_{p\theta,w(\cdot)}$ are trivial.

It was also proved there that if $p<\infty$ and $\|w(r)r^{\frac{n}{p}}\|_{L_\theta(0,t)} = \infty$ for all $t>0$, then the space $GM_{p\theta,w(\cdot)}$ is trivial. This also holds for $p=\infty$. Indeed, assume that f is not equivalent to 0 on $\mathbb{R}^n$. Then there exists $y\in\mathbb{R}^n$ such that $A_y \equiv \lim\limits_{\varrho\to 0^+} \|f\|_{L_\infty(B(y,\varrho))} > 0$. (This follows because otherwise for all $\varepsilon>0$ and for all $x\in\mathbb{R}^n$ there exists $r_{\varepsilon,x}>0$ such that $\|f\|_{L_\infty(B(x,r_{\varepsilon,x}))} < \varepsilon$. This implies that for all $\varepsilon > 0$

$$\|f\|_{L_\infty(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} \|f\|_{L_\infty(B(x,r_{\varepsilon,x}))} \le \varepsilon\,,$$

hence $\|f\|_{L_\infty(\mathbb{R}^n)} = 0$, which contradicts the assumption that f is not equivalent to 0.) Therefore

$$\|f\|_{GM_{\infty\theta,w(\cdot)}} \ge \|w(r)\|f\|_{L_\infty(B(y,r))}\|_{L_\theta(0,\infty)} \ge A_y\|w\|_{L_\theta(0,\infty)} = \infty\,.$$

We also note that if $0<p\le\infty$ and $\|w\|_{L_\theta(t,\infty)} = \infty$ for some $t>0$, then the space $GM_{p\theta,w(\cdot)}$ is trivial. Indeed, if f is not equivalent to 0, then $\sup\limits_{x\in\mathbb{R}^n}\|f\|_{L_p(B(x,t))} > 0$, hence

$$\|f\|_{GM_{\infty\theta,w(\cdot)}} \ge \sup_{x\in\mathbb{R}^n} \|w(r)\|f\|_{L_p(B(x,t))}\|_{L_\theta(t,\infty)}$$

$$\ge \sup_{x\in\mathbb{R}^n} \|f\|_{L_p(B(x,t))}\|w(r)\|_{L_\theta(t,\infty)} = \infty\,.$$

Finally assume that $\|w(r)r^{\frac{n}{p}}\|_{L_\theta(0,\tau)} = \infty$ for some $\tau>0$. Then there are two possibilities: 1) $\|w(r)r^{\frac{n}{p}}\|_{L_\theta(0,t)} = \infty$ for all $t>0$ or 2) there exists $s\in(0,\tau)$ such that $\|w(r)r^{\frac{n}{p}}\|_{L_\theta(0,s)} < \infty$ which implies that $\|w(r)r^{\frac{n}{p}}\|_{L_\theta(s,\tau)} = \infty$, hence

$$\|w\|_{L_\theta(s,\infty)} \ge \|w\|_{L_\theta(s,\tau)} \ge \tau^{-\frac{n}{p}}\|w(r)r^{\frac{n}{p}}\|_{L_\theta(s,\tau)} = \infty\,.$$

In both cases by the above the space $GM_{p\theta,w(\cdot)}$ is trivial.

2. If $w \in \Omega_\theta$, $f \in L_p(\mathbb{R}^n)$ and $f = 0$ on $B(0,t)$ for some $t > \tau$, then

$$\|f\|_{LM_{p\theta,w(\cdot)}} = \left\| w(r) \, \|f\|_{L_p(B(0,r))} \right\|_{L_\theta(t,\infty)} \le \|f\|_{L_p(\mathbb{R}^n)} \, \|w(r)\|_{L_\theta(t,\infty)} < \infty \, .$$

If $w \in \Omega_{p\theta}$ and $f \in L_p(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$, then [1]

$$\|f\|_{GM_{p\theta,w(\cdot)}} = \sup_{x\in\mathbb{R}^n} \left\| w(r) \, \|f\|_{L_p(B(x,r))} \right\|_{L_\theta(0,\infty)}$$

$$\le 2^{(\frac{1}{\theta}-1)_+} \Big(\sup_{x\in\mathbb{R}^n} \left\| w(r) \, \|f\|_{L_p(B(x,r))} \right\|_{L_\theta(0,1)}$$

$$+ \sup_{x\in\mathbb{R}^n} \left\| w(r) \, \|f\|_{L_p(B(x,r))} \right\|_{L_\theta(1,\infty)} \Big)$$

$$\le 2^{(\frac{1}{\theta}-1)_+} \Big(v_n^{\frac{1}{p}} \, \|f\|_{L_\infty(\mathbb{R}^n)} \left\| w(r) r^{\frac{n}{p}} \right\|_{L_\theta(0,1)} + \|f\|_{L_p(\mathbb{R}^n)} \, \|w(r)\|_{L_\theta(1,\infty)} \Big) < \infty \, .$$

□

In the sequel, keeping in mind Lemma 1, we always assume that $w \in \Omega_\theta$ for local Morrey-type spaces $LM_{p\theta,w(\cdot)}$ and $w \in \Omega_{p\theta}$ for global Morrey-type spaces $GM_{p\theta,w(\cdot)}$.

Remark 1. In [4-9] for the case of global Morrey-type spaces the class $\Omega_{p,\theta}$, wider than $\Omega_{p\theta}$, was considered. A function w belongs to this class if it is non-negative and measurable on $(0,\infty)$, not equivalent to 0, and there exist $t_1, t_2 > 0$ such that $\left\| w(r) r^{n/p} \right\|_{L_\theta(0,t_1)} < \infty$ and $\|w(r)\|_{L_\theta(t_2,\infty)} < \infty$. However this does not increase generality because for $w \in \Omega_{p,\theta} \setminus \Omega_{p\theta}$ the space $GM_{p\theta,w(\cdot)}$ is trivial.

Example 3. One can easily verify that $r^{-\lambda-\frac{1}{\theta}} \in \Omega_\theta$ if and only if $\lambda > 0$ for $\theta < \infty$ and $\lambda \ge 0$ for $\theta = \infty$; $r^{-\lambda-\frac{1}{\theta}} \in \Omega_{p\theta}$ if and only if $0 < \lambda < \frac{n}{p}$ for $\theta < \infty$ and $0 \le \lambda \le \frac{n}{p}$ for $\theta = \infty$.

For non-negative functions φ, ψ defined on $(0,\infty)$ we shall write $\varphi \ll \psi$ if there exists $c > 0$ such that $\varphi(t) \le c\psi(t)$ for all $t \in (0,\infty)$, and we shall write $\varphi \asymp \psi$ if $\varphi \ll \psi$ and $\psi \ll \varphi$.

Lemma 2 ([6]). *Let* $0 < p, \theta \le \infty$ *and* $w_1, w_2 \in \Omega_\theta$. *Then*

$$1) \quad LM_{p\theta,w_1(\cdot)} \subset LM_{p\theta,w_2(\cdot)} \Longleftrightarrow \|w_1\|_{L_\theta(t,\infty)} \ll \|w_2\|_{L_\theta(t,\infty)} \, ,$$

$$2) \quad LM_{p\theta,w_1(\cdot)} = LM_{p\theta,w_2(\cdot)} \Longleftrightarrow \|w_1\|_{L_\theta(t,\infty)} \asymp \|w_2\|_{L_\theta(t,\infty)} \, .$$

[1] Here and in the sequel a_+ denotes the positive part of the real number a.

3 Corollaries of weighted $L_{p,u(\cdot)}$-estimates

For a measurable set $\Omega \subset \mathbb{R}^n$ and a function u non-negative and measurable on Ω, let $L_{p,u(\cdot)}(\Omega)$ be the space of all functions f measurable on Ω for which

$$\|f\|_{L_{p,u(\cdot)}(\Omega)} = \|uf\|_{L_p(\Omega)} < \infty.$$

If $0 < p \le \theta \le \infty$, then

$$\|f\|_{LM_{p\theta,w(\cdot)}} \le \|f\|_{L_{p,W(\cdot)}}\,, \tag{3.1}$$

and if $0 < \theta \le p \le \infty$, then

$$\|f\|_{L_{p,W(\cdot)}} \le \|f\|_{LM_{p\theta,w(\cdot)}}\,, \tag{3.2}$$

where for all $x \in \mathbb{R}^n$

$$W(x) = \|w\|_{L_\theta(|x|,\infty)}\,.$$

These inequalities can be easily proved by applying the following inequality for Lebesgue spaces with mixed quasinorms: for $0 < p \le q \le \infty$ and for a function F measurable on $\mathbb{R}^{m+n}$

$$\Big\| \big\| F(x,y) \big\|_{L_{p,x}(\mathbb{R}^n)} \Big\|_{L_{q,y}(\mathbb{R}^m)} \le \Big\| \big\| F(x,y) \big\|_{L_{q,y}(\mathbb{R}^m)} \Big\|_{L_{p,x}(\mathbb{R}^n)}\,.$$

In particular, for $0 < p \le \infty$

$$\|f\|_{LM_{pp,w(\cdot)}} = \|f\|_{L_{p,V(\cdot)}}\,,$$

where for all $x \in \mathbb{R}^n$

$$V(x) = \|w\|_{L_p(|x|,\infty)}\,.$$

See, e.g., [6].

We shall use the following theorem stating necessary and sufficient conditions for the validity of the inequality

$$\|H_\alpha f\|_{L_{p_2,u_2(\cdot)}} \le c_1 \|f\|_{L_{p_1,u_1(\cdot)}}\,, \tag{3.3}$$

where

$$u_1(x) = \widehat{u}_1(|x|), \qquad u_2(x) = \widehat{u}_2(|x|),$$

$\widehat{u}_1$, $\widehat{u}_2$ are functions non-negative and measurable on $[0,\infty)$ and $c_1 > 0$ is independent of f.

Theorem 1 ([20], [21], [13]). *Let $1 \le p_1, p_2 \le \infty$. Then inequality (3.3) holds if and only if*

$$I(\widehat{u}_1, \widehat{u}_2) < \infty\,,$$

where for[2] $p_1 \le p_2$

$$I(\widehat{u}_1,\widehat{u}_2) = \Big\| \big\| \widehat{u}_2(\tau)\tau^{\alpha-n+\frac{n-1}{p_2}} \big\|_{L_{p_2}(t,\infty)} \big\| \widehat{u}_1(\tau)^{-1}\tau^{\frac{n-1}{p_1'}} \big\|_{L_{p_1'}(0,t)} \Big\|_{L_\infty(0,\infty)} \tag{3.4}$$

[2] If $p_1 = 1$, then the factor $\big\|\widehat{u}_1(\tau)^{-1}\tau^{\frac{n-1}{p_1'}}\big\|_{L_{p_1'}(0,t)}$ should be replaced by $\widehat{u}_1(t)^{-1}$ and if $p_2 = \infty$, then the factor $\big\|\widehat{u}_2(\tau)\tau^{\alpha-n+\frac{n-1}{p_2}}\big\|_{L_{p_2}(t,\infty)}$ should be replaced by $\widehat{u}_2(t)t^{\alpha-n}$.

