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CONSTRUCTIVE METHOD FOR SOLVING OF ONE CLASS OF CURVILINEAR INTEGRAL EQUATIONS OF THE FIRST KIND

E.H. Khalilov

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Key words: Dirichlet boundary value problems, Helmholtz equation, normal derivative, double-layer potential, integral equations of the first kind.

AMS Mathematics Subject Classification: 45E05; 31B10.

Abstract. A new method for the construction of a quadrature formula for the normal derivative of the double-layer potential is developed and a method for calculating the approximate solution of the integral equation of the first kind for Dirichlet boundary value problems for the Helmholtz equation in the two-dimensional space is presented in this work.

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1 Introduction and problem statement

It is known that in many cases it is impossible to find an exact solution of Dirichlet boundary value problems for the Helmholtz equation in the two-dimensional space. This generates interest for studying approximate solution of these problems with theoretical justification. One of the methods to solve Dirichlet boundary value problem for the Helmholtz equation in two-dimensional space is to reduce it to an integral equation of the first kind. Note that the main advantage of applying the method of integral equations to exterior boundary value problems is that this method allows reducing the problem for an unbounded domain to the one for a bounded domain of lower dimension.

Let $D \subset R^2$ be a bounded domain with twice continuously differentiable boundary L, and f be a given continuous function on L. Consider the Dirichlet boundary value problems for the Helmholtz equation:

Interior Dirichlet problem. Find a function u, which is twice continuously differentiable on D, continuous on $\bar{D}$, and satisfies the Helmholtz equation $\Delta u + k^2 u = 0$ in D and the boundary condition $u = f$ on L, where Δ is the Laplace operator, and k is a wave number with $Im\, k \geq 0$.

Exterior Dirichlet problem. Find a function u, which is twice continuously differentiable on $R^2 \backslash \bar{D}$, continuous on $R^2 \backslash D$, satisfies the Helmholtz equation in $R^2 \backslash \bar{D}$, Sommerfeld radiation condition

$$\left(\frac{x}{|x|}, \operatorname{grad} u(x) \right) - i\, k\, u(x) = o\left(\frac{1}{|x|^{1/2}} \right), \quad x \to \infty,$$

uniformly in all directions $x/|x|$ and the boundary condition $u = f$ on L.

It was shown in [3, p. 87] that the simple-layer potential

$$u(x) = \int_L \Phi(x, y)\, \varphi(y)\, dL_y, \quad x \in R^2 \backslash L,$$

with continuous density φ is a solution of the interior and exterior Dirichlet boundary value problems if φ is a solution of the integral equation of the first kind

$$S\varphi = 2\,f, \tag{1.1}$$

where

$$(S\varphi)(x) = 2\int_L \Phi(x,y)\,\varphi(y)\,dL_y, \quad x \in L,$$

$\Phi(x,\,y)$ is the fundamental solution of the Helmholtz equation, i.e.

$$\Phi(x,\,y) = \begin{cases} \frac{1}{2\pi}\ln\frac{1}{|x-y|} & for\ \ k=0, \\ \frac{i}{4}H_0^{(1)}(k\,|x-y|) & for\ \ k\neq 0, \end{cases}$$

where $H_0^{(1)}$ is the zero degree Hankel function of the first kind defined by the formula $H_0^{(1)}(z) = J_0(z) + i\,N_0(z)$,

$$J_0(z) = \sum_{m=0}^{\infty}\frac{(-1)^m}{(m!)^2}\left(\frac{z}{2}\right)^{2m}$$

is the Bessel function of zero degree,

$$N_0(z) = \frac{2}{\pi}\left(\ln\frac{z}{2} + C\right)J_0(z) + \sum_{m=1}^{\infty}\left(\sum_{l=1}^{m}\frac{1}{l}\right)\frac{(-1)^{m+1}}{(m!)^2}\left(\frac{z}{2}\right)^{2m}$$

is the Neumann function of zero degree, and $C = 0.57721...$ is Euler's constant.

Note that the integral equations of the first kind do not fit into the Riesz-Fredholm theory. But, it was proved in [3, p. 89–90] that if $Im\,k > 0$, then the operators S and

$$(Tf)(x) = 2\,\frac{\partial}{\partial\nu(x)}\left(\int_L \frac{\partial\Phi(x,\,y)}{\partial\nu(y)}\,f(y)dL_y\right), \quad x \in L,$$

are invertible, and

$$T^{-1} = -S\left(I - \tilde{K}\right)^{-1}\left(I + \tilde{K}\right)^{-1},$$

where

$$\left(\tilde{K}\rho\right)(x) = 2\int_L \frac{\partial\Phi(x,\,y)}{\partial\nu(x)}\,\rho(y)dL_y, \quad x \in L,$$

$\nu(x)$ is the outer unit normal at the point $x \in L$, and I is the unit operator in $C(L)$, the space of all continuous functions on L with the norm $\|\varphi\|_\infty = \max\limits_{x\in L}|\varphi(x)|$. Then the inverse operator S^{-1} is defined by

$$S^{-1} = -\left(I - \tilde{K}\right)^{-1}\left(I + \tilde{K}\right)^{-1}T.$$

Consequently, the solution of equation (1.1) has the form

$$\varphi = -2\left(I - \tilde{K}\right)^{-1}\left(I + \tilde{K}\right)^{-1}Tf. \tag{1.2}$$

Note that in spite of invertibility of the operators $I - \tilde{K}$ and $I + \tilde{K}$, the explicit forms of the inverse operators $\left(I - \tilde{K}\right)^{-1}$ and $\left(I + \tilde{K}\right)^{-1}$ are unknown. Besides, Lyapunov's counterexample shows ([6, p. 89–90]) that the derivatives of the double-layer potential with continuous density, in general, do not exist, i.e. the operator S^{-1}, inverse to the compact operator S, is unbounded in $\mathrm{N}(L)$, the space of all continuous functions φ, whose double-layer potential with the density φ has continuous normal derivatives on both sides of the curve L. Note that in [17], quadrature formulas for the simple-layer and double-layer potentials have been constructed using the asymptotic formula for the zero degree Hankel functions of the first kind, which does not allow to find the convergence rate of these quadrature formulas. But, in [11], quadrature formulas for the simple-layer and double-layer potentials have been constructed by using more practical method, and in [12], quadrature formulas for the normal derivative of the simple-layer potential have been constructed and the error estimates have been obtained for the constructed quadrature formulas. Further, in [2, 16], quadrature formulas for the normal derivative of the simple-layer and double-layer logarithmic potentials have been constructed and approximate solutions for integral equations of the exterior Dirichlet boundary value problem and the mixed problem for the Laplace equation have been studied in the two-dimensional space. In [10, 13], a new method for the construction of a cubature formula for the normal derivative of the acoustic double-layer potential has been proposed and justification of the collocation method for the integral equations of exterior Dirichlet and Neumann boundary value problems for the Helmholtz equation has been given in the three-dimensional space. However, it is known that the fundamental solution of the Helmholtz equation in three-dimensional space has the form

$$\Phi_k(x,\,y)=\frac{\exp\left(ik\left|x-y\right|\right)}{4\pi\left|x-y\right|},\;\;x,y\in R^3,\;\;x\neq y,$$

which differs essentially from the fundamental solution of the Helmholtz equation in the two-dimensional space. Also note that in [18, p. 115–116], considering normal derivative of the double-layer potential as a hypersingular integral, i.e. considering integral in the sense of finite value according to Hadamard, quadrature formula for the normal derivative of the double-layer potential has been constructed using subdomain method with an additional condition on the density of f ([18, p. 285–291]). It is known that with this condition the expression for the normal derivative of the double-layer potential can be represented in the form of singular integral ([3, p. 57], [18, p. 100]), i.e. the integral $(Tf)(x)$, $x\in L$, exists in the sense of the Cauchy principal value. Besides, it should be noted that the quadrature formula constructed in [18] is not practical, in other words, its coefficients are singular integrals.

