EURASIAN MATHEMATICAL JOURNAL
volume 16 , number 1, 2025

CONTENTS

EURASIAN NATIONAL UNIVERSITY

ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

EURASIAN MATHEMATICAL JOURNAL

VOLUME 16, NUMBER 1 2025

ISSN 2077-9879
9 772077 987003

ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2025, Volume 16, Number 1

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

EURASIAN MATHEMATICAL JOURNAL

Aims and Scope

The Eurasian Mathematical Journal (EMJ) publishes carefully selected original research papers in all areas of mathematics written by mathematicians, principally from Europe and Asia. However papers by mathematicians from other continents are also welcome.

From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

The contents of the EMJ are indexed in Scopus, Web of Science (ESCI), Mathematical Reviews, MathSciNet, Zentralblatt Math (ZMATH), Referativnyi Zhurnal – Matematika, Math-Net.Ru.

The EMJ is included in the list of journals recommended by the Committee for Control of Education and Science (Ministry of Education and Science of the Republic of Kazakhstan) and in the list of journals recommended by the Higher Attestation Commission (Ministry of Education and Science of the Russian Federation).

Information for the Authors

Submission. Manuscripts should be written in LaTeX and should be submitted electronically in DVI, PostScript or PDF format to the EMJ Editorial Office through the provided web interface (www.enu.kz).

When the paper is accepted, the authors will be asked to send the tex-file of the paper to the Editorial Office.

The author who submitted an article for publication will be considered as a corresponding author. Authors may nominate a member of the Editorial Board whom they consider appropriate for the article. However, assignment to that particular editor is not guaranteed.

Copyright. When the paper is accepted, the copyright is automatically transferred to the EMJ. Manuscripts are accepted for review on the understanding that the same work has not been already published (except in the form of an abstract), that it is not under consideration for publication elsewhere, and that it has been approved by all authors.

Title page. The title page should start with the title of the paper and authors' names (no degrees). It should contain the Keywords (no more than 10), the Subject Classification (AMS Mathematics Subject Classification (2010) with primary (and secondary) subject classification codes), and the Abstract (no more than 150 words with minimal use of mathematical symbols).

Figures. Figures should be prepared in a digital form which is suitable for direct reproduction.

References. Bibliographical references should be listed alphabetically at the end of the article. The authors should consult the Mathematical Reviews for the standard abbreviations of journals' names.

Authors' data. The authors' affiliations, addresses and e-mail addresses should be placed after the References.

Proofs. The authors will receive proofs only once. The late return of proofs may result in the paper being published in a later issue.

Offprints. The authors will receive offprints in electronic form.

Publication Ethics and Publication Malpractice

For information on Ethics in publishing and Ethical guidelines for journal publication see http://www.elsevier.com/publishingethics and http://www.elsevier.com/journal-authors/ethics.

Submission of an article to the EMJ implies that the work described has not been published previously (except in the form of an abstract or as part of a published lecture or academic thesis or as an electronic preprint, see http://www.elsevier.com/postingpolicy), that it is not under consideration for publication elsewhere, that its publication is approved by all authors and tacitly or explicitly by the responsible authorities where the work was carried out, and that, if accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright-holder. In particular, translations into English of papers already published in another language are not accepted.

No other forms of scientific misconduct are allowed, such as plagiarism, falsification, fraudulent data, incorrect interpretation of other works, incorrect citations, etc. The EMJ follows the Code of Conduct of the Committee on Publication Ethics (COPE), and follows the COPE Flowcharts for Resolving Cases of Suspected Misconduct (http://publicationethics.org/files/u2/NewCode.pdf). To verify originality, your article may be checked by the originality detection service CrossCheck http://www.elsevier.com/editors/plagdetect.

The authors are obliged to participate in peer review process and be ready to provide corrections, clarifications, retractions and apologies when needed. All authors of a paper should have significantly contributed to the research.

The reviewers should provide objective judgments and should point out relevant published works which are not yet cited. Reviewed articles should be treated confidentially. The reviewers will be chosen in such a way that there is no conflict of interests with respect to the research, the authors and/or the research funders.

The editors have complete responsibility and authority to reject or accept a paper, and they will only accept a paper when reasonably certain. They will preserve anonymity of reviewers and promote publication of corrections, clarifications, retractions and apologies when needed. The acceptance of a paper automatically implies the copyright transfer to the EMJ.

The Editorial Board of the EMJ will monitor and safeguard publishing ethics.

The procedure of reviewing a manuscript, established by the Editorial Board of the Eurasian Mathematical Journal

1. **Reviewing procedure**

1.1. All research papers received by the Eurasian Mathematical Journal (EMJ) are subject to mandatory reviewing.

1.2. The Managing Editor of the journal determines whether a paper fits to the scope of the EMJ and satisfies the rules of writing papers for the EMJ, and directs it for a preliminary review to one of the Editors-in-chief who checks the scientific content of the manuscript and assigns a specialist for reviewing the manuscript.

1.3. Reviewers of manuscripts are selected from highly qualified scientists and specialists of the L.N. Gumilyov Eurasian National University (doctors of sciences, professors), other universities of the Republic of Kazakhstan and foreign countries. An author of a paper cannot be its reviewer.

1.4. Duration of reviewing in each case is determined by the Managing Editor aiming at creating conditions for the most rapid publication of the paper.

1.5. Reviewing is confidential. Information about a reviewer is anonymous to the authors and is available only for the Editorial Board and the Control Committee in the Field of Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan (CCFES). The author has the right to read the text of the review.