(p_1' is the exponent conjugate to p_1), and for $p_2 < p_1$

$$I(\widehat{u}_1, \widehat{u}_2) = \Big\| \big\| \widehat{u}_2(\tau)\tau^{\alpha-n+\frac{n-1}{p_2}} \big\|_{L_{p_2}(t,\infty)} \Lambda(t) \Big\|_{L_s(0,\infty)}, \tag{3.5}$$

where[3]

$$\Lambda(t) = \Big\| \widehat{u}_1(\tau)^{-1}\tau^{\frac{n-1}{p_1'}} \Big\|_{L_{p_1'}(0,t)}^{\frac{p_1'}{p_2'}} \widehat{u}_1(t)^{-\frac{p_1'}{s}} t^{\frac{n-1}{s}}$$

and s is defined by

$$\frac{1}{s} = \frac{1}{p_2} - \frac{1}{p_1}. \tag{3.6}$$

Remark 2. Let for $1 \le p_1 \le p_2 \le \infty$

$$J(\widehat{u}_1, \widehat{u}_2) = \big\| t^{\alpha-n(\frac{1}{p_1}-\frac{1}{p_2})} \widehat{u}_1(t)^{-1}\widehat{u}_2(t) \big\|_{L_\infty(0,\infty)} \tag{3.4$'$}$$

and for $1 \le p_2 < p_1 \le \infty$

$$J(\widehat{u}_1, \widehat{u}_2) = \big\| t^{\alpha-n(\frac{1}{p_1}-\frac{1}{p_2})} \widehat{u}_1(t)^{-1}\widehat{u}_2(t) t^{-\frac{1}{s}} \big\|_{L_s(0,\infty)}. \tag{3.5$'$}$$

Assume that the functions $\widehat{u}_1$ and $\widehat{u}_2$ are non-increasing and that

$$\alpha < \frac{n}{p_2'} \quad \text{if} \quad p_2 < \infty \quad \text{and} \quad \alpha \le n \quad \text{if} \quad p_2 = \infty. \tag{3.7}$$

Then for any $\nu > 1$

$$c_2 J\Big(\widehat{u}_1\Big(\frac{t}{\nu}\Big), \widehat{u}_2(\nu t)\Big) \le I(\widehat{u}_1(t), \widehat{u}_2(t)) \le c_3 J(\widehat{u}_1(t), \widehat{u}_2(t)),$$

where $c_2 > 0$ and $c_3 > 0$ are independent of $\widehat{u}_1$ and $\widehat{u}_2$.

Indeed, if $p_1 \le p_2$, then

$$I(\widehat{u}_1(t), \widehat{u}_2(t)) \le \|\widehat{u}_2(t)\|\tau^{\alpha-n+\frac{n-1}{p_2}}\|_{L_{p_2}(t,\infty)} \widehat{u}_1(t)^{-1} \|\tau^{\frac{n-1}{p_1'}}\|_{L_{p_1'}(0,t)}\|_{L_\infty(0,\infty)}$$
$$= aJ(\widehat{u}_1(t), \widehat{u}_2(t))$$

and

$$I(\widehat{u}_1(t), \widehat{u}_2(t)) \ge \| \, \|\widehat{u}_2(\tau)\tau^{\alpha-n+\frac{n-1}{p_2}}\|_{L_{p_2}(t,\nu t)} \|\widehat{u}_1(\tau)^{-1}\tau^{\frac{n-1}{p_1'}}\|_{L_{p_1'}(\frac{t}{\nu},t)}\|_{L_\infty(0,\infty)}$$
$$\ge bJ\Big(\widehat{u}_1\Big(\frac{t}{\nu}\Big), \widehat{u}_2(\nu t)\Big),$$

where $a, b > 0$ are independent of $\widehat{u}_1, \widehat{u}_2$. Also similar estimates hold if $p_2 < p_1$.

Hence the condition $J(\widehat{u}_1, \widehat{u}_2) < \infty$ is sufficient for the validity of inequality (3.3).

---

[3] If $p_2 = 1$, then the factor $\Big\| \widehat{u}_1(\tau)^{-1}\tau^{\frac{n-1}{p_1'}} \Big\|_{L_{p_1'}(0,t)}^{\frac{p_1'}{p_2'}}$ should be omitted.

Moreover, if there exist $\mu > 1$ and $c_4 > 0$ such that

$$\widehat{u}_1(t) \le c_4 \widehat{u}_1(\mu t) \quad \text{or} \quad \widehat{u}_2 \le c_4 \widehat{u}_2(\mu t)\,, \quad t \in (0,\infty)\,, \tag{3.8}$$

then this condition is necessary and sufficient for the validity of inequality (3.3). This follows since in both cases

$$J\Big(\widehat{u}_1\Big(\frac{t}{\sqrt{\mu}}\Big), \widehat{u}_2(\sqrt{\mu} t)\Big) = \gamma_1 J\Big(\widehat{u}_1\Big(\frac{t}{\mu}\Big), \widehat{u}_2(t)\Big)$$

$$= \gamma_2 J(\widehat{u}_1(t), \widehat{u}_2(\mu t)) \ge \gamma J(\widehat{u}_1(t), \widehat{u}_2(t))$$

for some $\gamma_1, \gamma_2, \gamma > 0$ which are independent of $\widehat{u}_1$ and $\widehat{u}_2$ ($\gamma = \min\{\gamma_1,\gamma_2\} c_4^{-1}$).

It is necessary and sufficient, in particular, if $\widehat{u}_1(t) \asymp t^{-\beta}$ on $(0,\infty)$ where $\beta > 0$ and $\widehat{u}_2$ is non-increasing on $(0,\infty)$, or $\widehat{u}_2(t) \asymp t^{-\beta}$ on $(0,\infty)$ where $\beta > 0$ and $\widehat{u}_1$ is non-increasing on $(0,\infty)$. In the last case assumption (3.7) can be replaced by

$$\alpha < \frac{n}{p_2'} + \beta \quad \text{if} \quad p_2 < \infty \quad \text{and} \quad \alpha \le n + \beta \quad \text{if} \quad p_2 = \infty\,, \tag{3.9}$$

because in (3.4) $\|\widehat{u}_2(\tau)\tau^{\alpha+n+\frac{n-1}{p_2}}\|_{L_{p_2}(t,\infty)} < \infty$ if and only if this condition is satisfied.

Remark 3. It may happen that in conditions (3.4), (3.5) the first factor inside $\|\cdot\|_{L_\infty(0,\infty)}, \|\cdot\|_{L_s(0,\infty)}$ respectively, is equal to 0 and the second one is equal to ∞, or in conditions $(3.4'), (3.5')$ $\hat{u}_2(t) = 0$ and $\hat{u}_1(t)^{-1} = \infty$. In such cases it is assumed that $0 \cdot \infty = 0$.

Theorem 1, Remark 2, and inequalities (3.1) and (3.2) immediately imply the following result for the case of Morrey-type spaces.

Theorem 2. *Let $1 \le p_1, p_2 \le \infty$, $0 < \theta_1, \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$, and $w_2 \in \Omega_{\theta_2}$.*

1. *If $p_1 \ge \theta_1$ and $p_2 \le \theta_2$, then the condition*

$$I\Big(\|w_1\|_{L_{p_1}(t,\infty)}, \|w_2\|_{L_{p_2}(t,\infty)}\Big) < \infty \tag{3.10}$$

is sufficient for the boundedness of H_α from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$.

If $p_1 \le \theta_1$ and $p_2 \ge \theta_2$, then this conditions is necessary for the boundedness of H_α from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$.

In particular, if $\theta_1 = p_1$ and $\theta_2 = p_2$, then this conditions is necessary and sufficient for the boundedness of H_α from $LM_{p_1p_1,w_1(\cdot)}$ to $LM_{p_2p_2,w_2(\cdot)}$.

2. *If $p_1 \ge \theta_1$, $p_2 \le \theta_2$, and condition (3.7) is satisfied, then the condition*

$$\Big\|t^{\alpha-n(\frac{1}{p_1}-\frac{1}{p_2})}\,\|w_1\|^{-1}_{L_{\theta_1}(t,\infty)}\,\|w_2\|_{L_{\theta_2}(t,\infty)}\,t^{-\frac{1}{s}}\Big\|_{L_s(0,\infty)} < \infty\,, \tag{3.11}$$

where $s = \infty$ if $p_1 \le p_2$ and s is defined by equality (3.6), is sufficient for the boundedness of H_α from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$.

If $p_1 \le \theta_1$ and $p_2 \ge \theta_2$, then for any $\mu > 1$ both conditions

$$\Big\|t^{\alpha-n(\frac{1}{p_1}-\frac{1}{p_2})}\,\|w_1\|^{-1}_{L_{\theta_1}(\frac{t}{\mu},\infty)}\,\|w_2\|_{L_{\theta_2}(t,\infty)}\,t^{-\frac{1}{s}}\Big\|_{L_s(0,\infty)} < \infty\,, \tag{3.12}$$

and

$$\Big\|t^{\alpha-n(\frac{1}{p_1}-\frac{1}{p_2})}\,\|w_1\|^{-1}_{L_{\theta_1}(t,\infty)}\,\|w_2\|_{L_{\theta_2}(\mu t,\infty)}\,t^{-\frac{1}{s}}\Big\|_{L_s(0,\infty)}<\infty \tag{3.13}$$

are necessary for the boundedness of H_α *from* $LM_{p_1\theta_1,w_1(\cdot)}$ *to* $LM_{p_2\theta_2,w_2(\cdot)}$.

In particular, if $\theta_1=p_1$, $\theta_2=p_2$, *condition* (3.7) *is satisfied, and for some* $\mu>1$ *one of the conditions*

$$\|w_1\|_{L_{p_1}(t,\infty)}\ll\|w_1\|_{L_{p_1}(\mu t,\infty)}\quad\text{or}\quad\|w_1\|_{L_{p_2}(t,\infty)}\ll\|w_2\|_{L_{p_2}(\mu t,\infty)} \tag{3.14}$$

is satisfied, then condition (3.11) *is necessary and sufficient for the boundedness of* H_α *from* $LM_{p_1p_1,w_1(\cdot)}$ *to* $LM_{p_2p_2,w_2(\cdot)}$.

For example, if $p_1\ge\theta_1$, $p_2\le\theta_2$ and condition (3.10) is satisfied, then

$$\|H_\alpha f\|_{LM_{p_2\theta_2,w_2(\cdot)}}\le\|H_\alpha f\|_{L_{p_2,\|w_2\|_{L_{\theta_2}(|x|,\infty)}}}$$

$$\le c_1\|f\|_{L_{p_1,\|w_1\|_{L_{\theta_1}(|x|,\infty)}}}\le c_1\|f\|_{LM_{p_1\theta_1,w_1(\cdot)}}\,.$$

Corollary 1. *Let* $1\le p_1,p_2\le\infty$.

1. *If* $\lambda_1>0$ *for* $p_1<\infty$ *and* $\lambda_1\ge0$ *for* $p_1=\infty$, $w_2\in\Omega_{p_2}$, *and condition* (3.7) *is satisfied, then the condition*

$$\Big\|t^{\alpha-n\left(\frac{1}{p_1}-\frac{1}{p_2}\right)+\lambda_1}\|w_2\|_{L_{p_2}(t,\infty)}t^{-\frac{1}{s}}\Big\|_{L_s(0,\infty)}<\infty \tag{3.11$'$}$$

is necessary and sufficient for the boundedness of H_α *from* $LM^{\lambda_1}_{p_1p_1}$ *to* $LM_{p_2p_2,w_2(\cdot)}$.

2. *If* $w_1\in\Omega_{p_1}$, $\lambda_2>0$ *for* $p_2<\infty$ *and* $\lambda_2\ge0$ *for* $p_2=\infty$, *and condition* (3.9) *is satisfied, then the condition*

$$\Big\|t^{\alpha-n\left(\frac{1}{p_1}-\frac{1}{p_2}\right)-\lambda_2}\|w_1\|^{-1}_{L_{p_1}(t,\infty)}t^{-\frac{1}{s}}\Big\|_{L_s(0,\infty)}<\infty \tag{3.11$''$}$$

is necessary and sufficient for the boundedness of H_α *from* $LM_{p_1p_1,w_1(\cdot)}$ *to* $LM^{\lambda_2}_{p_2p_2}$.

3. *Let* $\alpha\in\mathbb{R}$, $\lambda_k>0$ *for* $p_k<\infty$ *and* $\lambda_k\ge0$ *for* $p_k=\infty$ $(k=1,2)$. *Then* H_α *is bounded from* $LM^{\lambda_1}_{p_1p_1}$ *to* $LM^{\lambda_2}_{p_2p_2}$ *if and only if*

$$p_1\le p_2\quad\text{and}\quad\alpha=\lambda_2-\lambda_1+n\Big(\frac{1}{p_1}-\frac{1}{p_2}\Big)\,. \tag{3.15}$$

Statement 3 of the corollary follows by Statement 2, because for the space $LM^{\lambda_1}_{p_1p_1}$ for all $t>0$

$$\|w_1\|_{L_{p_1}(t,\infty)}=\|r^{-\lambda_1-\frac{1}{p_1}}\|_{L_{p_1}(t,\infty)}=(\lambda_1p_1)^{-\frac{1}{p_1}}t^{-\lambda_1}$$

if $p_1<\infty$ and $\|w_1\|_{L_{p_1}(t,\infty)}=t^{-\lambda_1}$ if $p_1=\infty$, and condition (3.15) implies condition (3.9).