Despite important results in the field of numerical solution of integral equations of the first kind ([4, 5, 7, 8, 20]), due to the above reasons, approximate solving of Dirichlet boundary value problems for the Helmholtz equation in the two-dimensional space has not yet been studied by the method of integral equations of the first kind (1.1). In this work, considering the normal derivative of the double-layer potential as an integral in the sense of the Cauchy principal value, we construct a quadrature formula for the normal derivative of the double-layer potential by a more practical method, and, using formula (1.2), we give a method for calculating an approximate solution to equation (1.1) at some selected points.

2 Approximate solution to equation (1.1)

Assume that the curve L is defined by the parametric equation $x\left(t\right)=\left(x_1\left(t\right),x_2\left(t\right)\right)$, $t\in[a,b]$. Let us divide the interval $[a,b]$ into $n>2M_0\left(b-a\right)/d$ equal parts: $t_p=a+\frac{(b-a)\,p}{n}$, $p=\overline{0,n}$, where

$$M_0=\max_{t\in[a,b]}\sqrt{\left(x_1'\left(t\right)\right)^2+\left(x_2'\left(t\right)\right)^2}<+\infty$$

(see [19, p. 561]) and d is the standard radius ([21, p. 400]). As control points, we consider $x\left(\tau_p\right)$, $p=\overline{1,n}$, where $\tau_p=a+\frac{(b-a)\,(2p-1)}{2n}$. Then the curve L is divided into elementary parts: $L=\bigcup\limits_{p=1}^{n}L_p$, where $L_p=\{\,x\left(t\right):\;\;t_{p-1}\le t\le t_p\}$.

It is known ([14]) that

(1) $\forall p \in \{1, 2, ..., n\}$: $r_p(n) \sim R_p(n)$, where

$$r_p(n) = \min\{\, |x(\tau_p) - x(t_{p-1})|\,,\, |x(t_p) - x(\tau_p)|\,\},$$

$$R_p(n) = \max\{\, |x(\tau_p) - x(t_{p-1})|\,,\, |x(t_p) - x(\tau_p)|\,\},$$

and $a(n) \sim b(n)$ means $C_1 \leq \dfrac{a(n)}{b(n)} \leq C_2$, with the positive constants C_1 and C_2 independent of n.

(2) $\forall p \in \{1, 2, ..., n\} : R_p(n) \leq d/2$;

(3) $\forall p, j \in \{1, 2, ..., n\} : r_j(n) \sim r_p(n)$;

(4) $r(n) \sim R(n) \sim \frac{1}{n}$, where $R(n) = \max\limits_{p=\overline{1,n}} R_p(n)$, $r(n) = \min\limits_{p=\overline{1,n}} r_p(n)$.

The following lemma is true.

Lemma 2.1. [14]. *There exist constants $C_0' > 0$ and $C_1' > 0$, independent of n, such that the inequalities*

$$C_0'\ |y - x(\tau_p)| \leq |x(\tau_j) - x(\tau_p)| \leq C_1'\ |y - x(\tau_p)|$$

hold for $\forall p, j \in \{1,\ 2, ...,\ n\}\,, j \neq p$, and $\forall y \in L_j$.

Let

$$\Phi_n(x,\ y) = \frac{i}{4} H_{0,n}^{(1)}(k\,|x-y|)\,, \quad x, y \in L, \quad x \neq y,$$

where

$$H_{0,n}^{(1)}(z) = J_{0,n}(z) + i\,N_{0,n}(z)\,, J_{0,n}(z) = \sum_{m=0}^{n} \frac{(-1)^m}{(m!)^2} \left(\frac{z}{2}\right)^{2m}$$

and

$$N_{0,n}(z) = \frac{2}{\pi}\left(\ln\frac{z}{2} + C\right) J_{0,n}(z) + \sum_{m=1}^{n} \left(\sum_{l=1}^{m} \frac{1}{l}\right) \frac{(-1)^{m+1}}{(m!)^2} \left(\frac{z}{2}\right)^{2m}.$$

It is not difficult to show that

$$\frac{\partial \Phi_n(x,y)}{\partial \nu(x)} = \frac{i}{4}\left(\frac{\partial J_{0,n}(k\,|x-y|)}{\partial \nu(x)} + i\,\frac{\partial N_{0,n}(k\,|x-y|)}{\partial \nu(x)}\right),$$

where

$$\frac{\partial J_{0,n}(k\,|x-y|)}{\partial \nu(x)} = (x-y,\, \nu(x)) \sum_{m=1}^{n} \frac{(-1)^m k^{2m}\,|x-y|^{2m-2}}{2^{2m-1}(m-1)\,!\,m\,!}$$

and

$$\frac{\partial N_{0,n}(k\,|x-y|)}{\partial \nu(x)} =$$

$$= \frac{2}{\pi}\left(\ln\frac{k\ |x-y|}{2} + C\right) \frac{\partial J_{0,n}(k\,|x-y|)}{\partial \nu(x)} + \frac{2\ (x-y,\, \nu(x))}{\pi\ |x-y|^2} J_{0,n}(k\,|x-y|) +$$

$$+ (x-y,\, \nu(x)) \sum_{m=1}^{n} \left(\sum_{l=1}^{m} \frac{1}{l}\right) \frac{(-1)^{m+1} k^{2m}\,|x-y|^{2m-2}}{2^{2m-1}(m-1)\,!\,m\,!}.$$

Consider the matrix $\tilde{K}^n = \left(\tilde{k}_{pj}\right)_{p,j=1}^{n}$ with the elements

$$\tilde{k}_{pj} = \frac{2\ |sgn\,(p-j)|\ (b-a)}{n} \frac{\partial \Phi_n(x(\tau_p)\,,\, x(\tau_j))}{\partial \nu(x(\tau_p))} \sqrt{(x_1'(\tau_j))^2 + (x_2'(\tau_j))^2}.$$

It was proved in [12] that if $\varphi \in C(L)$, then the expression

$$\left(\tilde{K}_n \varphi\right)(x(\tau_p)) = \sum_{\substack{j=1 \\ j \neq p}}^{n} \tilde{k}_{pj}\, \varphi(x(\tau_j))$$

is a quadrature formula for the integral $\left(\tilde{K}\varphi\right)(x)$ at the control points $x(\tau_p)$, $p = \overline{1,\, n}$, with[1]

$$\max_{p=\overline{1,\, n}} \left| \left(\tilde{K}\varphi\right)(x(\tau_p)) - \left(\tilde{K}_n\varphi\right)(x(\tau_p)) \right| \leq M \left(\omega(\varphi,\, 1/n) + \|\varphi\|_\infty \frac{\ln n}{n} \right),$$

where $\omega(\varphi,\, \delta)$ is the modulus of continuity of the function φ, i.e.

$$\omega(\varphi,\, \delta) = \max_{\substack{|x-y| \leq \delta \\ x,\, y \in L}} |\varphi(x) - \varphi(y)|, \quad \delta > 0.$$

It is known that if $Im\, k > 0$, then for every right-hand side $g \in C(L)$ the integral equations ([3, p. 81])

$$\varphi \pm \tilde{K}\varphi = g$$

are uniquely solvable in the space $C(L)$. Then, proceeding in the same way as in [9], it is not difficult to prove the following lemmas.

Lemma 2.2. *If $Im k > 0$, then there exists the inverse matrix $\left(I^n + \tilde{K}^n\right)^{-1}$ with*

$$M_1 = \sup_n \left\| \left(I^n + \tilde{K}^n\right)^{-1} \right\| < +\infty$$

and

$$\max_{l=\overline{1,n}} \left| \left(\left(I + \tilde{K}\right)^{-1} g \right)(x(\tau_l)) - \sum_{j=1}^{n} \tilde{k}^{+}_{lj}\, g(x(\tau_l)) \right| \leq M \left(\omega(g, 1/n) + \|g\|_\infty \frac{\ln n}{n} \right),$$

where I^n is a unit operator in the space C^n, and $\tilde{k}^{+}_{lj}$ is the element of the matrix $\left(I^n + \tilde{K}^n\right)^{-1}$ in the $l-th$ row and $j-th$ column.