1.6. If required, the review is sent to the author by e-mail.

1.7. A positive review is not a sufficient basis for publication of the paper.

1.8. If a reviewer overall approves the paper, but has observations, the review is confidentially sent to the author. A revised version of the paper in which the comments of the reviewer are taken into account is sent to the same reviewer for additional reviewing.

1.9. In the case of a negative review the text of the review is confidentially sent to the author.

1.10. If the author sends a well reasoned response to the comments of the reviewer, the paper should be considered by a commission, consisting of three members of the Editorial Board.

1.11. The final decision on publication of the paper is made by the Editorial Board and is recorded in the minutes of the meeting of the Editorial Board.

1.12. After the paper is accepted for publication by the Editorial Board the Managing Editor informs the author about this and about the date of publication.

1.13. Originals reviews are stored in the Editorial Office for three years from the date of publication and are provided on request of the CCFES.

1.14. No fee for reviewing papers will be charged.

2. **Requirements for the content of a review**

2.1. In the title of a review there should be indicated the author(s) and the title of a paper.

2.2. A review should include a qualified analysis of the material of a paper, objective assessment and reasoned recommendations.

2.3. A review should cover the following topics:

- compliance of the paper with the scope of the EMJ;

- compliance of the title of the paper to its content;

- compliance of the paper to the rules of writing papers for the EMJ (abstract, key words and phrases, bibliography etc.);

- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);

- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

2.4. The final part of the review should contain an overall opinion of a reviewer on the paper and a clear recommendation on whether the paper can be published in the Eurasian Mathematical Journal, should be sent back to the author for revision or cannot be published.

Web-page

The web-page of the EMJ is www.emj.enu.kz. One can enter the web-page by typing Eurasian Mathematical Journal in any search engine (Google, Yandex, etc.). The archive of the web-page contains all papers published in the EMJ (free access).

Subscription

Subscription index of the EMJ 76090 via KAZPOST.

E-mail

eurasianmj@yandex.kz

The Eurasian Mathematical Journal (EMJ)
The Astana Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana, Republic of Kazakhstan

The Moscow Editorial Office
The Patrice Lumumba Peoples' Friendship University of Russia
(RUDN University)
Room 473
3 Ordzonikidze St
117198 Moscow, Russian Federation

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 16, Number 1 (2025), 49 – 59

MEASURE OF NONCOMPACTNESS APPROACH TO NONLINEAR FRACTIONAL PANTOGRAPH DIFFERENTIAL EQUATIONS

A. El Mfadel, S. Melliani

Communicated by V.I. Burenkov

Key words: Ψ-fractional integral, Ψ-Caputo fractional derivative, topological degree theory.

AMS Mathematics Subject Classification: 26A33, 34A08, 47H08.

Abstract. The aim of this manuscript is to explore the existence and uniqueness of solutions for a class of nonlinear Ψ-Caputo fractional pantograph differential equations subject to nonlocal conditions. The proofs rely on key results in topological degree theory for condensing maps, coupled with the method of measures of noncompactness and essential tools in Ψ-fractional calculus. To support the theoretical findings, a nontrivial example is presented as an application.

DOI: https://doi.org/10.32523/2077-9879-2025-16-1-49-59

1 Introduction

The fractional calculus which permits the integration and differentiation of functions with non-integer orders, is one of the fastest-growing fields of mathematics due to the discovery that fractional operators were utilized in mathematical modeling, see [10, 19, 20, 27, 29]. Fractional differential equations, which can be used to model and describe non-homogeneous physical events, have recently attracted a lot of attention, particularly initial and boundary value problems for nonlinear fractional differential equations. Different researchers have found some interesting solutions to initial and boundary value problems for fractional differential equations involving various fractional derivatives, including their existence and uniqueness, such as Riemann-Liouville [23], Caputo [3], Hilfer [24], Erdelyi-Kober [25] and Hadamard [2]. There is a certain type of kernel dependency included in all those definitions. Therefore, a fractional derivative with respect to another function known as the Ψ-Caputo derivative was introduced in order to study fractional differential equations in a general manner. For specific selections of Ψ, we can obtain some well-known fractional derivatives, such as the Caputo, Caputo-Hadamard, or Caputo-Erdelyi-Kober fractional derivatives, which depend on a kernel. From the viewpoint of applications, this approach also seems appropriate. With the help of a good selection of a "trial" function Ψ, the Ψ-Caputo fractional derivative allows some measure of control over the modeling of the phenomenon under consideration. Almeida et al. [5] investigated the existence and uniqueness results for nonlinear fractional differential equations involving a Ψ-Caputo-type fractional derivative by using fixed point theorems and Picard iteration method. For more details, the reader can also consult [7, 16, 17, 18, 28] and references therein. In particular, the pantograph equation was employed as a useful tool to shed light on some of the modern problems originating from several scientific disciplines, including electrodynamics, probability, quantum mechanics, and number theory. However, a substantial investigation on the characteristics of this type of fractional differential equation, both analytical and numerical, has been done, and intriguing findings have been published in [1, 6, 8, 9, 13, 14, 15].