4 Estimates over balls

In this section we first investigate assumptions on the parameters ensuring the validity of the inequality

$$\|H_\alpha f\|_{L_{p_2}(B(0,r))} \le c_5(r)\|f\|_{L_{p_1}(B(0,r))} \tag{4.1}$$

for all $r>0$ and for all $f \in L_{p_1}(B(0,r))$, where $c_5(r)>0$ depends only on r, n, α, p_1, p_2, and the dependence on r of the sharp constant $c_5^*(r)$ in this inequality. We start with noting that inequality (4.1) holds for all $r>0$ and for all $f \in L_{p_1}(B(0,r))$ if and only if it holds for $r=1$ for all $f \in L_{p_1}(B(0,1))$. Moreover,

$$c_5^*(r) = c_5^*(1) r^{\alpha - n\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\,. \tag{4.2}$$

This follows because by (1.11)

$$\|H_\alpha f\|_{L_{p_2}(B(0,r))} = r^\alpha \|\tau_{\frac{1}{r}} H_\alpha(\tau_r f)\|_{L_{p_2}(B(0,r))} = r^{\alpha+\frac{n}{p_2}}\|H_\alpha(\tau_r f)\|_{L_{p_2}(B(0,1))}$$

and

$$\|\tau_r f\|_{L_{p_1}(B(0,1))} = r^{-\frac{n}{p_1}}\|f\|_{L_{p_1}(B(0,r))}\,.$$

Lemma 3. *Let* $1 \le p_1, p_2 \le \infty$. *Then inequality* (4.1) *holds for all* $r>0$ *and for all* $f \in L_{p_1}(B(0,r))$ *if and only if*

$$\alpha \ge n\left(\frac{1}{p_1}-\frac{1}{p_2}\right) \text{ if } 1<p_1\le p_2\le\infty \text{ or } p_1=1 \text{ and } p_2=\infty \tag{4.3}$$

and

$$\alpha > n\left(\frac{1}{p_1}-\frac{1}{p_2}\right) \text{ if } p_1=1\le p_2<\infty \text{ or } 1\le p_2<p_1\le\infty\,. \tag{4.4}$$

Proof. By the above it suffices to assume that $r=1$. By Theorem 1 inequality (4.1) holds if and only if $K \equiv I(\chi_{(0,1)}, \chi_{(0,1)}) < \infty$.

1. First let $p_1 \le p_2$. If $p_1>1$ and $p_2<\infty$, then

$$K = \sup_{0<t<1}\Big(\int_t^1 \tau^{(\alpha-\frac{n}{p_2'})p_2-1}d\tau\Big)^{\frac{1}{p_2}}\Big(\int_0^t \tau^{n-1}d\tau\Big)^{\frac{1}{p_1'}}\,.$$

If $\alpha \ge \frac{n}{p_2'}$, then

$$K \le n^{-\frac{1}{p_1'}}\sup_{0<t<1}\Big(\int_t^1 \tau^{-1}d\tau\Big)^{\frac{1}{p_2}} t^{\frac{n}{p_1'}} = n^{-\frac{1}{p_1'}}\sup_{0<t<1}|\ln t|^{\frac{1}{p_2}}\, t^{\frac{n}{p_1'}} < \infty\,.$$

If $\alpha < \frac{n}{p_2'}$, then

$$K = \Big(\Big(\frac{n}{p_2'}-\alpha\Big)p_2\Big)^{-\frac{1}{p_2}} n^{-\frac{1}{p_1'}}\sup_{0<t<1}\Big(t^{\left(\alpha-\frac{n}{p_2'}\right)p_2}-1\Big)^{\frac{1}{p_2}} t^{\frac{n}{p_1'}} < \infty$$

if and only if $\alpha \geq n\big(\frac{1}{p_1} - \frac{1}{p_2}\big)$.

If $p_1 > 1$ and $p_2 = \infty$, then

$$K = n^{-\frac{1}{p_1'}} \sup_{0<t<1} t^{\alpha - \frac{n}{p_1}} < \infty$$

if and only if $\alpha \geq \frac{n}{p_1}$.

If $p_1 = 1$ and $p_2 < \infty$, then

$$K = \sup_{0<t<1} \Big(\int_t^1 \tau^{(\alpha - \frac{n}{p_2'})p_2 - 1} d\tau \Big)^{\frac{1}{p_2}} < \infty$$

if and only if $\alpha > n\big(1 - \frac{1}{p_2}\big)$.

If $p_1 = 1$ and $p_2 = \infty$, then

$$K = \sup_{0<t<1} t^{\alpha - n} < \infty$$

if and only if $\alpha \geq n$.

2. Next let $p_2 < p_1$. If $p_2 > 1$, then

$$K = \Big\| \Big(\int_t^1 \tau^{(\alpha - \frac{n}{p_2'})p_2 - 1} d\tau \Big)^{\frac{1}{p_2}} \Big(\int_0^t \tau^{n-1} d\tau \Big)^{\frac{1}{p_2'}} t^{\frac{n-1}{s}} \Big\|_{L_s(0,1)}.$$

If $\alpha \geq \frac{n}{p_2'}$, then

$$K \leq n^{-\frac{1}{p_1'}} \Big\| \Big(\int_t^1 \tau^{-1} d\tau \Big)^{\frac{1}{p_2}} t^{\frac{n}{p_1'} - \frac{1}{s}} \Big\|_{L_s(0,1)} = n^{-\frac{1}{p_1'}} \big\| |\ln t|\, t^{\frac{n}{p_1'} - \frac{1}{s}} \big\|_{L_s(0,1)} < \infty.$$

If $\alpha < \frac{n}{p_2'}$, then

$$K = \Big(\Big(\frac{n}{p_2'} - \alpha \Big) p_2 \Big)^{-\frac{1}{p_2}} n^{-\frac{1}{p_1'}} \Big\| \Big(t^{\left(\alpha - \frac{n}{p_2'}\right)p_2} - 1 \Big)^{\frac{1}{p_2}} t^{\frac{n}{p_1'} - \frac{1}{s}} \Big\|_{L_s(0,1)} < \infty$$

if and only if $\alpha > n\big(\frac{1}{p_1} - \frac{1}{p_2}\big)$.

If $p_2 = 1 < p_1 \leq \infty$, then

$$K = \Big\| \Big(\int_t^1 \tau^{\alpha - 1} d\tau \Big) t^{\frac{n-1}{p_1'}} \Big\|_{L_{p_1'}(0,1)}$$

and by a similar argument it again follows that $K < \infty$ if and only if $\alpha > n\big(\frac{1}{p_1} - 1\big)$. $\square$

Lemma 4. *Let* $1 \le p_1 \le \infty, 0 < p_2 \le \infty$ *and* $\alpha \in \mathbb{R}$ *is such that condition* (4.3) *and the condition*

$$\alpha > n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) \text{ if } p_1 = 1 \le p_2 < \infty \text{ or } 0 < p_2 < p_1 \le \infty\,. \tag{4.5}$$

are satisfied. Then there exists $c_6 > 0$ *such that for all* $x \in \mathbb{R}^n$*, for all* $r > 0$*, and for all* $f \in L_{p_1}(B(0,|x|+r))$

$$\|H_\alpha f\|_{L_{p_2}(B(x,r))} \le c_6\, r^{\frac{n}{p_2}}(|x|+r)^{\alpha-\frac{n}{p_1}}\|f\|_{L_{p_1}(B(0,|x|+r))}\,. \tag{4.6}$$

Proof. **1.** Assume first that $|x| \le 2r$. If $p_2 \ge 1$, then by Lemma 3

$$\|H_\alpha f\|_{L_{p_2}(B(x,r))} \le \|H_\alpha f\|_{L_{p_2}(B(0,|x|+r))}$$

$$\le c_5^*(1)(|x|+r)^{\alpha-n\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\|f\|_{L_{p_1}(B(0,|x|+r))}$$
$$\le c_5^*(1)3^{\frac{n}{p_2}}r^{\frac{n}{p_2}}(|x|+r)^{\alpha-\frac{n}{p_1}}\|f\|_{L_{p_1}(B(0,|x|+r))}\,.$$

If $0 < p_2 < 1$, then by Hölder's inequality and by the above with $p_2 = 1$

$$\|H_\alpha f\|_{L_{p_2}(B(x,r))} \le (v_n r^n)^{\frac{1}{p_2}-1}\|H_\alpha f\|_{L_1(B(x,r))}$$

$$\le v_n^{\frac{1}{p_2}-1}\tilde{c}_5^*(1)3^n r^{\frac{n}{p_2}}(|x|+r)^{\alpha-\frac{n}{p_1}}\|f\|_{L_{p_1}(B(0,|x|+r))}\,,$$

where $\tilde{c}_5^*(1)$ is the value of $c_5^*(1)$ (which depends on n, α, p_1, p_2) for $p_2 = 1$.

2. Next assume that $|x| \ge 2r$. Then for all $y \in B(x,r)$ $\frac{|x|+r}{3} \le |x|-r \le |y| \le |x|+r$ and by Hölder's inequality

$$\|H_\alpha f\|_{L_{p_2}(B(x,r))} = \left\|\frac{1}{|B(0,|y|)|^{1-\frac{\alpha}{n}}}\int\limits_{B(0,|y|)} f(z)dz\right\|_{L_{p_2}(B(x,r))}$$
$$\le \left\||B(0,|y|)|^{\frac{\alpha}{n}-\frac{1}{p_1}}\|f\|_{L_{p_1}(B(0,|y|))}\right\|_{L_{p_2}(B(x,r))}$$
$$\le v_n^{\frac{\alpha}{n}-\frac{1}{p_1}}\left\||y|^{\alpha-\frac{n}{p_1}}\right\|_{L_{p_2}(B(x,r))}\|f\|_{L_{p_1}(B(0,|x|+r))}$$
$$\le v_n^{\frac{\alpha}{n}-\frac{1}{p_1}+\frac{1}{p_2}}\,3^{((\frac{n}{p_1}-\alpha)_+)}r^{\frac{n}{p_2}}(|x|+r)^{\alpha-\frac{n}{p_1}}\|f\|_{L_{p_1}(B(0,|x|+r))}\,.$$

□

5 Necessary and sufficient conditions for general Morrey-type spaces

Lemma 5. *Let* $\alpha \in \mathbb{R}, 1 \le p_1 \le \infty, 0 < p_2, \theta_1, \theta_2 \le \infty$.

If $w_1 \in \Omega_{\theta_1}$ and $w_2 \in \Omega_{\theta_2}$, then the condition: for all $t > 0$

$$\|w_2(r) r^{\alpha - \frac{n}{p_2'}}\|_{L_{\theta_2}(t,\infty)} < \infty \tag{5.1}$$

is necessary for the boundedness of H_α from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$.

If $w_1 \in \Omega_{p_1\theta_1}$ and $w_2 \in \Omega_{p_2\theta_2}$, then this condition is also necessary for the boundedness of H_α from $GM_{p_1\theta_1,w_1(\cdot)}$ to $GM_{p_2\theta_2,w_2(\cdot)}$.