Lemma 2.3. *If $Im k > 0$, then there exists the inverse matrix $\left(I^n - \tilde{K}^n\right)^{-1}$ with*

$$M_2 = \sup_n \left\| \left(I^n - \tilde{K}^n\right)^{-1} \right\| < +\infty$$

and

$$\max_{l=\overline{1,n}} \left| \left(\left(I - \tilde{K}\right)^{-1} g \right)(x(\tau_l)) - \sum_{j=1}^{n} \tilde{k}^{-}_{lj}\, g(x(\tau_l)) \right| \leq M \left(\omega(g, 1/n) + \|g\|_\infty \frac{\ln n}{n} \right),$$

where $\tilde{k}^{-}_{lj}$ is the element of the matrix $\left(I^n - \tilde{K}^n\right)^{-1}$ in the $l-th$ row and $j-th$ column.

[1] Hereinafter M denotes different positive constants which can be different in different inequalities.

Now, let us construct a quadrature formula for the normal derivative of the double-layer potential. For this, let us first determine the conditions for the existence of the normal derivative of the double-layer potential and derive the formulas for calculating it.

Lemma 2.4. *Let a function ρ be continuously differentiable on L and*

$$\int_0^{\operatorname{diam} L} \frac{\omega\left(\operatorname{grad}\rho,\, t\right)}{t} dt < +\infty.$$

Then the double-layer potential

$$W\left(x\right) = \int_L \frac{\partial\Phi\left(x,\, y\right)}{\partial\nu\left(y\right)} \rho\left(y\right) dL_y, \quad x \in L,$$

has the normal derivative in L, with

$$\frac{\partial W\left(x\right)}{\partial\nu\left(x\right)} = \int_L \frac{\partial V\left(x, y\right)}{\partial\nu\left(x\right)} \rho\left(y\right) dL_y - \frac{1}{\pi}\int_L \frac{\left(x-y, \nu\left(y\right)\right)\,\left(x-y, \nu\left(x\right)\right)}{\left|x-y\right|^4}\left(\rho\left(y\right)-\rho\left(x\right)\right) dL_y +$$

$$+\frac{1}{2\pi}\int_L \frac{\left(\nu\left(y\right),\, \nu\left(x\right)\right)}{\left|x-y\right|^2}\left(\rho\left(y\right)-\rho\left(x\right)\right) dL_y, \;\; x \in L \tag{2.1}$$

and

$$\left|\frac{\partial W(x)}{\partial\nu\left(x\right)}\right| \le M\left(\left\|\rho\right\|_\infty + \left\|\operatorname{grad}\rho\right\|_\infty + \int_0^d \frac{\omega\left(\operatorname{grad}\rho, t\right)}{t} dt\right), \forall\, x \in L,$$

where

$$V\left(x, y\right) = \left(\frac{i}{4} - \frac{C}{2\pi} - \frac{1}{2\pi}\ln\frac{k\,\left|x-y\right|}{2}\right)\left(y-x,\, \nu\left(y\right)\right)\sum_{m=1}^{\infty}\frac{\left(-1\right)^m k^{2m}\left|x-y\right|^{2m-2}}{2^{2m-1}\left(m-1\right)!\,m!} -$$

$$-\left(y-x,\, \nu\left(y\right)\right)\sum_{m=1}^{\infty}\left(\sum_{l=1}^{m}\frac{1}{l}\right)\frac{\left(-1\right)^{m+1} k^{2m}\left|x-y\right|^{2m-2}}{2^{2m+1}\left(m-1\right)!\,m!} -$$

$$-\frac{1}{2\pi}\left(y-x,\, \nu\left(y\right)\right)\sum_{m=1}^{\infty}\frac{\left(-1\right)^m k^{2m}\left|x-y\right|^{2m-2}}{2^{2m}\left(m!\right)^2},$$

the first and the second integral terms in (2.1) are weakly singular, and the last integral exists in the sense of the Cauchy principal value.

Proof. It is easy to calculate that

$$\frac{\partial\Phi\left(x, y\right)}{\partial\nu\left(y\right)} = \frac{i}{4}\left(\frac{\partial J_0\left(k\left|x-y\right|\right)}{\partial\nu\left(y\right)} + i\,\frac{\partial N_0\left(k\left|x-y\right|\right)}{\partial\nu\left(y\right)}\right),$$

where

$$\frac{\partial J_0\left(k\left|x-y\right|\right)}{\partial\nu\left(y\right)} = \left(y-x,\, \nu\left(y\right)\right)\sum_{m=1}^{\infty}\frac{\left(-1\right)^m k^{2m}\left|x-y\right|^{2m-2}}{2^{2m-1}\left(m-1\right)!\,m!}$$

and

$$\frac{\partial N_0\left(k\left|x-y\right|\right)}{\partial\nu\left(y\right)} = \frac{2}{\pi}\left(\ln\frac{k\,\left|x-y\right|}{2} + C\right)\frac{\partial J_0\left(k\left|x-y\right|\right)}{\partial\nu\left(y\right)} + \frac{2\,\left(y-x,\, \nu\left(y\right)\right)}{\pi\,\left|x-y\right|^2} J_0\left(k\left|x-y\right|\right) +$$

$$+\left(y-x,\nu\left(y\right)\right)\sum_{m=1}^{\infty}\left(\sum_{l=1}^{m}\frac{1}{l}\right)\frac{\left(-1\right)^{m+1}k^{2m}\left|x-y\right|^{2m-2}}{2^{2m-1}\left(m-1\right)!\,m!}.$$

Then the expression $W\left(x\right)$ can be represented as

$$W\left(x\right)=\int_{L}\left(\frac{\left(x-y,\nu\left(y\right)\right)}{2\pi\left|x-y\right|^{2}}+V\left(x,y\right)\right)\rho\left(y\right)dL_{y},\quad x\in L.$$

It was shown in [15] that if a function ρ is continuously differentiable on L and

$$\int_{0}^{\mathrm{diam}\,L}\frac{\omega\left(\mathrm{grad}\rho,\,t\right)}{t}dt<+\infty,$$

then the function

$$W_{0}\left(x\right)=\frac{1}{2\pi}\int_{L}\frac{\left(x-y,\nu\left(y\right)\right)}{\left|x-y\right|^{2}}\rho\left(y\right)dL_{y},\quad x\in L,$$

has the normal derivative in L, with

$$\frac{\partial W_{0}\left(x\right)}{\partial\nu\left(x\right)}=-\frac{1}{\pi}\int_{L}\frac{\left(x-y,\nu\left(y\right)\right)\left(x-y,\nu\left(x\right)\right)}{\left|x-y\right|^{4}}\left(\rho\left(y\right)-\rho\left(x\right)\right)dL_{y}+$$

$$+\frac{1}{2\pi}\int_{L}\frac{\left(\nu\left(y\right),\nu\left(x\right)\right)}{\left|x-y\right|^{2}}\left(\rho\left(y\right)-\rho\left(x\right)\right)dL_{y},\;\;x\in L \tag{2.2}$$

and

$$\left|\frac{\partial W_{0}\left(x\right)}{\partial\nu\left(x\right)}\right|\leq M\left(\left\|\rho\right\|_{\infty}+\left\|\mathrm{grad}\rho\right\|_{\infty}+\int_{0}^{d}\frac{\omega\left(\mathrm{grad}\,\rho,t\right)}{t}dt\right),\forall\,x\in L.$$

The last integral in (2.2) exists in the sense of the Cauchy principal value.