Inspired by the above recent results, in this paper, we investigate the existence and uniqueness of solutions to the following nonlinear fractional pantograph differential equation with Ψ-Caputo type fractional derivatives of order $\beta \in (1,2)$:

$$\begin{cases} {}^{C}D_{0^+}^{\beta,\Psi}u(t) = h(t,u(t),u(\varepsilon t)), & t \in J = [0,T], \\ u'(0) = 0, \quad u(0) + \omega(u) = u_0, \end{cases} \tag{1.1}$$

where ${}^{C}D_{0^+}^{\beta,\Psi}$ is the $\Psi-$Caputo fractional derivative of u of order β, $\varepsilon \in (0,1)$, $T > 0$, $h \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $u_0 \in \mathbb{R}$ and $\omega : C(J) \to \mathbb{R}$ is a nonlocal term satisfying some given conditions, which will be stated in Section 3. For more details we refer the reader to [14, 15].

To the best of the authors' knowledge, topological degree theory for condensing maps has not been applied to nonlinear pantograph differential equations with Ψ-Caputo fractional derivatives.

The structure of this paper is as follows. In Section 2, we give some basic definitions and preliminary results that we will need to prove our main results. In Section 3, we prove the existence of solutions for pantograph equation (1.1). After that, we give a concrete example to illustrate our main results in Section 4 and the last Section 5 contains conclusions on the results obtained in the paper.

2 Basic concepts

This section deals with some preliminaries and notations which are used throughout this paper. For more details we refer the reader to [4].

Definition 1. [5] Let $q > 0$, $g \in L^1(J,\mathbb{R})$ and $\Psi \in C^n(J,\mathbb{R})$ be such that $\Psi'(t) > 0$ for all $t \in J$. The Ψ-Riemann-Liouville fractional integral of order q of a function g is given by

$$I_{0^+}^{q,\Psi}g(t) = \frac{1}{\Gamma(q)} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{q-1} g(s)ds, \tag{2.1}$$

where $\Gamma(.)$ is the Euler Gamma function.

Definition 2. [5] Let $q > 0$, $g \in C^{n-1}(J,\mathbb{R})$ and $\Psi \in C^n(J,\mathbb{R})$ such that $\Psi'(t) > 0$ for all $t \in J$. The Ψ-Caputo fractional derivative of order q of a function g is given by

$${}^{C}D_{0^+}^{q,\Psi}g(t) = \frac{1}{\Gamma(n-q)} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{n-q-1} g_{\Psi}^{\{n\}}(s)ds, \tag{2.2}$$

where $g_{\Psi}^{\{n\}}(s) = \left(\frac{1}{\Psi'(s)}\frac{d}{ds}\right)^n g(s)$ and $n = [q] + 1$ ($[q]$ denotes the integer part of the real number q).

Remark 1. In particular, if $q \in]0,1[$, then we have

$${}^{C}D_{0^+}^{q,\Psi}g(t) = \frac{1}{\Gamma(q)} \int_0^t (\Psi(t) - \Psi(s))^{q-1} g'(s)ds,$$

and

$${}^{C}D_{0^+}^{q,\Psi}g(t) = I_{0^+}^{1-q,\Psi}\left(\frac{g'(t)}{\Psi'(t)}\right).$$

Proposition 2.1. [5] *Let $q > 0$, if $g \in C^{n-1}(J, \mathbb{R})$, then we have*

1) ${}^{C}D_{0^+}^{q,\Psi} I_{0^+}^{q,\Psi} g(t) = g(t)$.

2) $I_{0^+}^{q,\Psi}\ {}^{C}D_{0^+}^{q,\Psi} g(t) = g(t) - \sum_{k=0}^{n-1} \frac{g_{\Psi}^{[k]}(0)}{k!}(\Psi(t) - \Psi(0))^k$.

3) $I_{0^+}^{q,\Psi}$ *is linear and bounded from* $\mathcal{C}$ *to* $\mathcal{C}$.

Proposition 2.2. [5] *Let $\mu > \nu > 0$ and $t \in J$, then*

1) $I_{0^+}^{\mu,\Psi}(\Psi(t) - \Psi(0))^{\nu-1} = \frac{\Gamma(\nu)}{\Gamma(\mu+\nu)}(\Psi(t) - \Psi(0))^{\mu+\nu-1}$.

2) $D_{0^+}^{\mu,\Psi}(\Psi(t) - \Psi(0))^{\nu-1} = \frac{\Gamma(\nu)}{\Gamma(\nu-\mu)}(\Psi(t) - \Psi(0))^{\nu-\mu-1}$.

3) $D_{0^+}^{\mu,\Psi}(\Psi(t) - \Psi(0))^{k} = 0, \quad \forall k \in \mathbb{N}$.

Definition 3. [11] Let X be a Banach space with the norm $\|.\|$ and $\mathfrak{B}_X$ be the family of all non-empty and bounded subsets of X. The Kuratowski measure of non-compactness is the mapping $\rho : \mathfrak{B}_X \to [0, +\infty[$ defined by: for any $A \in \mathfrak{B}_X$

$\rho(A) = \inf\{\ r > 0\text{: } A \text{ admits a finite cover by sets of diameter} \leq r\}$.

Proposition 2.3. [11] *The Kuratowski measure of noncompactness ρ satisfies the following assertions: for any $A, A_1, A_2 \in \mathfrak{B}_X$*

1. $\rho(A) = 0$ *if and only if A is relatively compact.*
2. $\rho(kA) = |k|\rho(A), \quad k \in \mathbb{R}$.
3. $\rho(A_1 + A_2) \leq \rho(A_1) + \rho(A_2)$.
4. *If $A_1 \subset A_2$ then $\rho(A_1) \leq \rho(A_2)$.*
5. $\rho(A_1 \cup A_2) = \max\{\rho(A_1),\ \rho(A_2)\}$.
6. $\rho(A) = \rho(\overline{A}) = \rho(\mathrm{conv}A)$ *where $\overline{A}$ and $\mathrm{conv}A$ denote the closure and the convex hull of A, respectively.*

Definition 4. [11] Let $\varphi : \Omega \subset X \to X$ be a continuous bounded map. We say that φ is ρ-Lipschitz if there exists $l \geq 0$ such that

$$\rho(\varphi(A)) \leq l\rho(A), \quad \text{for every} \quad A \subset \Omega.$$

Moreover, if $l < 1$ then we say that φ is a strict ρ-contraction.