Proof. It suffices to prove that if $\|w_2(r) r^{\alpha - \frac{n}{p_2'}}\|_{L_{\theta_2}(t,\infty)} = \infty$ for some $t > 0$, then there exists a function $f \in GM_{p_1\theta_1,w_1(\cdot)}$ (hence $f \in LM_{p_1\theta_1,w_1(\cdot)}$) such that $f \notin LM_{p_2\theta_2,w_2(\cdot)}$ (hence $f \notin GM_{p_2\theta_2,w_2(\cdot)}$). In fact this is true for any non-negative function $f \in GM_{p_1\theta_1,w_1(\cdot)}$ such that $\|f\|_{L_{p_1}(B(0,\frac{t}{2}))} > 0$. Indeed, if $|y| \ge \frac{t}{2}$, then

$$(H_\alpha f)(y) \ge v_n^{\frac{\alpha}{n}-1} |y|^{\alpha-n} \|f\|_{L_1(B(0,\frac{t}{2}))}\,.$$

Hence, if $r \ge t$, then

$$\|H_\alpha f\|_{L_{p_2}(B(0,r))} \ge \|H_\alpha f\|_{L_{p_2}(B(0,r)\setminus B(0,\frac{r}{2}))}$$

$$\ge k_1 r^{\alpha-n} |B(0,r) \setminus B(0,r/2)|^{\frac{1}{p_2}} = k_2\, r^{\alpha - \frac{n}{p_2'}}\,,$$

where

$$k_1 = v_n^{\frac{\alpha}{n}-1} \max\{1, 2^{n-\alpha}\} \|f\|_{L_1(B(0,\frac{t}{2}))}\,, \quad k_2 = v_n^{\frac{1}{p_2}} (1 - 2^{-n})^{\frac{1}{p_2}} k_1 > 0\,.$$

Therefore

$$\|H_\alpha f\|_{LM_{p_2\theta_2,w_2(\cdot)}} \ge \|w_2(r) \|H_\alpha f\|_{L_{p_2}(B(0,r))}\|_{L_{\theta_2}(t,\infty)}$$

$$\ge k_2 \|w_2(r) r^{\alpha - \frac{n}{p_2'}}\|_{L_{\theta_2}(t,\infty)} = \infty\,.$$

□

Remark 4. For $w_2 \in \Omega_{\theta_2}$ condition (5.1) implies that $\|w_2\|_{L_{\theta_2}(t,\infty)} < \infty$ not only for some $t > 0$ (which is the meaning of the condition $w_2 \in \Omega_{\theta_2}$) but also for all $t > 0$.

Corollary 2. *Let $\alpha \in \mathbb{R}, 1 \le p_1 \le \infty, 0 < p_2, \theta_1, \theta_2 \le \infty$.*

If $\lambda_2 > 0$ if $\theta_2 < \infty$, $\lambda_2 \ge 0$ if $\theta_2 = \infty$ and $w_1 \in \Omega_{\theta_1}$, then the condition

$$\alpha < \lambda_2 + \frac{n}{p_2'} \quad \text{if} \quad \theta_2 < \infty\,, \quad \alpha \le \lambda_2 + \frac{n}{p_2'} \quad \text{if} \quad \theta_2 = \infty \tag{5.2}$$

is necessary for the boundedness of H_α from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM^{\lambda_2}_{p_2\theta_2}$.

If $0 < \lambda_2 < \frac{n}{p_2}$ if $\theta_2 < \infty$, $0 \le \lambda_2 \le \frac{n}{p_2}$ if $\theta_2 = \infty$ and $w_1 \in \Omega_{p_1\theta_1}$, then this condition is also necessary for the boundedness of H_α from $GM_{p_1\theta_1,w_1(\cdot)}$ to $GM^{\lambda_2}_{p_2\theta_2}$. (In this case it implies, in particular, that $\alpha < n$ if $\theta_2 < \infty$ and $\alpha \le n$ if $\theta_2 = \infty$.)

Lemma 6. *Let $\alpha \in \mathbb{R}, 1 \le p_1 \le \infty, 0 < p_2, \theta_1, \theta_2 \le \infty$, $w_1 \in \Omega_{p_1\theta_1}$, and $w_2 \in \Omega_{p_2\theta_2}$.*

Then the condition

$$\alpha \le \frac{n}{p_1} \tag{5.3}$$

is necessary for the boundedness of H_α from $GM_{p_1\theta_1,w_1(\cdot)}$ to $GM_{p_2\theta_2,w_2(\cdot)}$.

Moreover, if in addition $\|w_2(r)r^{\frac{n}{p_2}}\|_{L_{\theta_2}(0,\infty)} = \infty$, then the condition

$$\alpha < \frac{n}{p_1} \tag{5.4}$$

is necessary for the boundedness of H_α from $GM_{p_1\theta_1,w_1(\cdot)}$ to $GM_{p_2\theta_2,w_2(\cdot)}$.

Proof. **1**. If $1 \le p_1 \le \infty$ and $\alpha > n$, then $H_\alpha f \notin GM_{p_2\theta_2,w_2(\cdot)}$ for any non-negative function $f \in GM_{p_1\theta_1,w_1(\cdot)}$ which is not equivalent to 0. Indeed, for all $x \in \mathbb{R}^n, x \ne 0$, and for all $y \in \mathbb{R}^n$ with $|y| \ge |x|$

$$(H_\alpha f)(y) = \frac{1}{(v_n|y|^n)^{1-\frac{\alpha}{n}}} \int\limits_{B(0,|y|)} f(z)\,dz \ge v_n^{\frac{\alpha}{n}-1}|x|^{\alpha-n}\|f\|_{L_1(B(0,|x|))}\,,$$

hence, since $|B(x,r) \setminus B(0,|x|)| \ge \frac{1}{2}|B(x,r)|$,

$$\|H_\alpha f\|_{L_{p_2}(B(x,r)\setminus B(0,|x|))} \ge v_n^{\frac{\alpha}{n}-1}|x|^{\alpha-n}\|f\|_{L_1(B(0,|x|))}|B(x,r) \setminus B(0,|x|)|^{\frac{1}{p_2}}$$

$$\ge 2^{-\frac{1}{p_2}} v_n^{\frac{\alpha}{n}-1+\frac{1}{p_2}}|x|^{\alpha-n}\|f\|_{L_1(B(0,|x|))} r^{\frac{n}{p_2}}\,.$$

and

$$\|H_\alpha f\|_{GM_{p_2\theta_2,w_2(\cdot)}} \ge \sup_{x\in\mathbb{R}^n} \|w_2(r)\|H_\alpha f\|_{L_{p_2}(B(x,r)\setminus B(0,|x|))}\|_{L_{\theta_2}(0,\infty)}$$

$$\ge 2^{-\frac{1}{p_2}} v_n^{\frac{\alpha}{n}-1+\frac{1}{p_2}}\Big(\lim_{x\to\infty}|x|^{\alpha-n}\|f\|_{L_1(B(0,|x|))}\Big)\|w_2(r)r^{\frac{n}{p_2}}\|_{L_{\theta_2}(0,\infty)} = \infty\,.$$

2. Assume that $p_1 > 1$ and $\frac{n}{p_1} < \alpha \le n$. Consider the function

$$f_\beta(y) = |y|^\beta \chi_{{}^c B(0,1)}(y)\,, \quad y \in \mathbb{R}^n\,,$$

where $\max\{-\alpha, -n\} < \beta < -\frac{n}{p_1}$. Note that for all $x \in \mathbb{R}^n$ and $r > 0$

$$\|f_\beta\|_{L_{p_1}(B(x,r))} \le \|1\|_{L_{p_1}(B(x,r))} = k_1 r^{\frac{n}{p_1}}\,,$$

where $k_1 = v_n^{\frac{1}{p_1}}$, and

$$\|f_\beta\|_{L_{p_1}(B(x,r))} \le \||y|^\beta\|_{L_{p_1}({}^c B(0,1))} = k_2 < \infty\,,$$

where $k_2 > 0$ depends only on n, p_1 and β. Therefore

$$\|f_\beta\|_{GM_{p_1\theta_1,w_1(\cdot)}} = \sup_{x\in\mathbb{R}^n}\|w_1(r)\|f_\beta\|_{L_{p_1}(B(x,r))}\|_{L_{\theta_1}(0,\infty)}$$

$$\le 2^{(\frac{1}{\theta_1}-1)_+}\Big(\sup_{x\in\mathbb{R}^n}\|w_1(r)\|f_\beta\|_{L_{p_1}(B(x,r))}\|_{L_{\theta_1}(0,1)}$$
$$+\sup_{x\in\mathbb{R}^n}\|w_1(r)\|f_\beta\|_{L_{p_1}(B(x,r))}\|_{L_{\theta_1}(1,\infty)}\Big)$$
$$\le 2^{(\frac{1}{\theta_1}-1)_+}\Big(k_1\|w_1(r)r^{\frac{n}{p_1}}\|_{L_{\theta_1}(0,1)}+k_2\|w_1(r)\|_{L_{\theta_1}(1,\infty)}\Big)<\infty\,.$$

Since $\beta>-n$, for all $y\in\mathbb{R}^n$ with $|y|\ge 2$

$$(H_\alpha f_\beta)(y)=\frac{1}{(v_n|y|^n)^{1-\frac{\alpha}{n}}}\int\limits_{B(0,|y|)\setminus B(0,1)}|z|^\beta\,dz$$
$$=nv_n^{\frac{\alpha}{n}}(\beta+n)^{-1}|y|^{\alpha-n}(|y|^{\beta+n}-1)\ge k_3|y|^{\alpha+\beta}\,,$$

where $k_3=nv_n^{\frac{\alpha}{n}}(\beta+n)^{-1}(1-2^{-(\beta+n)})$, because $|y|^{\beta+n}-1\ge(1-2^{-(\beta+n)})|y|^{\beta+n}$ for $|y|\ge 2$.

If $|x|\ge 2$, then for all $y\in B(x,r)\setminus B(0,|x|)$ $\quad|y|\ge|x|\ge 2$, hence

$$\|H_\alpha f_\beta\|_{GM_{p_2\theta_2,w_2(\cdot)}}\ge\sup_{|x|\ge 2}\|w_2(r)\|H_\alpha f_\beta\|_{L_{p_2}(B(x,r)\setminus B(0,|x|))}\|_{L_{\theta_2}(0,\infty)}$$
$$\ge\Big(\frac{v_n}{2}\Big)^{\frac{1}{p_2}}k_3\Big(\sup_{|x|\ge 2}|x|^{\alpha+\beta}\Big)\|w_2(r)r^{\frac{n}{p_2}}\|_{L_{\theta_2}(0,\infty)}=\infty\,.$$

3. The second statement of the lemma will be proved later as a corollary of Theorem 4. ☐

Lemma 7. *Let* $1\le p_1\le\infty$, $0<p_2,\theta_1,\theta_2\le\infty$, $w_1\in\Omega_{\theta_1}$, *and* $w_2\in\Omega_{\theta_2}$. *Assume that conditions* (4.3) *and* (4.5) *are satisfied.*

If $LM_{p_1\theta_1,w_1(\cdot)}$ *is continuously embedded in* $LM_{p_1\theta_2,v_2(\cdot)}$, *briefly*

$$LM_{p_1\theta_1,w_1(\cdot)}\hookrightarrow LM_{p_1\theta_2,v_2(\cdot)}\,,\tag{5.5}$$

where

$$v_2(r)=w_2(r)r^{\alpha-n\left(\frac{1}{p_1}-\frac{1}{p_2}\right)},\tag{5.6}$$

then the operator H_α *is bounded from* $LM_{p_1\theta_1,w_1(\cdot)}$ *to* $LM_{p_2\theta_2,w_2(\cdot)}$.

Moreover, if the function $w_2(r)r^{\frac{n}{p_2}}$ *is almost increasing, then the operator* H_α *is bounded from* $LM_{p_1\theta_1,w_1(\cdot)}$ *to* $GM_{p_2\theta_2,w_2(\cdot)}$ *with* $w_2\in\Omega_{p_2\theta_2}$, *hence also from* $GM_{p_1\theta_1,w_1(\cdot)}$ *to* $GM_{p_2\theta_2,w_2(\cdot)}$. *(In the last case it is also assumed that* $w_1\in\Omega_{p_1\theta_1}$.*)*

Proof. **1.** Let $c_7>0$ be the norm of the embedding operator corresponding to the embedding (5.5). Then by (4.6) with $x=0$

$$\begin{aligned}\|H_\alpha f\|_{LM_{p_2\theta_2,w_2(\cdot)}}&=\Big\|w_2(r)\,\|H_\alpha f\|_{L_{p_2}(B(0,r))}\Big\|_{L_{\theta_2}(0,\infty)}\\&\le c_6\Big\|v_2(r)\,\|f\|_{L_{p_1}(B(0,r))}\Big\|_{L_{\theta_2}(0,\infty)}\\&=c_6\,\|f\|_{LM_{p_1\theta_2,v_2(\cdot)}}\le c_6c_7\,\|f\|_{LM_{p_1\theta_1,w_1(\cdot)}}\,.\end{aligned}$$

Hence H_α is bounded from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$.