As ([21, p. 403])

$$\left|\left(x-y,\nu\left(x\right)\right)\right|\leq M\left|x-y\right|^{2},\forall\,x,y\in L, \tag{2.3}$$

taking into account the inequalities

$$\left|J_{0}\left(k\left|x-y\right|\right)\right|=\left|\sum_{m=0}^{\infty}\frac{\left(-1\right)^{m}}{\left(m\,!\right)^{2}}\left(\frac{k\left|x-y\right|}{2}\right)^{2m}\right|\leq\sum_{m=0}^{\infty}\frac{\left(\left|k\right|\,\mathrm{diam}L\right)^{2m}}{4^{m}\left(m\,!\right)^{2}},\forall x,y\in L, \tag{2.4}$$

and

$$\left|\sum_{m=1}^{\infty}\left(\sum_{l=1}^{m}\frac{1}{l}\right)\frac{\left(-1\right)^{m+1}k^{2m}\left|x-y\right|^{2m-2}}{2^{2m-1}\left(m-1\right)!\,m!}\right|\leq$$

$$\leq\sum_{m=1}^{\infty}\left(\sum_{l=1}^{m}\frac{1}{l}\right)\frac{\left|k\right|^{2m}\left(\mathrm{diam}L\right)^{2m-2}}{2^{2m-1}\left(m-1\right)!\,m!},\forall x,y\in L, \tag{2.5}$$

we obtain

$$\left|V\left(x,y\right)\right|\leq M\left|x-y\right|,\quad\forall x,y\in L.$$

Consequently, the function

$$W_{1}\left(x\right)=\int_{L}V\left(x,y\right)\rho\left(y\right)dL_{y},\quad x\in L,$$

has the normal derivative in L, with

$$\frac{\partial W_{1}\left(x\right)}{\partial\nu\left(x\right)}=\int_{L}\frac{\partial V\left(x,y\right)}{\partial\nu\left(x\right)}\rho\left(y\right)dL_{y}=$$

$$= \frac{1}{2\pi} \int_L \frac{(y-x,\nu(x))\,(y-x,\nu(y))}{|x-y|^2} \sum_{m=1}^{\infty} \frac{(-1)^m k^{2m} |x-y|^{2m-2}}{2^{2m-1}(m-1)\,!\,m\,!} \rho(y)\,dL_y -$$

$$- \int_L \left(\frac{i}{4} - \frac{C}{2\pi} - \frac{1}{2\pi} \ln \frac{k\,|x-y|}{2} \right) (\nu(y)\,,\nu(x)) \sum_{m=1}^{\infty} \frac{(-1)^m k^{2m} |x-y|^{2m-2}}{2^{2m-1}(m-1)\,!\,m\,!} \rho(y)\,dL_y +$$

$$+ \int_L \left(\frac{i}{4} - \frac{C}{2\pi} - \frac{1}{2\pi} \ln \frac{k\,|x-y|}{2} \right) (y-x,\nu(y))\ (x-y,\nu(x)) \times$$

$$\times \sum_{m=2}^{\infty} \frac{(-1)^m k^{2m} |x-y|^{2m-4}}{2^{2m-2}(m-2)\,!\,m\,!} \rho(y)\,dL_y +$$

$$+ \int_L (\nu(y)\,,\nu(x)) \sum_{m=1}^{\infty} \left(\sum_{l=1}^{m} \frac{1}{l} \right) \frac{(-1)^{m+1} k^{2m} |x-y|^{2m-2}}{2^{2m+1}(m-1)\,!\,m\,!} \rho(y)\,dL_y -$$

$$- \int_L (x-y,\nu(x))\ (y-x,\nu(y)) \sum_{m=2}^{\infty} \left(\sum_{l=1}^{m} \frac{1}{l} \right) \frac{(-1)^{m+1} k^{2m} |x-y|^{2m-4}}{2^{2m}(m-2)\,!\,m\,!} \rho(y)\,dL_y +$$

$$+ \frac{1}{2\pi} \int_L (\nu(y)\,,\nu(x))\ (y-x,\nu(y)) \sum_{m=1}^{\infty} \frac{(-1)^m k^{2m} |x-y|^{2m-2}}{2^{2m}(m!)^2} \rho(y)\,dL_y -$$

$$- \frac{1}{2\pi} \int_L (x-y,\nu(x))\ (y-x,\nu(y)) \sum_{m=2}^{\infty} \frac{(-1)^m (m-1) k^{2m} |x-y|^{2m-4}}{2^{2m-1}(m!)^2} \rho(y)\,dL_y$$

and

$$\left| \frac{\partial V(x,y)}{\partial \nu(x)} \right| \le M\ |\ln|x-y||\,, \quad \forall\, x,y \in L. \tag{2.6}$$

Hence, we have

$$\left| \frac{\partial W_1(x)}{\partial \nu(x)} \right| \le M\ \|\rho\|_\infty\,, \quad \forall\, x \in L.$$

$\square$

Obviously, there exists a positive integer n_0 such that

$$\sqrt{R(n)} \le \min\{1\,,\ d/2\}\,, \quad \forall n > n_0.$$

Let

$$P_l = \left\{ j \mid 1 \le j \le n\ ,\ |x(\tau_l) - x(\tau_j)| \le \sqrt{R(n)} \right\},$$

$$Q_l = \left\{ j \mid 1 \le j \le n\ ,\ |x(\tau_l) - x(\tau_j)| > \sqrt{R(n)} \right\}$$

and

$$V_n(x,y) = \left(\frac{i}{4} - \frac{C}{2\pi} - \frac{1}{2\pi} \ln \frac{k\,|x-y|}{2} \right) (y-x,\nu(y)) \sum_{m=1}^{n} \frac{(-1)^m k^{2m} |x-y|^{2m-2}}{2^{2m-1}(m-1)\,!\,m\,!} -$$

$$- (y-x,\nu(y)) \sum_{m=1}^{n} \left(\sum_{l=1}^{m} \frac{1}{l} \right) \frac{(-1)^{m+1} k^{2m} |x-y|^{2m-2}}{2^{2m+1}(m-1)\,!\,m\,!} -$$

$$- \frac{1}{2\pi} (y-x,\nu(y)) \sum_{m=1}^{n} \frac{(-1)^m k^{2m} |x-y|^{2m-2}}{2^{2m}(m!)^2}.$$

It is easy to see that

$$\frac{\partial V_n(x,y)}{\partial\nu(x)}=\frac{1}{2\pi}\frac{(y-x,\nu(x))(y-x,\nu(y))}{|x-y|^2}\sum_{m=1}^{n}\frac{(-1)^m k^{2m}|x-y|^{2m-2}}{2^{2m-1}(m-1)!\,m!}-$$

$$-\left(\frac{i}{4}-\frac{C}{2\pi}-\frac{1}{2\pi}\ln\frac{k\,|x-y|}{2}\right)(\nu(y),\nu(x))\sum_{m=1}^{n}\frac{(-1)^m k^{2m}|x-y|^{2m-2}}{2^{2m-1}(m-1)!\,m!}+$$

$$+\left(\frac{i}{4}-\frac{C}{2\pi}-\frac{1}{2\pi}\ln\frac{k\,|x-y|}{2}\right)(y-x,\nu(y))\,(x-y,\nu(x))\sum_{m=2}^{n}\frac{(-1)^m k^{2m}|x-y|^{2m-4}}{2^{2m-2}(m-2)!\,m!}+$$

$$+(\nu(y),\nu(x))\sum_{m=1}^{n}\left(\sum_{l=1}^{m}\frac{1}{l}\right)\frac{(-1)^{m+1}k^{2m}|x-y|^{2m-2}}{2^{2m+1}(m-1)!\,m!}-$$

$$-(x-y,\nu(x))\,(y-x,\nu(y))\sum_{m=2}^{n}\left(\sum_{l=1}^{m}\frac{1}{l}\right)\frac{(-1)^{m+1}k^{2m}|x-y|^{2m-4}}{2^{2m}(m-2)!\,m!}+$$

$$+\frac{1}{2\pi}(\nu(y),\nu(x))\,(y-x,\nu(y))\sum_{m=1}^{n}\frac{(-1)^m k^{2m}|x-y|^{2m-2}}{2^{2m}(m!)^2}-$$

$$-\frac{1}{2\pi}(x-y,\nu(x))\,(y-x,\nu(y))\sum_{m=2}^{n}\frac{(-1)^m(m-1)k^{2m}|x-y|^{2m-4}}{2^{2m-1}(m!)^2}.$$

The following theorem is true.