Definition 5. [11] We say that a function ω is ρ-condensing if

$$\rho(\omega(A)) < \rho(A),$$

for every bounded subset A of Ω with $\rho(A) > 0$. In other words

$$\rho(\omega(A)) \geq \rho(A) \Rightarrow \rho(A) = 0.$$

Definition 6. [11] We say that a function $g : \Omega \to X$ is Lipschitz if there exists $l > 0$ such that

$$\| g(u) - g(v) \| \leq l \| u - v \|, \quad \text{for all} \quad u, v \in \Omega.$$

Moreover, if $l < 1$ then we say that g is a strict contraction.

Lemma 2.1. [11] *If $L, F : \Omega \to X$ are ρ-Lipschitz mappings with the constants l_1 respectively l_2, then the mapping $F + L : \Omega \to X$ is ρ-Lipschitz with the constant $l_1 + l_2$.*

Lemma 2.2. [11] *If $g : \Omega \to X$ is compact, then g is ρ-Lipschitz with constant $c = 0$.*

Lemma 2.3. [11] *If $g : \Omega \to X$ is Lipschitz with constant l, then g is $\rho-$Lipschitz with the same constant l.*

Theorem 2.1. (See Isaia [22]). *Let $\mathcal{H} : X \to X$ be ρ-condensing and*

$$\mathcal{E}_\gamma = \{x \in X : x = \gamma \mathcal{H} x \text{ for some } 0 \leq \gamma \leq 1\}.$$

If $\mathcal{E}_\gamma$ is a bounded set in X, then there exists $r > 0$ such that $\mathcal{S}_\gamma \subset B_r = \{x \in X : ||x|| \leq r\}, r > 0$, and we have

$$\deg(I - \delta\mathcal{H}, B_r, 0) = 1, \quad \forall \delta \in [0, 1],$$

where $\deg(\cdot, \cdot, \cdot)$ denotes the topological degree in the sense of Leray-Schauder.

As a consequence, the operator $\mathcal{H}$ has at least one fixed point and the set of all fixed points of $\mathcal{H}$ lies in B_r .

3 Main results

We start this section by introducing necessary notations and hypotheses on the functions $\omega \in C(\mathbb{R}, \mathbb{R})$ and $h \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, entering equation (1.1).
• We denote by $\mathcal{C} := C(J, \mathbb{R})$ the space of continuous real-valued functions defined on J provided with the maximum norm

$$\| u \| = \max_{t \in J} |u(t)|.$$

• We denote by B_η the closed ball in $\mathcal{C}$ centred at 0 of radius $\eta > 0$.

(H_1) There exists a constant $L_\omega > 0$ such that

$$|\omega(u) - \omega(v)| \leq L_\omega \|u - v\|, \text{ for each } u, v \in \mathcal{C}.$$

(H_2) There exist two constants $K_\omega, M_\omega > 0$ and $q \in (0, 1)$ such that

$$|\omega(u)| \leq K_\omega \|u\|^q + M_\omega \text{ for each } u \in \mathcal{C}.$$

(H_3) There exist two constants $K_h, M_h > 0$ and $p \in (0, 1)$ such that

$$|h(t, u(t), u(\varepsilon t))| \leq K_h \mid u(t) \mid^p + M_h \text{ for each } u \in \mathcal{C}, \epsilon \in (0, 1).$$

Lemma 3.1. *A function $u \in \mathcal{C}$ is a solution of (1.1) if and only if u satisfies the following fractional integral equation*

$$u(t) = u_0 - \omega(u) + \frac{1}{\Gamma(\beta)} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\beta-1} h(s, u(s), u(\varepsilon s)) ds, t \in J. \tag{3.1}$$

Proof. Let u be a solution to (1.1), then by applying Ψ-fractional integral $I_{0^+}^{\beta,\Psi}$ to both sides of (1.1) we obtain

$$I_{0^+}^{\beta,\Psi}\ {}^C D_{0^+}^{\beta,\Psi} u(t) = I_{0^+}^{\beta,\Psi} h(t, u(t), u(\varepsilon t)),$$

and by employing Proposition 2.1 we get

$$u(t) = c_0 + (\Psi(t) - \Psi(0))c_1 + I_{0^+}^{\beta,\Psi} h(t, u(t), u(\varepsilon t)),$$

where $c_0, c_1 \in \mathbb{R}$, hence,

$$u'(t) = c_1\Psi'(t) + \frac{1}{\Gamma(\beta)}\int_0^t \left(\Psi'(s)(\Psi(t) - \Psi(s))^{\beta-1} h(s, u(s), u(\varepsilon s))\right)_t' ds.$$

Since $u(0) + \omega(u) = u_0$ and $u'(0) = 0$, then $c_0 = u_0 - \omega(u)$ and $c_1 = 0$. Hence, (3.1) holds.