2. Next let the function $w_2(r)r^{\frac{n}{p_2}}$ be almost increasing, i. e. there exists $c_8>0$ such that $w_2(r)r^{\frac{n}{p_2}} \le c_8 w_2(\varrho)\varrho^{\frac{n}{p_2}}$ for all $0<r\le\varrho<\infty$. Then by (4.6)

$$\begin{aligned}
&\|H_\alpha f\|_{GM_{p_2\theta_2,w_2(\cdot)}}\\
=\;& \sup_{x\in\mathbb{R}^n}\left\|w_2(r)\,\|H_\alpha f\|_{L_{p_2}(B(x,r))}\right\|_{L_{\theta_2}(0,\infty)}\\
\le\;& c_6\sup_{x\in\mathbb{R}^n}\left\|w_2(r)r^{\frac{n}{p_2}}(|x|+r)^{\alpha-\frac{n}{p_1}}\,\|f\|_{L_{p_1}(B(0,|x|+r))}\right\|_{L_{\theta_2}(0,\infty)}\\
=\;& c_6\sup_{x\in\mathbb{R}^n}\left\|w_2(\varrho-|x|)(\varrho-|x|)^{\frac{n}{p_2}}\varrho^{\alpha-\frac{n}{p_1}}\,\|f\|_{L_{p_1}(B(0,\varrho))}\right\|_{L_{\theta_2}(|x|,\infty)}\\
\le\;& c_6c_8\sup_{x\in\mathbb{R}^n}\left\|w_2(\varrho)\varrho^{\alpha-n\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\,\|f\|_{L_{p_1}(B(0,\varrho))}\right\|_{L_{\theta_2}(|x|,\infty)}\\
=\;& c_6c_8\,\|f\|_{LM_{p_1\theta_2,v_2(\cdot)}}\le c_6c_7c_8\,\|f\|_{LM_{p_1\theta_1,w_1(\cdot)}}\le c_6c_7c_8\,\|f\|_{GM_{p_1\theta_1,w_1(\cdot)}}\,.
\end{aligned}$$

Hence H_α is bounded from $GM_{p_1\theta_1,w_1(\cdot)}$ to $GM_{p_2\theta_2,w_2(\cdot)}$. □

Lemma 8 ([19])**.** *Let $0<\theta_1,\theta_2\le\infty$, $u_1\in\Omega_{\theta_1}$, and $u_2\in\Omega_{\theta_2}$. Then the inequality*

$$\|u_2 g\|_{L_{\theta_2}(0,\infty)}\le c_9\,\|u_1 g\|_{L_{\theta_1}(0,\infty)} \tag{5.7}$$

holds for some $c_9>0$ for all functions g non-negative and non-decreasing on $(0,\infty)$ if and only if

1. for $\theta_1\le\theta_2$

$$\left\|\|u_2\|_{L_{\theta_2}(t,\infty)}\,\|u_1\|^{-1}_{L_{\theta_1}(t,\infty)}\,\right\|_{L_\infty(0,\infty)}<\infty\,, \tag{5.8}$$

2. for $\theta_2<\theta_1<\infty$

$$\left\|\|u_2\|_{L_{\theta_2}(t,\infty)}\|u_1\|^{-\frac{\theta_1}{\theta_2}}_{L_{\theta_1}(t,\infty)}u_1(t)^{\frac{\theta_1}{\sigma}}\right\|_{L_\sigma(0,\infty)}<\infty\,, \tag{5.9}$$

where σ is defined by

$$\frac{1}{\sigma}=\frac{1}{\theta_2}-\frac{1}{\theta_1}. \tag{5.10}$$

Theorem 3. *Let $1\le p_1\le\infty$, $0<p_2,\theta_1,\theta_2\le\infty$, and conditions* (4.3), (4.5) *be satisfied.*

1. *Assume that $w_1\in\Omega_{\theta_1}$, $w_2\in\Omega_{\theta_2}$ and condition* (5.1) *is satisfied. Then for $\theta_1\le\theta_2$ the condition* [4]

$$\left\|\|v_2\|_{L_{\theta_2}(t,\infty)}\|w_1\|^{-1}_{L_{\theta_1}(t,\infty)}\right\|_{L_\infty(0,\infty)}<\infty \tag{5.11}$$

[4] If $\alpha=n\left(\frac{1}{p_1}-\frac{1}{p_2}\right)$, then it coincides with condition (3.11).

and for $\theta_2 < \theta_1 < \infty$ the condition

$$\Big\| \|v_2\|_{L_{\theta_2}(t,\infty)} \|w_1\|_{L_{\theta_1}(t,\infty)}^{-\frac{\theta_1}{\theta_2}} w_1(t)^{\frac{\theta_1}{\sigma}} \Big\|_{L_\sigma(0,\infty)} < \infty \,, \tag{5.12}$$

where the function v_2 is defined by equality (5.6)*, are sufficient for the boundedness of the operator H_α from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$.*

2. *Assume that $w_1 \in \Omega_{p_1\theta_1}$, $w_2 \in \Omega_{p_2\theta_2}$, condition* (5.1) *is satisfied, the function $w_2(r)r^{\frac{n}{p_2}}$ is almost increasing, $\alpha \le \frac{n}{p_1}$, and $\alpha < \frac{n}{p_1}$ if $\|w_2(r)r^{\frac{n}{p_2}}\|_{L_{\theta_2}(0,\infty)} = \infty$. Then conditions* (2.2) *and* (2.8) *are sufficient for the boundedness of the operator H_α also from $GM_{p_1\theta_1,w_1(\cdot)}$ to $GM_{p_2\theta_2,w_2(\cdot)}$.*

Proof. Follows by Lemmas 5 – 8. □

Remark 5. Condition (2.2) implies that for all $\varepsilon \ge 0$ and for all $\gamma \ge 1$

$$\Big\| t^\varepsilon \|v_2(r)r^{-\varepsilon}\|_{L_{\theta_2}(\gamma t,\infty)} \|w_1\|_{L_{\theta_1}(t,\infty)}^{-1} \Big\|_{L_\infty(0,\infty)} < \infty. \tag{5.13}$$

Theorem 4. *Let $1 \le p_1 \le \infty$, $0 < p_2, \theta_1, \theta_2 \le \infty$.*

Then

1) *For $w_1 \in \Omega_{\theta_1}$ and $w_2 \in \Omega_{\theta_2}$ condition* (5.13) *for $\varepsilon = 0$ and for all $\gamma > 1$ if $p_1 = 1$, and for all $\varepsilon > 0$ and for all $\gamma > 1$ if $p_1 > 1$ is necessary for the boundedness of the operator H_α from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$.*

2) *For $w_1 \in \Omega_{p_1\theta_1}$ and $w_2 \in \Omega_{p_2\theta_2}$ the condition: for $\varepsilon = 0$ and for all $\gamma > 1$ if $p_1 = 1$, and for all $\varepsilon > 0$ and for all $\gamma > 1$ if $p_1 > 1$*

$$\left\| \frac{t^{\alpha-\frac{n}{p_1}} \|w_2(r)r^{\frac{n}{p_2}}\|_{L_{\theta_2}(0,\gamma t)} + t^\varepsilon \|v_2(r)r^{-\varepsilon}\|_{L_{\theta_2}(\gamma t,\infty)}}{t^{-\frac{n}{p_1}} \|w_1(r)r^{\frac{n}{p_1}}\|_{L_{\theta_1}(0,t)} + \|w_1(r)\|_{L_{\theta_1}(t,\infty)}} \right\|_{L_\infty(0,\infty)} < \infty \tag{5.14}$$

is necessary for the boundedness of the operator H_α from $GM_{p_1\theta_1,w_1(\cdot)}$ to $GM_{p_2\theta_2,w_2(\cdot)}$.

Proof. **1**. First let $1 < p_1 \le \infty$. We note that due to monotonicity in ε it suffices to prove the statements for sufficiently small $\varepsilon > 0$, say for $0 < \varepsilon < \frac{n}{p_1'}$. Consider the family of functions

$$f_{t,\varepsilon}(y) = |y|^{-\frac{n}{p_1}-\varepsilon} \chi_{{}^cB(0,t)}(y), \qquad y \in \mathbb{R}^n,$$

where $t > 0$ and $0 < \varepsilon < \frac{n}{p_1'}$. Then for all $x \in \mathbb{R}^n$ and $t > 0$

$$\|f_{t,\varepsilon}\|_{L_{p_1}(B(x,r))} = \Big\| |y|^{-\frac{n}{p_1}-\varepsilon} \Big\|_{L_{p_1}(B(x,r)\cap {}^cB(0,t))} .$$

Hence

$$\|f_{t,\varepsilon}\|_{L_{p_1}(B(0,r))} = 0 \qquad \text{if } r \le t \,, \tag{5.15}$$

$$\|f_{t,\varepsilon}\|_{L_{p_1}(B(x,r))} \le \Big\| |y|^{-\frac{n}{p_1}-\varepsilon} \Big\|_{L_{p_1}({}^cB(0,t))} = c_{10} t^{-\varepsilon} \,, \tag{5.16}$$

where $c_{10} = \left(\frac{nv_n}{\varepsilon p_1}\right)^{\frac{1}{p_1}}$, and

$$\|f_{t,\varepsilon}\|_{L_{p_1}(B(x,r))} \le t^{-\frac{n}{p_1}-\varepsilon} \|1\|_{L_{p_1}((B(x,r))} = c_{11} t^{-\frac{n}{p_1}-\varepsilon} r^{\frac{n}{p_1}}, \tag{5.17}$$

where $c_{11} = v_n^{\frac{1}{p_1}}$.

Also

$$(H_\alpha f_{t,\varepsilon})(y) = 0 \quad \text{if } |y| \le t,$$
$$(H_\alpha f_{t,\varepsilon})(y) = c_{12}|y|^{\alpha-n}\left(|y|^{\frac{n}{p_1'}-\varepsilon} - t^{\frac{n}{p_1'}-\varepsilon}\right) \quad \text{if } |y| \ge t,$$

where $c_{12} = n\left(\frac{n}{p_1'}-\varepsilon\right)^{-1} v_n^{\frac{\alpha}{n}}$. Note that, if $\gamma > 1$, $r \ge \gamma t$, and $\frac{\gamma+1}{2\gamma} r \le |y| \le r$, then

$$(H_\alpha f_{t,\varepsilon})(y) \ge c_{12}|y|^{\alpha-n}\Big(|y|^{\frac{n}{p_1'}-\varepsilon} - \Big(\frac{r}{\gamma}\Big)^{\frac{n}{p_1'}-\varepsilon}\Big) \ge c_{13} r^{\alpha-\frac{n}{p_1}-\varepsilon},$$

where

$$c_{13} = c_{12}\Big(\frac{\gamma+1}{2\gamma}\Big)^{(n-\alpha)_+}\Big(\Big(\frac{\gamma+1}{2\gamma}\Big)^{\frac{n}{p_1'}-\varepsilon} - \Big(\frac{1}{\gamma}\Big)^{\frac{n}{p_1'}-\varepsilon}\Big) > 0.$$

Therefore, if $\gamma > 1$ and $r \ge \gamma t$, then

$$\|H_\alpha f_{t,\varepsilon}\|_{L_{p_2}(B(0,r))} \ge \|H_\alpha f_{t,\varepsilon}\|_{L_{p_2}(B(0,r))\setminus(B(0,\frac{\gamma+1}{2\gamma}r))}$$

$$\ge c_{13}\left\|r^{\alpha-\frac{n}{p_1}-\varepsilon}\right\|_{L_{p_2}(B(0,r))\setminus(B(0,\frac{\gamma+1}{2\gamma}r))} \ge c_{14} r^{\alpha-n\left(\frac{1}{p_1}-\frac{1}{p_2}\right)-\varepsilon}, \tag{5.18}$$

where $c_{14} = c_{13} v_n^{\frac{1}{p_2}}\left(1-\left(\frac{\gamma+1}{2\gamma}\right)^n\right)^{\frac{1}{p_2}} > 0$.