Theorem 2.1. *Let a function ρ be continuously differentiable on L and*

$$\int_0^{\operatorname{diam}L}\frac{\omega(\operatorname{grad}\rho,\,t)}{t}dt<+\infty.$$

Then the expression

$$(T_n\rho)(x(\tau_p))=\frac{2\,(b-a)}{n}\sum_{\substack{j=1\\ j\neq p}}^{n}\frac{\partial V_n(x(\tau_p),x(\tau_j))}{\partial\nu(x(\tau_p))}\sqrt{(x_1'(\tau_j))^2+(x_2'(\tau_j))^2}\,\rho(x(\tau_j))\;-$$

$$-\frac{2\,(b-a)}{\pi\,n}\sum_{\substack{j=1\\ j\neq p}}^{n}\frac{(x(\tau_p)-x(\tau_j),\,\nu(x(\tau_j)))\,(x(\tau_p)-x(\tau_j),\,\nu(x(\tau_p)))}{|x(\tau_p)-x(\tau_j)|^4}\times$$

$$\times\sqrt{(x_1'(\tau_j))^2+(x_2'(\tau_j))^2}\,(\rho(x(\tau_j))-\rho(x(\tau_p)))+$$

$$+\frac{b-a}{\pi\,n}\sum_{j\in Q_p}\frac{(\nu(x(\tau_j)),\,\nu(x(\tau_p)))}{|x(\tau_j)-x(\tau_p)|^2}\sqrt{(x_1'(\tau_j))^2+(x_2'(\tau_j))^2}\,(\rho(x(\tau_j))-\rho(x(\tau_p)))$$

is a quadrature formula for $(T\rho)(x)$ at the control points $x(\tau_p)$, $p=\overline{1,n}$, and the following estimate holds:

$$\max_{p=\overline{1,n}}|(T\rho)(x(\tau_p))-(T_n\rho)(x(\tau_p))|\leq$$

$$\leq M\left[\frac{\|\rho\|_\infty\ln n}{n}+\frac{\|\operatorname{grad}\rho\|_\infty}{\sqrt{n}}+\int_0^{1/\sqrt{n}}\frac{\omega(\operatorname{grad}\rho,\,t)}{t}dt\right].$$

Proof. It was proved in [1] that if a function ρ is continuously differentiable on L and

$$\int_0^{\operatorname{diam} L} \frac{\omega(\operatorname{grad}\rho,\, t)}{t}\, dt < +\infty,$$

then the expression

$$\left(\frac{\partial W_0}{\partial \nu}\right)^n (x(\tau_p)) = -\frac{b-a}{\pi\, n} \sum_{\substack{j=1\\ j\neq p}}^{n} \frac{\left(x(\tau_p) - x(\tau_j)\,,\, \nu(x(\tau_j))\right)\left(x(\tau_p) - x(\tau_j)\,,\, \nu(x(\tau_p))\right)}{\left|x(\tau_p) - x(\tau_j)\right|^4} \times$$

$$\times \sqrt{\left(x_1'(\tau_j)\right)^2 + \left(x_2'(\tau_j)\right)^2}\, \left(\rho(x(\tau_j)) - \rho(x(\tau_p))\right) +$$

$$+\frac{b-a}{2\pi\, n} \sum_{j\in Q_p} \frac{\left(\nu(x(\tau_j))\,,\, \nu(x(\tau_p))\right)}{\left|x(\tau_j) - x(\tau_p)\right|^2} \sqrt{\left(x_1'(\tau_j)\right)^2 + \left(x_2'(\tau_j)\right)^2}\, \left(\rho(x(\tau_j)) - \rho(x(\tau_p))\right)$$

is a quadrature formula for the integral $\frac{\partial W_0(x)}{\partial \nu(x)}$ at the control points $x(\tau_p)\,,\ p = \overline{1,\, n}$, and the following estimate holds:

$$\max_{p=\overline{1,\, n}} \left| \frac{\partial W_0(x(\tau_p))}{\partial \nu(x(\tau_p))} - \left(\frac{\partial W_0}{\partial \nu}\right)^n (x(\tau_p)) \right| \leq$$

$$\leq M \left[\frac{\|\rho\|_\infty \ln n}{n} + \frac{\|\operatorname{grad}\rho\|_\infty}{\sqrt{n}} + \int_0^{1/\sqrt{n}} \frac{\omega(\operatorname{grad}\rho,\, t)}{t}\, dt \right].$$

Now, let us show that the expression

$$\left(\frac{\partial W_1}{\partial \nu}\right)^n (x(\tau_l)) = \frac{b-a}{n} \sum_{\substack{j=1\\ j\neq l}}^{n} \frac{\partial V_n\left(x(\tau_l)\,, x(\tau_j)\right)}{\partial \nu(x(\tau_l))} \sqrt{\left(x_1'(\tau_j)\right)^2 + \left(x_2'(\tau_j)\right)^2}\, \rho(x(\tau_j))$$

is a quadrature formula for the integral $\frac{\partial W_1(x)}{\partial \nu(x)}$ at the control points $x(\tau_l)\,,\ l = \overline{1,\, n}$. It is not difficult to see that

$$\frac{\partial W_1(x(\tau_p))}{\partial \nu(x(\tau_p))} - \left(\frac{\partial W_1}{\partial \nu}\right)^n (x(\tau_p)) = \int_{L_p} \frac{\partial V\left(x(\tau_p)\,,\, y\right)}{\partial \nu(x)}\, \rho(y)\, dL_y +$$

$$+\sum_{\substack{j=1\\ j\neq p}}^{n} \int_{L_j} \left(\frac{\partial V\left(x(\tau_p)\,,\, y\right)}{\partial \nu(x(\tau_p))} - \frac{\partial V_n\left(x(\tau_p)\,,\, x(\tau_j)\right)}{\partial \nu(x(\tau_p))} \right) \rho(y)\, dL_y +$$

$$+\sum_{\substack{j=1\\ j\neq p}}^{n} \int_{L_j} \frac{\partial V_n\left(x(\tau_p)\,,\, x(\tau_j)\right)}{\partial \nu(x(\tau_p))} \left(\rho(y) - \rho(x(\tau_j))\right) dL_y +$$

$$+\sum_{\substack{j=1\\ j\neq p}}^{n} \int_{t_{j-1}}^{t_j} \frac{\partial V_n\left(x(\tau_p)\,,\, x(\tau_j)\right)}{\partial \nu(x(\tau_p))} \times$$

$$\times \left(\sqrt{\left(x_1'(t)\right)^2 + \left(x_2'(t)\right)^2} - \sqrt{\left(x_1'(\tau_j)\right)^2 + \left(x_2'(\tau_j)\right)^2} \right) \rho(x(\tau_j))\, dt.$$

Denote the terms in the last equality by $\delta_1^n(x(\tau_p))$, $\delta_2^n(x(\tau_p))$, $\delta_3^n(x(\tau_p))$ and $\delta_4^n(x(\tau_p))$, respectively.