Conversely, by simple calculus, it is clear that if u satisfies (3.1), then (1.1) holds. □

To prove that (3.1) has at least one solution $u \in \mathcal{C}$, we consider two operators $\mathcal{B}, \mathcal{A} : \mathcal{C} \to \mathcal{C}$ defined by

$$\mathcal{A}u(t) = u_0 - \omega(u(t)), \quad t \in J, \tag{3.2}$$

and

$$\mathcal{B}u(t) = \frac{1}{\Gamma(\beta)}\int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\beta-1} h(s, u(s), u(\varepsilon s))ds, \quad t \in J, \tag{3.3}$$

thus (3.1) can be formulated as follows

$$u(t) = \mathcal{F}u(t) := \mathcal{A}u(t) + \mathcal{B}u(t), \quad t \in J. \tag{3.4}$$

Theorem 3.1. *Assume that hypotheses $(H_1) - (H_3)$ are satisfied, then fractional pantograph differential equation (1.1) has at least one solution $u \in C(J, \mathbb{R})$. Moreover, the set of all solutions to (1.1) is bounded in $C(J, \mathbb{R})$.*

In order to prove the Theorem 3.1, we will need to show some lemmas and preliminary results under the assumption that hypotheses $(H_1) - (H_3)$ are satisfied.

Lemma 3.2. *The operator $\mathcal{A}$ is ρ-Lipschitz with the constant L_ω. Moreover, $\mathcal{A}$ satisfies the following inequality:*

$$\|\mathcal{A}u\| \leq |u_0| + K_\omega\|u\|^q + M_\omega, \text{for every } u \in \mathcal{C}. \tag{3.5}$$

Proof. To prove that the operator $\mathcal{A}$ is Lipschitz with the constant L_ω,we argue as follows.

Let $u, v \in \mathcal{C}$, then we have

$$|\mathcal{A}u(t) - \mathcal{A}v(t)| \leq |\omega(u) - \omega(v)|,$$

by using hypothesis (H_1) we get

$$|\mathcal{A}u(t) - \mathcal{A}v(t)| \leq L_\omega\|u - v\|.$$

Taking supremum over t, we obtain

$$\|\mathcal{A}u - \mathcal{A}v\| \leq L_\omega\|u - v\|,$$

hence, $\mathcal{A}$ is Lipschitz with the constant L_ω. By using Lemma 2.3, it follows that $\mathcal{A}$ is ρ-Lipschitz with the same constant L_ω.

To prove (3.5), let $u \in \mathcal{C}$, then we have

$$|\mathcal{A}u(t)| = |u_0 - \omega(u)| \leq |u_0| + |\omega(u)|,$$

by using assumption (H_2) we get

$$\|\mathcal{A}u\| \leq |u_0| + K_\omega \|u\|^q + M_\omega.$$

□

Lemma 3.3. *The operator $\mathcal{B}$ is continuous and the following inequality holds*

$$\|\mathcal{B}u\| \leq \frac{1}{\Gamma(\beta+1)}(K_\omega\|u\|^p + M_\omega)(\Psi(T)-\Psi(0))^\beta, \quad \forall u \in \mathcal{C}. \tag{3.6}$$

Proof. To prove that the operator $\mathcal{B}$ is continuous, let a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{C}$ converge to u in $\mathcal{C}$, it follows that there exists $\delta > 0$ such that $\|u_n\| \leq \delta$ and $\|u\| \leq \delta$. Now let $t \in J$, then we have

$$|\mathcal{B}u_n(t) - \mathcal{B}u(t)| \leq \frac{1}{\Gamma(\beta)} \int_0^t \Psi'(s)(\Psi(t)-\Psi(s))^{\beta-1} \left|h(s,u_n(s),u_n(\varepsilon s)) - h(s,u(s),u(\varepsilon s))\right| ds.$$

Since h is continuous, we have

$$\lim_{n\to\infty} h(s,u_n(s),u_n(\varepsilon s)) = h(s,u(s),u(\varepsilon s)).$$

On the other hand, by using (H_3) we obtain

$$\begin{aligned}\frac{1}{\Gamma(\beta)}(\Psi'(s)(\Psi(t)-\Psi(s))^{\beta-1} \left\|h(s,u_n(s),u_n(\varepsilon s)) - h(s,u(s),u(\varepsilon s))\right\| &\leq (K_\omega \delta^p + M_\omega) \\ &\times \frac{1}{\Gamma(\beta)}(\Psi'(s)(\Psi(t)-\Psi(s))^{\beta-1},\end{aligned}$$

since $s \mapsto \frac{1}{\Gamma(\beta)}(\Psi'(s)(\Psi(t)-\Psi(s))^{\beta-1}$ is an integrable function on $[0,t]$, then Lebesgue dominated convergence theorem implies that

$$\lim_{n\mapsto+\infty} \frac{1}{\Gamma(\beta)} \int_0^t \Psi'(s)(\Psi(t)-\Psi(s))^{\beta-1} \left\|h(s,u_n(s),u_n(\varepsilon s)) - h(s,u(s),u(\varepsilon s))\right\| ds = 0.$$

It follows that

$$\lim_{n\mapsto+\infty} \| \mathcal{B}u_n - \mathcal{B}u \| = 0,$$

hence, $\mathcal{B}$ is continuous .