Also, if $\gamma > 1, z \in \mathbb{R}^n, |z| = 2\gamma t$ and $r \le \gamma t$, then for all $y \in B(z,r)$ we have $\gamma t \le |y| \le 3\gamma t$ and

$$(H_\alpha f_{t,\varepsilon})(y) \ge c_{15} t^{\alpha-\frac{n}{p_1}-\varepsilon},$$

where $c_{15} = c_{12}\gamma^{\alpha-n}3^{((n-\alpha)_+)}\left(\gamma^{\frac{n}{p_1'}-\varepsilon}-1\right) > 0$, hence

$$\|H_\alpha f_{t,\varepsilon}\|_{L_{p_2}(B(z,r))} \ge c_{16} t^{\alpha-\frac{n}{p_1}-\varepsilon} r^{\frac{n}{p_2}}, \tag{5.19}$$

where $c_{16} = c_{15} v_n^{\frac{1}{p_2}}$.

If the operator $H_\alpha : LM_{p_1\theta_1,w_1(\cdot)} \to LM_{p_2\theta_2,w_2(\cdot)}$ is bounded, i.e. for some $c_{17} > 0$

$$\left\|w_2(r)\,\|H_\alpha f\|_{L_{p_2}(B(0,r))}\right\|_{L_{\theta_2(0,\infty)}} \le c_{17}\left\|w_1(r)\,\|f\|_{L_{p_1}(B(0,r))}\right\|_{L_{\theta_1(0,\infty)}}$$

for all $f \in LM_{p_1\theta_1,w_1(\cdot)}$, then by taking here $f = f_{t,\varepsilon}$ and applying (5.18), (5.15) and (5.16) we get that for all $t > 0$

$$c_{14}\left\|w_2(r) r^{\alpha-n\left(\frac{1}{p_1}-\frac{1}{p_2}\right)-\varepsilon}\right\|_{L_{\theta_2(\gamma t,\infty)}} \le c_{17} c_{10} t^{-\varepsilon}\|w_1(r)\|_{L_{\theta_1(t,\infty)}}, \tag{5.20}$$

hence the first statement follows.

If the operator $H_\alpha : GM_{p_1\theta_1,w_1(\cdot)} \to GM_{p_2\theta_2,w_2(\cdot)}$ is bounded, i.e. for some $c_{18} > 0$

$$\sup_{x\in\mathbb{R}^n} \left\| w_2(r) \, \|H_\alpha f\|_{L_{p_2}(B(x,r))} \right\|_{L_{\theta_2}(0,\infty)}$$
$$\le c_{18} \sup_{x\in\mathbb{R}^n} \left\| w_1(r) \, \|f\|_{L_{p_1}(B(x,r))} \right\|_{L_{\theta_1}(0,\infty)}$$

for all $f \in LM_{p_1\theta_1,w_1(\cdot)}$, then

$$\left\| w_2(r) \, \|H_\alpha f\|_{L_{p_2}(B(z,r))} \right\|_{L_{\theta_2}(0,\gamma t)} + \left\| w_2(r) \, \|H_\alpha f\|_{L_{p_2}(B(0,r))} \right\|_{L_{\theta_2}(\gamma t,\infty)}$$
$$\le \sup_{x\in\mathbb{R}^n} \left\| w_2(r) \, \|H_\alpha f\|_{L_{p_2}(B(x,r))} \right\|_{L_{\theta_2}(0,\gamma t)}$$
$$+ \sup_{x\in\mathbb{R}^n} \left\| w_2(r) \, \|H_\alpha f\|_{L_{p_2}(B(x,r))} \right\|_{L_{\theta_2}(\gamma t,\infty)}$$
$$\le 2 \sup_{x\in\mathbb{R}^n} \left\| w_2(r) \, \|H_\alpha f\|_{L_{p_2}(B(x,r))} \right\|_{L_{\theta_2}(0,\infty)}$$
$$\le c_{19} \Big(\sup_{x\in\mathbb{R}^n} \left\| w_1(r) \, \|f\|_{L_{p_1}(B(x,r))} \right\|_{L_{\theta_1}(0,t)}$$
$$+ \sup_{x\in\mathbb{R}^n} \left\| w_1(r) \, \|f\|_{L_{p_1}(B(x,r))} \right\|_{L_{\theta_1}(t,\infty)} \Big),$$

where $c_{19} = c_{18} 2^{((\frac{1}{\theta}_1 - 1)_+)+1}$. By taking here$f = f_{t,\varepsilon}$ and applying (5.19), (5.18), (5.17) and (5.16) we get that for all $t > 0$

$$c_{16} t^{\alpha - \frac{n}{p_1} - \varepsilon} \left\| w_2(r) r^{\frac{n}{p_2}} \right\|_{L_{\theta_2}(0,\gamma t)} + c_{14} \left\| v_2(r) r^{-\varepsilon} \right\|_{L_{\theta_2}(\gamma t,\infty)}$$
$$\le c_{19} \Big(c_{11} t^{-\frac{n}{p_1} - \varepsilon} \left\| w_1(r) r^{\frac{n}{p_1}} \right\|_{L_{\theta_1}(0,t)} + c_{10} t^{-\varepsilon} \left\| w_1 \right\|_{L_{\theta_1}(t,\infty)} \Big), \tag{5.21}$$

hence the second statement follows.

2. Let $p_1 = 1$ and $\gamma > 1$. Consider the family of functions

$$g_{t,\gamma}(y) = \chi_{B(0,\gamma_1 t) \setminus B(0,t)}, \quad t > 0,$$

where $\gamma_1 = \frac{\gamma+1}{2}$.

Then for all $x \in \mathbb{R}^n$ and $t > 0$

$$\|g_{t,\gamma}\|_{L_1(B(x,r))} = \|1\|_{L_1(B(x,r) \cap (B(0,\gamma_1 t) \setminus B(0,t)))}.$$

Hence

$$\|g_{t,\gamma}\|_{L_1(B(0,r))} = 0 \qquad \text{if } r \le t, \tag{5.22}$$

$$\|g_{t,\gamma}\|_{L_1(B(x,r))} \le \|1\|_{L_1(B(0,\gamma_1 t) \setminus B(0,t))} = c_{20} t^n, \tag{5.23}$$

where $c_{20} = v_n(\gamma_1^n - 1)$, and

$$\|g_{t,\gamma}\|_{L_1(B(x,r))} \le \|1\|_{L_1((B(x,r))} = c_{21} r^n, \tag{5.24}$$

where $c_{21} = v_n$.

Also

$$\left(H_\alpha g_{t,\gamma}\right)(y) = 0 \quad \text{if } |y| \le t,$$
$$\left(H_\alpha g_{t,\gamma}\right)(y) = c_{22}|y|^{\alpha-n}t^n \quad \text{if } |y| \ge \gamma_1 t,$$

where $c_{22} = v_n^{\frac{\alpha}{n}}(\gamma_1^n - 1)$.

Note that, if $r \ge \gamma t$ and $\frac{\gamma+1}{2\gamma}\, r \le |y| \le r$, then $|y| \ge \gamma_1 r$ and

$$\|H_\alpha g_{t,\gamma}\|_{L_{p_2}(B(0,r))} \ge \|H_\alpha g_{t,\gamma}\|_{L_{p_2}(B(0,r))\setminus(B(0,\frac{\gamma+1}{2\gamma}r))}$$
$$\ge c_{23} t^n r^{\alpha-n\left(1-\frac{1}{p_2}\right)} \tag{5.25}$$

where $c_{23} = c_{22}\gamma_1^{((n-\alpha)_+)} v_n^{\frac{1}{p_2}}\left(1 - \left(\frac{\gamma+1}{2\gamma}\right)^n\right)^{\frac{1}{p_2}} > 0$.

Also, if $z \in \mathbb{R}^n$, $|z| = 2\gamma t$ and $r \le \gamma t$, then for all $y \in B(z,r)$ we have $\gamma t \le |y| \le 3\gamma t$ and

$$\left(H_\alpha g_{t,\gamma}\right)(y) \ge c_{24} t^\alpha\,,$$

where $c_{24} = c_{22}\gamma^{\alpha-n}3^{(n-\alpha)_+}$, hence

$$\|H_\alpha g_{t,\gamma}\|_{L_{p_2}(B(z,r))} \ge c_{25} t^\alpha r^{\frac{n}{p_2}}\,, \tag{5.26}$$

where $c_{25} = c_{24} v_n^{\frac{1}{p_2}}$.

If the operator $H_\alpha : LM_{p_1\theta_1,w_1(\cdot)} \to LM_{p_2\theta_2,w_2(\cdot)}$ is bounded, then similarly to how inequality (5.20) was obtained, by taking $f = g_{t,\gamma}$ and applying (5.25), (5.22) and (5.23) we get that for all $t > 0$

$$c_{23} t^n \left\| w_2(r) r^{\alpha-n\left(1-\frac{1}{p_2}\right)} \right\|_{L_{\theta_2(\gamma t,\infty)}} \le c_{17} c_{20} t^n \left\|w_1(r)\right\|_{L_{\theta_1(t,\infty)}}\,,$$

hence the first statement with $\varepsilon = 0$ follows.

If the operator $H_\alpha : GM_{p_1\theta_1,w_1(\cdot)} \to GM_{p_2\theta_2,w_2(\cdot)}$ is bounded, then similarly to how inequality (5.21) was obtained, by taking $f = g_{t,\gamma}$ and applying (5.26), (5.25), (5.24) and (5.23) we get that for all $t > 0$

$$c_{25} t^\alpha \big\|w_2(r) r^{\frac{n}{p_2}}\big\|_{L_{\theta_2}(0,\gamma t)} + c_{23} t^n \big\|v_2(r)\big\|_{L_{\theta_2}(\gamma t,\infty)}$$
$$\le c_{19}\Big(c_{22}\big\|w_1(r) r^n\big\|_{L_{\theta_1}(0,t)} + c_{20} t^n \big\|w_1\big\|_{L_{\theta_1}(t,\infty)}\Big)\,,$$

hence the second statement with $\varepsilon = 0$ follows. □

Second proof of Lemma 6 (including the proof of the second statement of the lemma). If $\alpha > \frac{n}{p_1}$, or $\alpha = \frac{n}{p_1}$ and $\|w_2(r) r^{\frac{n}{p_2}}\|_{L_{\theta_2}(0,\infty)} = \infty$, then the numerator of the fraction in (5.14) tends to infinity as $t \to \infty$ whilst the denominator is bounded because for $t \ge 1$

$$t^{-\frac{n}{p_1}}\big\|w_1(r) r^{\frac{n}{p_1}}\big\|_{L_{\theta_1}(0,t)} + \big\|w_1(r)\big\|_{L_{\theta_1}(t,\infty)}$$
$$\le 2^{(\frac{1}{\theta_1}-1)_+}\left(\big\|w_1(r) r^{\frac{n}{p_1}}\big\|_{L_{\theta_1}(0,1)} + \big\|w_1(r)\big\|_{L_{\theta_1}(1,\infty)}\right) + \big\|w_1(r)\big\|_{L_{\theta_1}(1,\infty)}\,.$$

Hence $\alpha \le \frac{n}{p_1}$, and $\alpha < \frac{n}{p_1}$ if $\|w_2(r) r^{\frac{n}{p_2}}\|_{L_{\theta_2}(0,\infty)} = \infty$. □

Theorem 5. *Let* $1 \le p_1 \le \infty$, $0 < p_2 \le \infty, 0 < \theta_1 \le \theta_2 \le \infty$, *and conditions* (4.3), (4.5) *be satisfied.*

1. *Assume that* $w_1 \in \Omega_{\theta_1}$, $w_2 \in \Omega_{\theta_2}$ *and condition* (5.1) *is satisfied. If for* $p_1 = 1$ *for some* $\gamma > 1$ *and* $c_{26} > 0$ *for all* $t > 0$

$$\|v_2(r)\|_{L_{\theta_2(\gamma t,\infty)}} \ge c_{26}\|v_2(r)\|_{L_{\theta_2}(t,\infty)}\,, \tag{5.27}$$

or for $p_1 > 1$ *for some* $\varepsilon > 0, \gamma > 1$ *and* $c_{27} > 0$ *for all* $t > 0$

$$t^{\varepsilon}\|v_2(r)r^{-\varepsilon}\|_{L_{\theta_2(\gamma t,\infty)}} \ge c_{27}\|v_2(r)\|_{L_{\theta_2}(t,\infty)}\,, \tag{5.28}$$

where the function v_2 *is defined by equality* (5.6), *then condition* (2.2) *is necessary and sufficient for the boundedness of the operator* H_α *from* $LM_{p_1\theta_1,w_1(\cdot)}$ *to* $LM_{p_2\theta_2,w_2(\cdot)}$.