Taking into account (2.6) and the formula for calculating a curvilinear integral, we obtain

$$\left|\delta_1^n\left(x\left(\tau_p\right)\right)\right| \leq M\ \|\rho\|_\infty \int_0^{R(n)} |\ln \tau|\, d\tau \leq M\ \|\rho\|_\infty\, R\left(n\right)\ \left|\ln R\left(n\right)\right|.$$

Let $y \in L_j$ and $j \neq p$. From Lemma 2.1 and inequality (2.3) it is obvious that

$$\left|\left|x\left(\tau_p\right)-y\right|^q-\left|x\left(\tau_p\right)-x\left(\tau_j\right)\right|^q\right| \leq M\, q\, R\left(n\right)\ \left(\mathrm{diam} L\right)^{q-1},$$

$$\left|\left(\nu\left(y\right), \nu\left(x\left(\tau_p\right)\right)\right)-\left(\nu\left(x\left(\tau_j\right)\right), \nu\left(x\left(\tau_p\right)\right)\right)\right| \leq M\, R\left(n\right),$$

$$\left|\left(x\left(\tau_p\right)-y, \nu\left(y\right)\right)-\left(x\left(\tau_p\right)-x\left(\tau_j\right), \nu\left(y\right)\right)\right| = \left|\left(x\left(\tau_j\right)-y, \nu\left(y\right)\right)\right| \leq M\ \left(R\left(n\right)\right)^2,$$

$$\left|\left(x\left(\tau_p\right)-y, \nu\left(x\left(\tau_p\right)\right)\right)-\left(x\left(\tau_p\right)-x\left(\tau_j\right), \nu\left(x\left(\tau_p\right)\right)\right)\right| = \left|\left(x\left(\tau_j\right)-y, \nu\left(x\left(\tau_p\right)\right)\right)\right| \leq$$
$$\leq \left|\left(x\left(\tau_j\right)-y, \nu\left(x\left(\tau_j\right)\right)\right)\right| + \left|\left(x\left(\tau_j\right)-y, \nu\left(x\left(\tau_p\right)\right)-\nu\left(x\left(\tau_j\right)\right)\right)\right| \leq M\ \left|y-x\left(\tau_p\right)\right|\ R\left(n\right)$$

and

$$\left|\ln\left(k\left|x\left(\tau_p\right)-y\right|\right)-\ln\left(k\left|x\left(\tau_p\right)-x\left(\tau_j\right)\right|\right)\right| = \left|\ln\frac{\left|x\left(\tau_p\right)-x\left(\tau_j\right)\right|}{\left|x\left(\tau_p\right)-y\right|}\right| =$$
$$= \left|\ln\left(1+\frac{\left|x\left(\tau_p\right)-x\left(\tau_j\right)\right|-\left|x\left(\tau_p\right)-y\right|}{\left|x\left(\tau_p\right)-y\right|}\right)\right| \leq \left|\ln\left(1+\frac{\left|x\left(\tau_j\right)-y\right|}{\left|x\left(\tau_p\right)-y\right|}\right)\right| \leq M\frac{R\left(n\right)}{\left|x\left(\tau_p\right)-y\right|},$$

where $q \in \mathrm{N}$. Then, taking into account inequalities (2.4) and (2.5), it is not difficult to show that

$$\left|\frac{\partial V\left(x\left(\tau_p\right), y\right)}{\partial \nu\left(x\left(\tau_p\right)\right)}-\frac{\partial V\left(x\left(\tau_p\right), x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_p\right)\right)}\right| \leq M\ \left(R\left(n\right)\ \left|\ln\left|x\left(\tau_p\right)-y\right|\right|+\frac{R\left(n\right)}{\left|x\left(\tau_p\right)-y\right|}\right).$$

Also, by the inequality

$$\left|\frac{\partial V\left(x\left(\tau_p\right), x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_p\right)\right)}-\frac{\partial V_n\left(x\left(\tau_p\right), x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_p\right)\right)}\right| \leq M\frac{\left|\ln\left|x\left(\tau_p\right)-y\right|\right|}{n\,!}, \tag{2.7}$$

we have

$$\left|\frac{\partial V\left(x\left(\tau_p\right), y\right)}{\partial \nu\left(x\left(\tau_p\right)\right)}-\frac{\partial V_n\left(x\left(\tau_p\right), x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_p\right)\right)}\right| \leq$$
$$\leq M\ \left(R\left(n\right)\ \left|\ln\left|x\left(\tau_p\right)-y\right|\right|+\frac{R\left(n\right)}{\left|x\left(\tau_p\right)-y\right|}+\frac{\left|\ln\left|x\left(\tau_p\right)-y\right|\right|}{n\,!}\right).$$

So, we obtain

$$\left|\delta_2^n\left(x\left(\tau_p\right)\right)\right| \leq$$
$$\leq M\ \|\rho\|_\infty \left(R\left(n\right)\int_{r(n)}^{\mathrm{diam}\, L} |\ln \tau|\, d\tau + R\left(n\right)\int_{r(n)}^{\mathrm{diam}\, L}\frac{d\tau}{\tau}+\frac{1}{n\,!}\int_{r(n)}^{\mathrm{diam}\, L}|\ln \tau|\, d\tau\right) \leq$$
$$\leq M\ \|\rho\|_\infty \left(R\left(n\right)\ \left|\ln R\left(n\right)\right|+\frac{1}{n\,!}\right).$$

Let $y \in L_j$ and $j \neq p$. From Lemma 2.1 and inequalities (2.6), (2.7) it is obvious that

$$\left|\frac{\partial V_n\left(x\left(\tau_p\right), x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_p\right)\right)}\right| \leq$$
$$\leq \left|\frac{\partial V\left(x\left(\tau_p\right), x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_p\right)\right)}\right| + \left|\frac{\partial V\left(x\left(\tau_p\right), x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_p\right)\right)}-\frac{\partial V_n\left(x\left(\tau_p\right), x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_p\right)\right)}\right| \leq$$

$$\leq M\left(\left|\ln\left|x\left(\tau_p\right)-x\left(\tau_j\right)\right|\right|+\frac{1}{n!}\right),\forall n\in\mathrm{N}. \tag{2.8}$$

Then,

$$\left|\delta_3^n\left(x\left(\tau_p\right)\right)\right|\leq 2\,\omega\left(\rho,R\left(n\right)\right)\sum_{\substack{j=1\\ j\neq p}}^{n}\int_{L_j}\left|\frac{\partial V_n\left(x\left(\tau_p\right),\,x\left(\tau_j\right)\right)}{\partial\nu\left(x\left(\tau_p\right)\right)}\right|dL_y\leq$$

$$\leq 2\,\omega\left(\rho,R\left(n\right)\right)\int_{L}\left|\frac{\partial V_n\left(x\left(\tau_p\right),\,x\left(\tau_j\right)\right)}{\partial\nu\left(x\left(\tau_p\right)\right)}\right|dL_y\leq M\omega\left(\rho,R\left(n\right)\right).$$

Besides, taking into account Lemma 2.1, inequality (2.8) and

$$\left|\sqrt{\left(x_1'\left(t\right)\right)^2+\left(x_2'\left(t\right)\right)^2}-\sqrt{\left(x_1'\left(\tau_j\right)\right)^2+\left(x_2'\left(\tau_j\right)\right)^2}\right|\leq M\,R\left(n\right),\forall t\in\left[t_{j-1},t_j\right],$$

we obtain

$$\left|\delta_4^n\left(x\left(\tau_p\right)\right)\right|\leq M\left\|\rho\right\|_\infty R\left(n\right)\sum_{\substack{j=1\\ j\neq p}}^{n}\int_{t_{j-1}}^{t_j}\left|\frac{\partial V_n\left(x\left(\tau_p\right),\,x\left(\tau_j\right)\right)}{\partial\nu\left(x\left(\tau_p\right)\right)}\right|dt\leq$$

$$\leq M\left\|\rho\right\|_\infty R\left(n\right)\sum_{\substack{j=1\\ j\neq p}}^{n}\int_{L_j}\left|\frac{\partial V_n\left(x\left(\tau_p\right),\,x\left(\tau_j\right)\right)}{\partial\nu\left(x\left(\tau_p\right)\right)}\right|dL_y\leq$$

$$\leq M\left\|\rho\right\|_\infty R\left(n\right)\int_{L}\left|\frac{\partial V_n\left(x\left(\tau_p\right),\,x\left(\tau_j\right)\right)}{\partial\nu\left(x\left(\tau_p\right)\right)}\right|dL_y\leq M\left\|\rho\right\|_\infty R\left(n\right).$$