To show (3.6), let $u \in \mathcal{C}$, then we have

$$|\mathcal{B}u(t)| \leq \frac{1}{\Gamma(\beta)} \int_0^t \Psi'(s)(\Psi(t)-\Psi(s))^{\beta-1} \left|h(s,u(s),u(\varepsilon s))\right| ds,$$

from (H_3) we obtain

$$|\mathcal{B}u(t)| \leq \frac{(K_\omega\|u\|^p + M_\omega)}{\Gamma(\beta)} \int_0^t \Psi'(s)(\Psi(t)-\Psi(s))^{\beta-1} ds,$$

Finally, we obtain

$$\| \mathcal{B}u \| \leq \frac{(K_\omega \|u\|^p + M_\omega)(\Psi(T) - \Psi(0))^\beta}{\Gamma(\beta+1)}.$$

$\square$

Lemma 3.4. *$\mathcal{B} : \mathcal{C} \to \mathcal{C}$ is a compact operator.*

Proof. In order to demonstrate the compactness of $\mathcal{B}$ we need to show that $\mathcal{B}B_\eta$ is relatively compact in $\mathcal{C}$ and we use the Arzela-Ascoli Theorem [21]. Let $u \in B_\eta$, then from (3.6) we get

$$\| \mathcal{B}u \| \leq \frac{(K_\omega \eta^p + M_\omega)(\Psi(T) - \Psi(0))^\beta}{\Gamma(\beta+1)} := \xi.$$

So, it follows that $\mathcal{B}B_\eta \subset B_\xi$. Hence $\mathcal{B}B_\eta$ is bounded.

To prove that $\mathcal{B}B_\eta$ is is uniformly equicontinuous, let $u \in \mathcal{B}B_\eta$ and $t_1, t_2 \in J$ such that $t_1 < t_2$, then we have

$$|\mathcal{B}u(t_2) - \mathcal{B}u(t_1)| \leq \frac{K_\omega \mid u \mid^p + M_\omega}{\Gamma(\beta)} \int_{t_1}^{t_2} \Psi'(s)(\Psi(t_2) - \Psi(s))^{\beta-1} ds,$$

$$|\mathcal{B}u(t_2) - \mathcal{B}u(t_1)| \leq \frac{K_\omega \eta^p + M_\omega}{\Gamma(\beta)} \int_{t_1}^{t_2} \Psi'(s)(\Psi(t_2) - \Psi(s))^{\beta-1} ds,$$

$$|\mathcal{B}u(t_2) - \mathcal{B}u(t_1)| \leq \frac{K_\omega \eta^p + M_\omega}{\Gamma(\beta+1)} (\Psi(t_2) - \Psi(t_1))^\beta,$$

$$\sup_{u \in \mathcal{B}B_\gamma} \sup_{|t_1 - t_2| \leq \delta} |\mathcal{B}u(t_2) - \mathcal{B}u(t_1)| \leq \frac{K_\omega \eta^\beta + M_\omega}{\Gamma(\beta+1)} \sup_{|t_1 - t_2| \leq \delta} |\Psi(t_2) - \Psi(t_1)|^\beta.$$

Since Ψ is a continuous function on the closed interval J, then we obtain

$$\lim_{\delta \to 0^+} \sup_{u \in \mathcal{B}B_\gamma} \sup_{|t_1 - t_2| \leq \delta} |\mathcal{B}u(t_1) - \mathcal{B}u(t_2)| = 0.$$

which shows that $\mathcal{B}B_\eta$ is uniformly equicontinuous.

Hence, $\mathcal{B}B_\eta$ is uniformly bounded and is uniformly equicontinuous. Arzelà–Ascoli Theorem [21] permits us to conclude that $\mathcal{B}B_\eta$ is relatively compact, thus $\mathcal{B}$ is compact. $\square$

Corollary 3.1. *$\mathcal{B} : \mathcal{C} \to \mathcal{C}$ is ρ-Lipschitz with zero constant.*

Proof. Since the operator $\mathcal{B}$ is compact and by Lemma 2.2 it follows that $\mathcal{B}$ is ρ-Lipschitz with zero constant. $\square$

Now, we have all tools to establish the proof of Theorem 3.1.

Proof of Theorem 3.1.

Let $\mathcal{A}, \mathcal{B}, \mathcal{F}$: $\mathcal{C} \to \mathcal{C}$ be the operators given by equations (3.2),(3.3) and (3.4) respectively.

The operators $\mathcal{A}, \mathcal{B}, \mathcal{F}$ are continuous and bounded. Moreover, by using Lemma 3.2 we have that $\mathcal{A}$ is ρ-Lipschitz with constant $L_\omega \in [0,1)$ and by using Corollary 3.1 we have that $\mathcal{A}$ is ρ-Lipschitz with zero constant. It follows from Lemma 2.1 that $\mathcal{F}$ is a strict ρ-contraction with constant L_ω.

We consider the following set

$$\mathcal{S}_\gamma = \{u \in \mathcal{C} : u = \gamma \mathcal{F} u \ \text{ for some } \ \gamma \in [0,1]\}.$$

Let us show that $\mathcal{S}_\gamma$ is bounded in $\mathcal{C}$. For this purpose let $u \in \mathcal{S}_\gamma$, then $u = \gamma\mathcal{F}u = \gamma(\mathcal{A}u + \mathcal{B}u)$. It follows that

$$\|u\| = \gamma\|\mathcal{F}u\| \leq \gamma(\|\mathcal{A}u\| + \|\mathcal{B}u\|),$$

by using Lemmas 3.2 and 3.3 we get

$$\|u\| \leq |u_0| + K_\omega\|u\|^q + M_\omega + \frac{(K_h\|u\|^p + M_h)(\Psi(T) - \Psi(0))^\beta}{\Gamma(\beta+1)}. \tag{3.7}$$

From inequality (3.7) we deduce that $\mathcal{S}_\gamma$ is bounded in $\mathcal{C}$ with $p < 1$ and $q < 1$.