2. *Assume that* $w_1 \in \Omega_{p_1\theta_1}$, $w_2 \in \Omega_{p_2\theta_2}$, *condition* (5.1) *is satisfied, the function* $w_2(r)r^{\frac{n}{p_2}}$ *is almost increasing,* $\alpha \le \frac{n}{p_1}$, *and* $\alpha < \frac{n}{p_1}$ *if* $\|w_2(r)r^{\frac{n}{p_2}}\|_{L_{\theta_2}(0,\infty)} = \infty$. *If, in addition to* (5.27) *and* (5.28), *for some* $c_{28} > 0$ *for all* $t > 0$

$$t^{-\frac{n}{p_1}}\|w_1(r)r^{\frac{n}{p_1}}\|_{L_{\theta_1(0,t)}} \le c_{28}\|w_1(r)\|_{L_{\theta_1}(t,\infty)}\,, \tag{5.29}$$

then condition (2.2) *is also necessary and sufficient for the boundedness of the operator* H_α *from* $GM_{p_1\theta_1,w_1(\cdot)}$ *to* $GM_{p_2\theta_2,w_2(\cdot)}$.

Proof. The sufficiency follows by Theorem 3, the necessity follows by Theorem 4, (5.13), (5.27), (5.28), and (5.29). ☐

Corollary 3. *Let* $1 \le p_1 \le \infty$, $0 < p_2 \le \infty$, $0 < \theta_1 \le \theta_2 \le \infty$, *and conditions* (4.3), (4.5) *be satisfied.*

1. *Assume that* $w_1 \in \Omega_{\theta_1}$, $\lambda_2 > 0$ *and* $\alpha < \lambda_2 + \frac{n}{p_2'}$ *if* $\theta_2 < \infty$, $\lambda_2 \ge 0$ *and* $\alpha \le \lambda_2 + \frac{n}{p_2'}$ *if* $\theta_2 = \infty$. *Then the operator* H_α *is bounded from* $LM_{p_1\theta_1,w_1(\cdot)}$ *to* $LM^{\lambda_2}_{p_2\theta_2}$ *if and only if* [5]

$$\left\|t^{\alpha-n\left(\frac{1}{p_1}-\frac{1}{p_2}\right)-\lambda_2}\|w_1\|^{-1}_{L_{\theta_1}(t,\infty)}\right\|_{L_\infty(0,\infty)} < \infty\,. \tag{5.30}$$

2. *Assume that* $w_1 \in \Omega_{p_1\theta_1}$, *condition* (5.29) *holds,* $0 < \lambda_2 \le \frac{n}{p_2} - \frac{1}{\theta_2}$ *and* $\alpha < \lambda_2 + \frac{n}{p_2'}$ *if* $\theta_2 < \infty$, $0 \le \lambda_2 \le \frac{n}{p_2}$ *and* $\alpha \le \lambda_2 + \frac{n}{p_2'}$ *if* $\theta_2 = \infty$. *Then the operator* H_α *is bounded from* $GM_{p_1\theta_1,w_1(\cdot)}$ *to* $GM^{\lambda_2}_{p_2\theta_2}$ *also if and only if condition* (5.30) *is satisfied.*

Proof. Immediately follows by Theorem 5 if to take into account that the function $w_2(r)r^{\frac{n}{p_2}} = r^{\frac{n}{p_2}-\lambda_2-\frac{1}{\theta_2}}$ is non-decreasing since it is assumed that $\frac{n}{p_2} - \lambda_2 - \frac{1}{\theta_2} \ge 0$. ☐

Theorem 6. *Let* $1 \le p_1 \le \infty$, $0 < p_2 \le \infty, 0 < \theta_1 \le \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$, *and conditions* (4.3), (4.5), (5.1) *be satisfied.*

Assume that condition (5.27) *is satisfied if* $p_1 = 1$ *and condition* (5.28) *is satisfied if* $p_1 > 1$ *and let for all* $t > 0$

$$\|v_2\|_{L_{\theta_2}(t,\infty)} < \infty\,, \tag{5.31}$$

[5] For $\theta_1 = p_1$ this condition coincides with condition (3.11″).

where the function v_2 is defined by equality (5.6)*. Moreover, if $\theta_2 = \infty$ and $\theta_1 < \infty$ it is also assumed that*

$$\lim_{t\to\infty} \|v_2\|_{L_\infty(t,\infty)} = 0\,. \tag{5.32}$$

Then

1) H_α is bounded from $LM_{p_1\theta_1,w_1^*(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$, where w_1^* is a non-negative measurable function on $(0,\infty)$ defined by

$$\|w_1^*\|_{L_{\theta_1}(t,\infty)} = \|v_2\|_{L_{\theta_2}(t,\infty)}\,, \quad t\in(0,\infty). \tag{5.33}$$

2) If $w_1 \in \Omega_{\theta_1}$ and H_α is bounded from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$, then

$$LM_{p_1\theta_1,w_1(\cdot)} \subset LM_{p_1\theta_1,w_1^*(\cdot)}\,. \tag{5.34}$$

Thus the space $LM_{p_1\theta_1,w_1^*(\cdot)}$ is the maximal domain space for the operator H_α with the target space $LM_{p_2\theta_2,w_2(\cdot)}$ in the scale of spaces $\{LM_{p_1\theta_1,w_1(\cdot)},\ w_1\in\Omega_{\theta_1}\}$, i.e. the maximal among spaces $LM_{p_1\theta_1,w_1(\cdot)}$ for which H_α is bounded from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$.

Remark 6. If $0<\theta_1\le\theta_2<\infty$, then equality (5.33) defines the function w_1^* uniquely up to equivalence:

$$w_1^*(t) = \Big\{-\Big[\Big(\int_t^\infty v_2^{\theta_2}(r)\,dr\Big)^{\frac{\theta_1}{\theta_2}}\Big]'\Big\}^{\frac{1}{\theta_1}} = \Big(\frac{\theta_1}{\theta_2}\Big)^{\frac{1}{\theta_1}} \|v_2\|_{L_{\theta_2}(t,\infty)}^{1-\frac{\theta_2}{\theta_1}} (v_2(t))^{\frac{\theta_2}{\theta_1}} \tag{5.35}$$

for almost all $t\in(0,\infty)$. In particular, if $0<\theta_1=\theta_2<\infty$, then $w_1^*(t)=v_2(t)$ for almost all $t\in(0,\infty)$.

If $0<\theta_1<\theta_2=\infty$, then equality (5.33) also defines the function w_1^* uniquely up to equivalence:

$$w_1^*(t) = \Big\{-\Big[\|v_2\|_{L_\infty(t,\infty)}^{\theta_1}\Big]'\Big\}^{\frac{1}{\theta_1}} = \theta_1^{\frac{1}{\theta_1}} \|v_2\|_{L_\infty(t,\infty)}^{1-\frac{1}{\theta_1}} \big|(\|v_2\|_{L_\infty(t,\infty)})'\big|^{\frac{1}{\theta_1}} \tag{5.36}$$

for almost all $t\in(0,\infty)$.

If $0<\theta_1=\theta_2=\infty$, then equality (5.33) does not define the function w_1^* uniquely. However, this is not important because for different functions w_1^* satisfying (5.33) by Lemma 2 the spaces $LM_{p_1\theta_1,w_1^*(\cdot)}$ are the same. Under the additional assumptions that the function w_1^* is non-increasing and continuous on the right, equality (5.33) implies that

$$w_1^*(t) = \|v_2\|_{L_\infty(t,\infty)} \tag{5.37}$$

for all $t\in(0,\infty)$. [6]

[6] Indeed, if functions φ and ψ are non-increasing on $(0,\infty)$ and $\psi(t) = \operatorname{esssup}_{\tau>t}\varphi(\tau)$ for all $t>0$, then $\psi(t) = \lim\limits_{\tau\to t^+}\varphi(\tau) \equiv \varphi(t^+)$. (This follows since $\psi(t)\le\sup_{\tau>t}\varphi(\tau)$ and $\psi(t) = \sup_{\xi>t}\operatorname{esssup}_{t<\tau<\xi}\varphi(\tau) \ge \sup_{\xi>t}\varphi(\xi)$, hence $\psi(t) = \operatorname{esssup}_{\tau>t}\varphi(\tau) = \sup_{\tau>t}\varphi(\tau)$.) Therefore, if φ is continuous on the right at the point $t>0$, then $\varphi(t)=\varphi(t^+)=\psi(t)$.

Proof. 1) Statement 1 of the theorem follows by Theorem 5.

2) Let $w_1 \in \Omega_{\theta_1}$ and let H_α be bounded from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$. By Theorem 5 and equality (5.33) there exists $c_{29} > 0$ such that for all $t > 0$

$$\|w_1^*\|_{L_{\theta_1}(t,\infty)} = \|v_2\|_{L_{\theta_2}(0,\infty)} \le c_{29}\|w_1\|_{L_{\theta_1}(t,\infty)}.$$

Therefore, by the first statement of Lemma 2 inclusion (5.34) follows. □

Remark 7. Let us compare the necessary and sufficient conditions ensuring the boundedness of the operators H_α, M_α, and I_α in general local Morrey-type spaces.

This can be done if

$$1 < p_1 < p_2 < \infty, \quad 0 < \theta_1 \le \theta_2 \le \infty, \quad \alpha = n\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big),$$

$w_1 \in \Omega_{\theta_1}$, $w_2 \in \Omega_{\theta_2}$ and conditions (5.1), (5.28) are satisfied, when the necessary and sufficient conditions for all three operators H_α, M_α, and I_α are known.

Under these assumptions by Theorem 5 $H_{n(\frac{1}{p_1}-\frac{1}{p_2})}$ is bounded from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$ if and only if

$$\sup_{t>0} \|w_2\|_{L_{\theta_2}(t,\infty)}\|w_1\|^{-1}_{L_{\theta_1}(t,\infty)} < \infty\,,$$

by [6], [4] $M_{n(\frac{1}{p_1}-\frac{1}{p_2})}$ is bounded from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$ if and only if

$$\sup_{t>0} \Big(t^{-\frac{n}{p_2}}\|w_2(r)r^{\frac{n}{p_2}}\|_{L_{\theta_1}(t,\infty)} + \|w_2\|_{L_{\theta_2}(t,\infty)}\Big)\|w_1\|^{-1}_{L_{\theta_1}(t,\infty)} < \infty\,,$$

and by [7], [5] this condition is also necessary and sufficient for the boundedness of $I_{n(\frac{1}{p_1}-\frac{1}{p_2})}$ from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$.