Summing up the estimates obtained for the expressions $\delta_1^n\left(x\left(\tau_p\right)\right)$, $\delta_2^n\left(x\left(\tau_p\right)\right)$, $\delta_3^n\left(x\left(\tau_p\right)\right)$ and $\delta_4^n\left(x\left(\tau_p\right)\right)$, and considering the relation $R\left(n\right)\sim\frac{1}{n}$, we obtain

$$\max_{p=\overline{1,n}}\left|\frac{\partial W_1\left(x\left(\tau_p\right)\right)}{\partial\nu\left(x\left(\tau_p\right)\right)}-\left(\frac{\partial W_1}{\partial\nu}\right)^n\left(x\left(\tau_p\right)\right)\right|\leq M\left(\omega\left(\rho,\,1/n\right)+\left\|\rho\right\|_\infty\frac{\ln n}{n}\right).$$

As a result, summing up the quadrature formulas constructed for the integrals $\frac{\partial W_0(x)}{\partial\nu(x)}$ and $\frac{\partial W_1(x)}{\partial\nu(x)}$ at the control points $x\left(\tau_p\right),\ p=\overline{1,\,n}$, we get the validity of Theorem 2.1. □

Now, let us state the main result of this work. Let

$$t_{ll}=\frac{2\left(b-a\right)}{\pi\,n}\sum_{\substack{j=1\\ j\neq l}}^{n}\frac{\left(x\left(\tau_l\right)-x\left(\tau_j\right),\,\nu\left(x\left(\tau_j\right)\right)\right)\,\left(x\left(\tau_l\right)-x\left(\tau_j\right),\,\nu\left(x\left(\tau_l\right)\right)\right)}{\left|x\left(\tau_l\right)-x\left(\tau_j\right)\right|^4}\times$$

$$\times\sqrt{\left(x_1'\left(\tau_j\right)\right)^2+\left(x_2'\left(\tau_j\right)\right)^2}-$$

$$-\frac{b-a}{\pi\,n}\sum_{j\in Q_l}\frac{\left(\nu\left(x\left(\tau_j\right)\right),\,\nu\left(x\left(\tau_l\right)\right)\right)}{\left|x\left(\tau_j\right)-x\left(\tau_l\right)\right|^2}\sqrt{\left(x_1'\left(\tau_j\right)\right)^2+\left(x_2'\left(\tau_j\right)\right)^2}\ \ \text{for} l=\overline{1,n};$$

$$t_{lj}=\frac{2\,\left(b-a\right)}{n}\sqrt{\left(x_1'\left(\tau_j\right)\right)^2+\left(x_2'\left(\tau_j\right)\right)^2}\left(\frac{\partial V_n\left(x\left(\tau_l\right),x\left(\tau_j\right)\right)}{\partial\nu\left(x\left(\tau_l\right)\right)}-\right.$$

$$\left.-\frac{\left(x\left(\tau_l\right)-x\left(\tau_j\right),\,\nu\left(x\left(\tau_j\right)\right)\right)\,\left(x\left(\tau_l\right)-x\left(\tau_j\right),\,\nu\left(x\left(\tau_l\right)\right)\right)}{\left|x\left(\tau_l\right)-x\left(\tau_j\right)\right|^4}\right)\ \text{for}\ j\in P_l\,,\,j\neq l;$$

$$t_{lj} = \frac{2\,(b-a)}{n}\sqrt{\left(x_1'\left(\tau_j\right)\right)^2 + \left(x_2'\left(\tau_j\right)\right)^2}\,\left(\frac{\partial V_n\left(x\left(\tau_l\right), x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_l\right)\right)} - \right.$$

$$\left. - \frac{\left(x\left(\tau_l\right) - x\left(\tau_j\right),\, \nu\left(x\left(\tau_j\right)\right)\right)\,\left(x\left(\tau_l\right) - x\left(\tau_j\right),\, \nu\left(x\left(\tau_l\right)\right)\right)}{\left|x\left(\tau_l\right) - x\left(\tau_j\right)\right|^4} + \frac{\left(\nu\left(x\left(\tau_j\right)\right),\, \nu\left(x\left(\tau_l\right)\right)\right)}{2\,\left|x\left(\tau_j\right) - x\left(\tau_l\right)\right|^2}\right) \text{ for } j \in Q_l.$$

From Theorem 2.1 it follows that

$$\left(T_n \rho\right)\left(x\left(\tau_l\right)\right) = \sum_{j=1}^{n} t_{lj}\, \rho\left(x\left(\tau_l\right)\right), l = \overline{1, n}.$$

Theorem 2.2. *Let $Im k > 0$, a function f be continuously differentiable on L and*

$$\int_0^{\operatorname{diam} L} \frac{\omega(\operatorname{grad} f,\, t)}{t} dt < +\infty.$$

Then the expression

$$\varphi_n\left(x\left(\tau_l\right)\right) = -2\sum_{j=1}^{n} \tilde{k}_{lj}^{-}\left(\sum_{p=1}^{n} \tilde{k}_{jp}^{+}\left(\sum_{m=1}^{n} t_{pm}\, f\left(x\left(\tau_m\right)\right)\right)\right)$$

is an approximate value of the solution $\varphi(x)$ to equation (1.1) at the points $x\left(\tau_l\right)$, $l = \overline{1, n}$, with

$$\max_{l=\overline{1,n}} \left|\, \varphi\,\left(x\left(\tau_l\right)\right) - \varphi_n\left(x\left(\tau_l\right)\right)\right| \leq$$

$$\leq M\left[\frac{1}{\sqrt{n}} + \omega\left(\operatorname{grad} f, 1/n\right) + \int_0^{1/\sqrt{n}} \frac{\omega\left(\operatorname{grad} f,\, t\right)}{t}\, dt + \frac{1}{n}\int_{1/n}^{\operatorname{diam} L} \frac{\omega\left(\operatorname{grad} f,\, t\right)}{t^2}\, dt\right].$$

Proof. From Lemmas 2.2 and 2.3 we obtain

$$\max_{j=\overline{1,n}} \sum_{l=1}^{n}\left|\tilde{k}_{jl}^{+}\right| \leq M_1,\ \max_{j=\overline{1,n}} \sum_{l=1}^{n}\left|\tilde{k}_{jl}^{-}\right| \leq M_2.$$

Besides, taking into account the error estimate for the quadrature formula for $(Tf)\,(x)$,$x \in L$, at the control points $x\left(\tau_l\right)$, $l = \overline{1,\, n}$, we have

$$\left|\, \varphi\left(x\left(\tau_l\right)\right) - \varphi_n\left(x\left(\tau_l\right)\right)\right| \leq$$

$$\leq 2\left|\left(\left(I - \tilde{K}\right)^{-1}\left(I + \tilde{K}\right)^{-1} T f\right)\,\left(x\left(\tau_l\right)\right) - \sum_{j=1}^{n} \tilde{k}_{lj}^{-}\left(\left(I + \tilde{K}\right)^{-1} T f\right)\,\left(x\left(\tau_j\right)\right)\right| +$$

$$+2\left|\sum_{j=1}^{n} \tilde{k}_{lj}^{-}\left[\left(\left(I + \tilde{K}\right)^{-1} T f\right)\,\left(x\left(\tau_j\right)\right) - \sum_{p=1}^{n} \tilde{k}_{jp}^{+}\,\left(T f\right)\left(x\left(\tau_p\right)\right)\right]\right| +$$

$$+2\,\left|\sum_{j=1}^{n} \tilde{k}_{lj}^{-}\left(\sum_{p=1}^{n} \tilde{k}_{jp}^{+}\left[\left(T f\right)\,\left(x\left(\tau_p\right)\right) - \sum_{m=1}^{n} t_{pm}\, f\left(x\left(\tau_m\right)\right)\right]\right)\,\right| \leq$$

$$\leq M\,\left[\left\|\left(I + \tilde{K}\right)^{-1}\right\|\, \left\|T f\,\right\|_\infty R\left(n\right)\,\left|\,\ln R\left(n\right)\right| + \omega\left(\left(I + \tilde{K}\right)^{-1} T f, R\left(n\right)\right)\right] +$$

$$+M\, M_2\,\left[\,\left\|\, T f\,\right\|_\infty R\left(n\right)\,\left|\,\ln R\left(n\right)\right| + \omega\,\left(T f, R\left(n\right)\right)\,\right]\, +$$

$$+ M\, M_1 M_2 \left[\frac{\|\rho\|_\infty \ln n}{n} + \frac{\|\operatorname{grad} \rho\|_\infty}{\sqrt{n}} + \int_0^{1/\sqrt{n}} \frac{\omega(\operatorname{grad} \rho,\, t)}{t}\, dt \right]. \tag{2.9}$$