If this is not the case, we suppose that $\xi := \|u\| \longrightarrow \infty$. Dividing both sides of (3.7) by ξ, and taking $\xi \to \infty$, it follows that

$$1 \leq \lim_{\xi\to\infty} \frac{|u_0| + K_\omega\xi^q + M_\omega + \dfrac{(\mathbb{K}_\omega\xi^p + M_\omega)(\Psi(T) - \Psi(0))^\beta}{\Gamma(\beta+1)}}{\xi} = 0,$$

which is a contradiction. By using Theorem 2.1 we conclude that $\mathcal{F}$ has at least one fixed point which is a solution of (1.1) and the set of the fixed points of $\mathcal{F}$ is bounded in $\mathcal{C}$. $\square$

Remark 2. Note that if assumptions (H_2) and (H_3) are formulated for $q = 1$ and $p = 1$, then the conclusions of Theorem 3.1 remain valid provided that

$$K_\omega + \frac{K_h(\Psi(T) - \Psi(0))^\beta}{\Gamma(\beta+1)} < 1.$$

4 An illustrative example

In this section, we give an example to illustrate the usefulness of our main result.
Consider the following problem:

$$\begin{cases} {}^C D_{0^+}^{\frac{3}{2}, e^t} u(t) = h(t, u(t), u(\varepsilon t)), \quad t \in J = [0,1], \\[2ex] u'(0) = 0, \quad u(0) = \displaystyle\sum_{j=1}^{20} \theta_j |u(t_j)|, \quad \theta_j > 0, \quad 0 < t_j < 1, \quad j = 1, 2, .., 20, \end{cases} \tag{4.1}$$

where $h(t, u(t), u(\varepsilon t)) = \frac{\sin\left(u\left(\frac{t}{\sqrt{2}}\right)\right)}{(9+e^t)\sqrt{2}} \left(\frac{|u(t)|}{1+\left|u\left(\frac{t}{\sqrt{2}}\right)\right|}\right)$.

Here $\varepsilon = \frac{1}{\sqrt{2}}$, $\beta = \frac{3}{2}$, $T = 1$, $\Psi(t) = e^t$, and $\omega(u) = \displaystyle\sum_{j=1}^{20} \theta_j|u(t_j)|$ with $\sum\limits_{j=1}^{20} \theta_j < 1$. Clearly $h \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and (H_1), (H_2) hold with $K_\omega = L_\omega = \sum\limits_{i=j}^{20} \theta_j$, $M_\omega = 0$ and $q = 1$. Indeed, we can write

$$|\omega(u)| \leq \sum_{j=1}^{20} \theta_j \|u\|,$$

thus, $K_\omega = \sum_{j=1}^{20} \theta_j, \quad M_\omega = 0$ and $q = 1$.

Moreover, we have

$$|\omega(u(t)) - \omega(v(t))| = \Big| \sum_{j=1}^{20} \theta_j |u(t_j)| - \sum_{j=1}^{20} \theta_j |v(t_j)| \Big|,$$

hence,

$$|\omega(u) - \omega(v)| \leq \sum_{j=1}^{20} \theta_j \|u - v\|,$$

thus, in (H_2), $L_\omega = \sum_{j=1}^{20} \theta_j$.

To prove (H_3), let $t \in J$ and $u \in \mathbb{R}$, then we have

$$|h(t, u(t), u(\varepsilon t))| = \left| \frac{sin\left(u\left(\frac{t}{\sqrt{2}}\right)\right)}{(9+e^t)\sqrt{2}} \left(\frac{|u(t)|}{1 + \left|u\left(\frac{t}{\sqrt{2}}\right)\right|} \right) \right|,$$

$$|h(t, u(t), u(\varepsilon t))| \leq \frac{1}{10\sqrt{2}} (|u| + 1).$$

Thus, (H_3) holds with $K_h = M_h = \dfrac{1}{10\sqrt{2}}$ and $p = 1$.

Consequently, Theorem 3.1 implies that problem (4.1) has at least one solution. Moreover, by inequality (3.7) we have

$$\|u\| \leq \frac{(e-1)^{(3/2)}}{10\sqrt{2}\Gamma(8/3) - 1} \approx 0.19,$$

thus, the set of all solutions to (4.1) is bounded.

5 Conclusion

In this paper, we studied the existence of solutions to nonlinear pantograph differential equations involving Caputo type fractional derivative with respect to another function Ψ. The proofs of our proposed model are based on the topological degree theory for condensing maps. We also provided an example to make our results clear.

Acknowledgement

The authors are grateful to the anonymous referee for his/her valuable remarks that led to the improvement of the original manuscript.

References

[1] M.S. Abdo, T. Abdeljawad, K.D. Kucche, M.A. Alqudah, S.M. Ali, M.B. Jeelani, *On nonlinear pantograph fractional differential equations with Atangana-Baleanu-Caputo derivative*. Adv. Differ. Equ.(65) (2021), 1-17.

[2] R.P. Agarwal, S.K. Ntouyas, B. Ahmad, A.K. Alzahrani, *Hadamard-type fractional functional differential equations and inclusions with retarded and advanced arguments*. Advances in Difference Equations, 1 (2016), 1–15.

[3] R.P. Agarwal, S. Hristova, D. O'Regan, *A survey of Lyapunov functions, stability and impulsive Caputo fractional differential equations*. Fract. Calc. Appl. Anal., 19 (2016), 290–318.