Moreover, if

$$p_1 = 1, \quad 0 < p_2 < \infty, \quad 0 < \theta_1 \le \theta_2 < \infty, \quad n\Big(1 - \frac{1}{p_2}\Big)_+ < \alpha < n\,,$$

$w_1 \in \Omega_{\theta_1}$, $w_2 \in \Omega_{\theta_2}$ and conditions (5.1), (5.27) are satisfied, then by Theorem 5 H_α is bounded from $LM_{1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$ if and only if

$$\sup_{t>0} \|w_2(r)r^{\alpha-n(1-\frac{1}{p_2})}\|_{L_{\theta_2}(t,\infty)}\|w_1\|^{-1}_{L_{\theta_1}(t,\infty)} < \infty$$

and by [4] M_α is bounded from $LM_{1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$ if and only if

$$\sup_{t>0} \Big(t^{\alpha-n}\|w_2(r)r^{\frac{n}{p_2}}\|_{L_{\theta_2}(t,\infty)} + \|w_2\|_{L_{\theta_2}(t,\infty)}\Big)\|w_1\|^{-1}_{L_{\theta_1}(t,\infty)} < \infty\,. \tag{5.38}$$

If $0 < \theta_1 \le 1$, then condition (5.38) is also necessary and sufficient for the boundedness of I_α from $LM_{1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$. If $\theta_1 > 1$, then I_α is bounded from $LM_{1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$ if and only if apart from condition (5.38) also

$$\sup_{t>0} \|w_2(r)r^{\frac{n}{p_2}}\|_{L_{\theta_2}(0,t)} \left\| \frac{w_1^{\theta_1-1}(r)r^{\alpha-\frac{n}{p_1}}}{\|w_1\|^{\theta_1}_{L_{\theta_1}(r,\infty)}} \right\|_{L_{\theta_1'}(t,\infty)} < \infty\,.$$

(See [5].)

Clearly the conditions for the boundedness of H_α are in general weaker than for M_α and the conditions for the boundedness of M_α are in general weaker than for I_α which conforms with inequalities (1.4) and (1.5), though sometimes they coincide.

6 The case of Morrey spaces

Theorem 7. *Let* $1 \le p_1 \le \infty$, $0 < p_2 \le \infty$, $0 < \theta_1 \le \theta_2 \le \infty$. *Assume that conditions* (4.3) *and* (4.5) *are satisfied.*

1. *Assume that* $\lambda_i > 0$ *if* $\theta_i < \infty$; $\lambda_i \ge 0$ *if* $\theta_i = \infty$ $(i = 1, 2)$. *Then the operator* H_α *is bounded from* $LM^{\lambda_1}_{p_1\theta_1}$ *to* $LM^{\lambda_2}_{p_2\theta_2}$ *if and only if*

$$\alpha = \lambda_2 - \lambda_1 + n\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big)\,. \tag{6.1}$$

2. *Assume that* $\alpha < \frac{n}{p_1}$; $0 < \lambda_1 < \frac{n}{p_1}$ *if* $\theta_1 < \infty$; $0 \le \lambda_1 \le \frac{n}{p_1}$ *if* $\theta_1 = \infty$; $0 < \lambda_2 \le \frac{n}{p_2} - \frac{1}{\theta_2}$ *if* $\theta_2 < \infty$; $0 \le \lambda_2 \le \frac{n}{p_2}$ *if* $\theta_2 = \infty$. *Then the operator* H_α *is bounded from* $GM^{\lambda_1}_{p_1\theta_1}$ *to* $GM^{\lambda_2}_{p_2\theta_2}$ *also if and only if condition* (6.1) *is satisfied.*

Proof. Note that, for the spaces $LM^{\lambda_i}_{p_i\theta_i}$ and $GM^{\lambda_i}_{p_i\theta_i}$, $w_i(r) = r^{-\lambda_i - \frac{1}{\theta_i}}$ and by (6.1) for all $\varepsilon \ge 0$, $\gamma > 1$, and $t > 0$

$$t^\varepsilon \|v_2(r) r^{-\varepsilon}\|_{L_{\theta_2}(\gamma t,\infty)} = t^\varepsilon \|r^{-\lambda_2 - \frac{1}{\theta_2} + \alpha - n(\frac{1}{p_1} - \frac{1}{p_2}) - \varepsilon}\|_{L_{\theta_2}(\gamma t,\infty)}$$

$$= c_{30}\, \|r^{-\lambda_2 - \frac{1}{\theta_2} + \alpha - n(\frac{1}{p_1} - \frac{1}{p_2})}\|_{L_{\theta_2}(t,\infty)} = c_{30}\, \|v_2\|_{L_{\theta_2}(t,\infty)}\,,$$

where $c_{30} > 0$ is independent of t. Hence regularity conditions (5.27) and (5.28) are satisfied.

Also the function $w_2(r) r^{\frac{n}{p_2}} = r^{\frac{n}{p_2} - \lambda_2 - \frac{1}{\theta_2}}$ is non-decreasing since it is assumed that $\frac{n}{p_2} - \lambda_2 - \frac{1}{\theta_2} \ge 0$ and in the case of global Morrey spaces domination condition (5.29) is satisfied because for all $t > 0$

$$t^{-\frac{n}{p_1}} \|w_1(r) r^{\frac{n}{p_1}}\|_{L_{\theta_1}(0,t)} = t^{-\frac{n}{p_1}} \|r^{-\lambda_1 - \frac{1}{\theta_1} + \frac{n}{p_1}}\|_{L_{\theta_1}(0,t)}$$

$$= c_{31} \|r^{-\lambda_1 - \frac{1}{\theta_1}}\|_{L_{\theta_1}(t,\infty)} = c_{31} \|w_1(r)\|_{L_{\theta_1}(t,\infty)}\,,$$

where $c_{31} > 0$ is independent of t, since in this case $0 < \lambda_1 < \frac{n}{p_1}$ if $\theta_1 < \infty$ and $0 \le \lambda_1 \le \frac{n}{p_1}$ if $\theta_1 = \infty$.

Therefore the statements of the theorem follow by Corollary 3 because for $w_1(r) = r^{-\lambda_1 - \frac{1}{\theta_1}}$ condition (5.30) is equivalent to condition (6.1).

□

Remark 8. Note that if equation (6.1) is satisfied, then conditions (4.3) and (4.5) are equivalent to

$$\lambda_1 \le \lambda_2 \quad \text{for} \quad 1 < p_1 \le p_2 \le \infty \quad \text{or} \quad p_1 = 1 \quad \text{and} \quad p_2 = \infty\,, \tag{6.2}$$

and

$$\lambda_1 < \lambda_2 \quad \text{for} \quad p_1 = 1 \le p_2 < \infty \quad \text{or} \quad 0 < p_2 < p_1 \le \infty\,. \tag{6.3}$$

Also it follows that

$$\alpha < \lambda_2 + n\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big) \quad \text{if} \quad \theta_1 < \infty\,, \quad \alpha \le \lambda_2 + n\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big) \quad \text{if} \quad \theta_1 = \infty\,, \tag{6.4}$$

which is a stronger condition than the necessary condition (5.2) (since in Theorem 7 $p_1 \geq 1$ and $\theta_1 \leq \theta_2$).

In the case of global Morrey-type spaces the assumptions on λ_1 and λ_2 imply some further restrictions on the parameters:

$$\lambda_2 - \frac{n}{p_2} < \alpha \leq \frac{n}{p_1} - \lambda_1 - \frac{1}{\theta_2} \quad \text{if} \quad \theta_1 < \infty$$

or

$$\lambda_2 - \frac{n}{p_2} \leq \alpha \leq \frac{n}{p_1} - \lambda_1 - \frac{1}{\theta_2} \quad \text{if} \quad \theta_1 = \infty\,.$$

If $p_2 < p_1$, it may happen that $\alpha < 0$ for both local and global Morrey spaces (say, if $\lambda_1 = \frac{n}{2p_1}, \lambda_2 = \frac{n}{2p_2}$; in the case of global Morrey spaces it should also be assumed that $\theta_2 \geq \frac{2p_2}{n}$). In the case of local Morrey spaces it may also happen that $\alpha \geq n$ (say, $\lambda_1 = n, \lambda_2 = 3n, p_1 = p_2$). In such cases the boundedness of the Hardy operator does not follow by inequality (1.4) and the boundedness of the fractional maximal operator. For example, if $1 < p \leq \infty$ and $\lambda \geq 0$, then by Theorem 7

$$H_\alpha : LM^{\lambda}_{p,\infty} \to LM^{\lambda+\alpha}_{p,\infty}$$

for all $0 \leq \alpha < \infty$, but by applying inequality (1.4) and the boundedness of the fractional maximal operator M_α this can be proved only for $0 \leq \alpha < n$.

Corollary 4. *Let* $1 \leq p_1 \leq \infty$, $0 < p_2 \leq \infty$, $0 < \theta_1 \leq \theta_2 \leq \infty$. *Assume that conditions* (4.3) *and* (4.5) *are satisfied.*

Assume that $\lambda_2 > 0$ *if* $\theta_2 < \infty$, $\lambda_2 \geq 0$ *if* $\theta_2 = \infty$ *and that condition* (6.4) *is satisfied. Then the maximal domain space for the operator* H_α *with the power type target space* $LM^{\lambda_2}_{p_2\theta_2}$ *in the scale of the general local Morrey-type spaces* $\{LM_{p_1\theta_1,w_1(\cdot)}, w_1 \in \Omega_{\theta_1}\}$ *is the power type space* $LM^{\lambda_1}_{p_1\theta_1}$ *with* $\lambda_1 = \lambda_2 + n(\frac{1}{p_1} - \frac{1}{p_2}) - \alpha$.

Proof. Follows immediately by Theorem 6 and Remark 5. □

Remark 9. The necessity of condition (6.1) can be proved by using the 'dilation' argument. Indeed, let the operator H_α be bounded from $LM^{\lambda_1}_{p_1\theta_1}$ to $LM^{\lambda_2}_{p_2\theta_2}$, i.e., for all $f \in M^{\lambda_1}_{p_1\theta_1}$

$$\|H_\alpha f\|_{LM^{\lambda_2}_{p_2\theta_2}} \leq c_{32}\,\|f\|_{LM^{\lambda_1}_{p_1\theta_1}}\,, \tag{6.5}$$

where $c_{32} > 0$ is independent of f. Since $\tau_\varepsilon f \in LM^{\lambda_1}_{p_1\theta_1}$ for all $\varepsilon > 0$, this inequality also holds for $\tau_\varepsilon f$ for all $\varepsilon > 0$:

$$\|H_\alpha(\tau_\varepsilon f)\|_{LM^{\lambda_2}_{p_2\theta_2}} \leq c_{32}\,\|(\tau_\varepsilon f)\|_{LM^{\lambda_1}_{p_1\theta_1}}\,.$$

By (1.10) and (1.12), it follows that

$$\varepsilon^{\lambda_2 - \frac{n}{p_2} - \alpha - (\lambda_1 - \frac{n}{p_1})}\|H_\alpha f\|_{LM^{\lambda_2}_{p_2\theta_2}} \leq c_{32}\,\|f\|_{LM^{\lambda_1}_{p_1\theta_1}}$$

for all $\varepsilon > 0$ which is only possible if equality (6.1) holds.

A similar argument works if the spaces $LM^{\lambda_1}_{p_1\theta_1}$ and $LM^{\lambda_2}_{p_2\theta_2}$ are replaced by the spaces $GM^{\lambda_1}_{p_1\theta_1}$, $GM^{\lambda_2}_{p_2\theta_2}$ respectively.

Remark 10. Equality (6.1), rewritten in the form

$$\lambda_2 - \frac{n}{p_2} - \alpha = \lambda_1 - \frac{n}{p_1},$$

has a simple meaning: the sums of the differential dimension of the space and of the order of homogeneity of the operator in the left hand side and in the right hand side of inequality (6.5) coincide. (In the right hand side the operator is the identity operator whose order of homogeneity is equal to 0.)

Remark 11. In Theorem 7 it is assumed that $\theta_1 \le \theta_2$. It may happen that this condition is necessary for the boundedness of H_α from $LM^{\lambda_1}_{p_1\theta_1}$ to $LM^{\lambda_2}_{p_2\theta_2}$, at least, by Corollary 1, this is so if $\theta_1 = p_1$ and $\theta_2 = p_2$.

Remark 12. Let $1 \le p_1, p_2 \le \infty$. By Lemma 3 conditions (4.3) and (4.4) are necessary and sufficient for the validity of inequality (4.1). However, it may happen that they are not necessary for the boundedness of H_α from $LM^{\lambda_1}_{p_1\theta_1}$ to $LM^{\lambda_2}_{p_2\theta_2}$. (By Corollary 1, this is so if $\theta_1 = p_1$ and $\theta_2 = p_2$.) We note that the method developed in this paper does not allow investigating the case in which conditions (4.3) and (4.4) are not satisfied, because it is based on inequality (4.1), point-wise in $r > 0$, and a different approach in this case will be required.

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