From Lemma 2.4 it follows

$$\|Tf\|_\infty \le M \left(\|f\|_\infty + \|\operatorname{grad} f\|_\infty + \int_0^{\operatorname{diam} L} \frac{\omega(\operatorname{grad} f,\, t)}{t}\, dt \right).$$

Further, as the integral $\frac{\partial W_1(x)}{\partial \nu(x)}$, $x \in L$, is weakly singular, it is not difficult to show that

$$\omega\left(\frac{\partial W_1}{\partial \nu}, \delta \right) \le M\ \|\rho\|_\infty\ \delta\ |\ln \delta|\,, \delta > 0.$$

It was shown in [15] that if a function f is continuously differentiable on L and

$$\int_0^{\operatorname{diam} L} \frac{\omega(\operatorname{grad} f,\, t)}{t} dt < +\infty,$$

then

$$\omega\left(\frac{\partial W_0}{\partial \nu}, \delta \right) \le$$

$$\le M \left(\delta\ |\ln \delta| + \omega\left(\operatorname{grad} f, \delta\right) + \int_o^\delta \frac{\omega\left(\operatorname{grad} f, t\right)}{t} dt + \delta \int_\delta^{\operatorname{diam} L} \frac{\omega\left(\operatorname{grad} f, t\right)}{t^2} dt \right),$$

where $\delta > 0$. Hence, it follows that

$$\omega\ (Tf, \delta) \le 2 \left(\omega\left(\frac{\partial W_0}{\partial \nu}, \delta \right) + \omega\left(\frac{\partial W_1}{\partial \nu}, \delta \right) \right) \le$$

$$\le M \left(\delta\ |\ln \delta| + \omega\left(\operatorname{grad} f, \delta\right) + \int_o^\delta \frac{\omega\left(\operatorname{grad} f, t\right)}{t} dt + \delta \int_\delta^{\operatorname{diam} L} \frac{\omega\left(\operatorname{grad} f, t\right)}{t^2} dt \right), \delta > 0.$$

It is known ([3, p. 53–54]) that

$$\omega\left(\tilde{K} \rho, \delta \right) \le M\ \|\rho\|_\infty\ \delta\ |\ln \delta|\,, \delta > 0.$$

Then, if a function ρ_* is a solution of the equation $\rho + \tilde{K}\rho = Tf$, we have

$$\omega\left(\left(I + \tilde{K} \right)^{-1} Tf, \delta \right) = \omega\left(\rho_*, \delta\right) = \omega\left(Tf - \tilde{K}\rho_*, \delta \right) \le \omega\left(Tf, \delta\right) + \omega\left(\tilde{K}\rho_*, \delta \right) \le$$

$$\le \omega\left(Tf, \delta\right) + M\ \|\rho_*\|_\infty\ \delta\ |\ln \delta| = \omega\left(Tf, \delta\right) + M \left\| \left(I + \tilde{K} \right)^{-1} Tf \right\|_\infty \delta\ |\ln \delta| \le$$

$$\le \omega\left(Tf, \delta\right) + M \left\| \left(I + \tilde{K} \right)^{-1} \right\| \|Tf\|_\infty\, \delta\ |\ln \delta| \le$$

$$\le M \left(\delta\ |\ln \delta| + \omega\left(\operatorname{grad} f, \delta\right) + \int_o^\delta \frac{\omega\left(\operatorname{grad} f, t\right)}{t} dt + \delta \int_\delta^{\operatorname{diam} L} \frac{\omega\left(\operatorname{grad} f, t\right)}{t^2} dt \right), \delta > 0.$$

So, taking into account the above obtained inequalities in (2.9) and the relation $R\,(n) \sim \frac{1}{n}$, we get the validity of the theorem. $\square$

Theorem 2.2 has the following corollaries.

Corollary 2.1. *Let $Im\,k>0$, a function f be continuously differentiable on L and*

$$\int_0^{\operatorname{diam} L} \frac{\omega(\operatorname{grad} f,\, t)}{t} dt < +\infty.$$

Then the sequence

$$u_n(x_*) = \frac{b-a}{n} \sum_{l=1}^{n} \Phi_n\left(x_*, x\left(\tau_l\right)\right)\, \varphi_n\left(x\left(\tau_l\right)\right)\, \sqrt{\left(x_1'\left(\tau_l\right)\right)^2 + \left(x_2'\left(\tau_l\right)\right)^2}, \quad x_* \in D,$$

converges to the value $u\left(x_*\right)$ of the solution $u\left(x\right)$ to the interior Dirichlet boundary value problem for the Helmholtz equation at the point x_*, with

$$\left|u_n\left(x_*\right) - u\left(x_*\right)\right| \le$$

$$\le M\,\left[\frac{1}{\sqrt{n}} + \omega\left(\operatorname{grad} f, 1/n\right) + \int_0^{1/\sqrt{n}} \frac{\omega\left(\operatorname{grad} f\,,\, t\right)}{t}\, dt + \frac{1}{n} \int_{1/n}^{\operatorname{diam} L} \frac{\omega\left(\operatorname{grad} f,\, t\right)}{t^2}\, dt\right],$$

where

$$\varphi_n\left(x\left(\tau_l\right)\right) = -2 \sum_{j=1}^{n} \tilde{k}_{l\,j}^{-} \left(\sum_{p=1}^{n} \tilde{k}_{j\,p}^{+} \left(\sum_{m=1}^{n} t_{pm}\, f\left(x\left(\tau_m\right)\right)\right)\right).$$

Corollary 2.2. *Let $Im\,k>0$, a function f be continuously differentiable on L and*

$$\int_0^{\operatorname{diam} L} \frac{\omega(\operatorname{grad} f,\, t)}{t} dt < +\infty.$$

Then the sequence

$$u_n\left(x^*\right) = \frac{b-a}{n} \sum_{l=1}^{n} \Phi_n\left(x^*, x\left(\tau_l\right)\right)\, \varphi_n\left(x\left(\tau_l\right)\right)\, \sqrt{\left(x_1'\left(\tau_l\right)\right)^2 + \left(x_2'\left(\tau_l\right)\right)^2}, \quad x^* \in R^2 \backslash \bar{D},$$

converges to the value $u\left(x^*\right)$ of the solution $u\left(x\right)$ to the exterior Dirichlet boundary value problem for the Helmholtz equation at the point x^*, with

$$\left|u_n\left(x^*\right) - u\left(x^*\right)\right| \le$$

$$\le M\,\left[\frac{1}{\sqrt{n}} + \omega\left(\operatorname{grad} f, 1/n\right) + \int_0^{1/\sqrt{n}} \frac{\omega\left(\operatorname{grad} f\,,\, t\right)}{t}\, dt + \frac{1}{n} \int_{1/n}^{\operatorname{diam} L} \frac{\omega\left(\operatorname{grad} f,\, t\right)}{t^2}\, dt\right],$$

where

$$\varphi_n\left(x\left(\tau_l\right)\right) = -2 \sum_{j=1}^{n} \tilde{k}_{l\,j}^{-} \left(\sum_{p=1}^{n} \tilde{k}_{j\,p}^{+} \left(\sum_{m=1}^{n} t_{pm}\, f\left(x\left(\tau_m\right)\right)\right)\right).$$

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