[4] R. Almeida, *Caputo fractional derivative of a function with respect to another function*. Commun. Nonlinear Sci. Numer. Simul., 44 (2017), 460–481.

[5] R. Almeida, A.B. Malinowska, M.T.T. Monteiro, *Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications*. Mathematical Methods in the Applied Sciences, 41 (2018), 336–352.

[6] K. Balachandran, S. Kiruthika, J. Trujillo, *Existence of solutions of nonlinear fractional pantograph equations*. Acta Math. Sci., 33 (2013), 712–720.

[7] D. Baleanu, H. Jafari, H. Khan, S.J. Johnston, *Results for mild solution of fractional coupled hybrid boundary value problems*. Open Math., 13(2015), 601–608.

[8] S. Bhalekar, J. Patade, *Series solution of the pantograph equation and its properties*. Fractal Fract., 1(2017), 1–6.

[9] S. Bouriah, D. Foukrach, M. Benchohra, J. Graef, *Existence and uniqueness of periodic solutions for some nonlinear fractional pantograph differential equations with $\Psi-$Caputo derivative*. Arabian Journal of Mathematics, 10 (2021), 575–587.

[10] M. Caputo, *Linear models of dissipation whose Q is almost frequency independent*. International Journal of Geographical Information Science, 13 (1967), 529–539.

[11] K. Deimling, *Nonlinear functional analysis*. Springer-Verlag, New York, (1985).

[12] B.C. Dhage, *On β-condensing mappings in Banach algebras*. Math. Student, 63(1994), 146–152.

[13] A. El Mfadel, F.E. Bourhim, M. Elomari, *Existence of mild solutions for semilinear Ψ-Caputo-type fractional evolution equations with nonlocal conditions in Banach spaces*. Results in Nonlinear Analysis, 5 (2022), 459–472.

[14] A. El Mfadel, S. Melliani, M. Elomari, *On the initial value problem for fuzzy nonlinear fractional differential equations*. 48(2024), 547–554.

[15] A. El Mfadel, S. Melliani, M. Elomari, *Existence of solutions for nonlinear Ψ-Caputo-type fractional hybrid differential equations with periodic boundary conditions*. Asia Pac. J. Math. (2022).

[16] A. El Mfadel, S. Melliani, M. Elomari, *Existence and uniqueness results for Caputo fractional boundary value problems involving the p-Laplacian operator*. U.P.B. Sci. Bull. Series A, 84 (2022), 37–46.

[17] A. El Mfadel, S. Melliani, M. Elomari, *New existence results for nonlinear functional hybrid differential equations involving the Ψ-Caputo fractional derivative*. Results in Nonlinear Analysis, 5 (2022), 78–86.

[18] A. El Mfadel, S. Melliani, M. Elomari, *Existence results for nonlocal Cauchy problem of nonlinear Ψ-Caputo type fractional differential equations via topological degree methods*. Advances in the Theory of Nonlinear Analysis and its Applications, 6 (2022), 270–279.

[19] A. El Mfadel, S. Melliani, M. Elomari, *On the existence and uniqueness results for fuzzy linear and semilinear fractional evolution equations involving Caputo fractional derivative*. Journal of Function Spaces (2021).

[20] A. El Mfadel, S. Melliani, M. Elomari, *A note on the stability analysis of fuzzy nonlinear fractional differential equations involving the Caputo fractional derivative*. International Journal of Mathematics and Mathematical Sciences (2021).

[21] J.W. Green, F.A. Valentine. *On the ArzelΓ -Ascoli theorem*. Mathematics Magazine, 34 (1961), 199–202.

[22] F. Isaia, *On a nonlinear integral equation without compactness*. Acta. Math. Univ. Comenianae, 75 (2006), 233–240.

[23] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and applications of fractional differential equations*. North-Holland Mathematical Studies 204. Ed van Mill. Amsterdam. Elsevier Science B.V. Amsterdam, (2006).

[24] R. Hilfer, *Applications of fractional calculus in physics*. Singapore, (2000).

[25] Y. Luchko, J.J. Trujillo, *Caputo-type modification of the Erdelyi-Kober fractional derivative*. Fract. Calc. Appl. Anal., 10 (2007), 249–267.

[26] M. Liu, D. Li, *Properties of analytic solution and numerical solution of multi-pantograph equation*. Appl. Math. Comput., 155(2004), 3853–871

[27] F. Mainardi, *Fractals and fractional calculus continuum mechanics*. Springer Verlag (1997).

[28] S.K. Ntouyas, *Boundary value problems for nonlinear fractional differential equations and inclusions with nonlocal and fractional integral boundary conditions*. Opuscula Math., 33 (2013), 117–138.

[29] I. Podlubny, *Fractional differential equations*. Mathematics in Science and Engineering. Academic Press, New York (1999).

[30] J. Schauder, *Der fixpunktsatz in functionalrΓ¤umen*. Studia Math., 2 (1930), 171–180.

Ali El Mfadel
Laboratory of Applied Mathematics and Scientific Computing,
Sultan Moulay Slimane University, Beni Mellal, Morocco.
E-mails:a.elmfadel@usms.ma
and
Higher School of Technology,
Sultan Moulay Slimane University, Khenifra, Morocco.
E-mail: elmfadelali@gmail.com

Said Melliani
Laboratory of Applied Mathematics and Scientific Computing,
Sultan Moulay Slimane University,Beni Mellal, Morocco.
E-mail: s.melliani@usms.ma

Received: 20.12.2023