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SOME NEW APPROACHES IN THE THEORY OF TRIGONOMETRIC SERIES WITH MONOTONE COEFFICIENTS

M.I. Dyachenko, A.P. Solodov

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Abstract. In this paper, we observe the latest results concerning the trigonometric series whose coefficients are monotone or fractional monotone. We study the asymptotic properties of the sums for both classes of series and also the problems of convergence and integrability for series with fractional monotone coefficients.

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1 Introduction and problem statement

Everywhere below in this paper we denote by

$$f(\boldsymbol{a}, x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{1.1}$$

and

$$g(\boldsymbol{a}, x) = \sum_{n=1}^{\infty} a_n \sin nx \tag{1.2}$$

for all x for which the corresponding series converges.

It is known that one of the most important classes of trigonometric series is the class of series with monotone coefficients, i.e. the sequence $\boldsymbol{a} = \{a_n\}_{n=0}^{\infty}$ is such that $a_n \downarrow 0$ as $n \to \infty$. In this case series (1.1) and (1.2) have a lot of very good properties. For instance, the following theorem was proved by G. Hardy and J. Littlewood.

Theorem 1.1 ([14]). *Let $p \in (1, \infty)$. Then $f(\boldsymbol{a}, x) \in L_p([0, \pi])$ (or $g(\boldsymbol{a}, x) \in L_p([0, \pi])$) if and only if*

$$J_p(\boldsymbol{a}) \equiv \sum_{n=1}^{\infty} a_n^p n^{p-2} < \infty.$$

We mention also the well-known theorems of G. Lorentz (Theorem 1.2) and T. Chaundy and A. Jolliffe (Theorem 1.3).

Theorem 1.2 ([16]). *Let $\alpha \in (0, 1)$. Then $f(\boldsymbol{a}, x) \in \operatorname{Lip} \alpha$ (or $g(\boldsymbol{a}, x) \in \operatorname{Lip} \alpha$) if and only if for some $C > 0$ we have $a_n \leqslant C/n^{1+\alpha}$ for all $n \in \mathbb{N}$.*

Theorem 1.3 ([7]). *Series* (1.2) *uniformly converges if and only if* $na_n \to 0$ *as* $n \to \infty$.

One of the main topics of the present paper is the so-called asymptotic behaviour of the sums of trigonometric series with monotone coefficients in a neighbourhood of zero. The first results in this direction were obtained by R. Salem [22], [23] (see also [6]). His research was continued by S. Izumi [15], S.A. Telyakovskii [28], [29], A.Yu. Popov and A.P. Solodov (see [18]–[21], [24]–[27]), and others. Note that the properties of sine and cosine series differ significantly in this problem. In Section 2 we discuss in detail the asymptotics of the sums of sine series and new approaches to this problem.

As series with monotone coefficients are very interesting because of their properties, many authors introduced the classes of trigonometric series with generalized monotone coefficients. In Section 3 we discuss fractional monotone sequences and the corresponding trigonometric series. M.I. Dyachenko introduced this class in paper [8] and proved some convergence and smoothness properties of cosine and sine series with coefficients belonging to this class. It is necessary to say that many important auxiliary results essential for the study of monotonicity of fractional order were established by A. Andersen [4]. A number of new results in this direction were obtained by M.I. Dyachenko, E.D. Nursultanov, A.P. Solodov, A.B. Mukanov, and E.D. Alferova (see [8]–[13], [17], [2]). Similar questions were also considered in the works [1], [5], [30].

2 New approaches to asymptotic properties

This section is devoted to the study of the asymptotic behavior in the right half-neighbourhood of zero of sums of a sine series with monotone coefficients.

To obtain a two-sided estimate of the sum of a series (1.2), R. Salem [22] defined the following function:

$$v(\boldsymbol{a}, x) = x \sum_{n=1}^{m(x)} n a_n, \quad m(x) = [\pi/x] .$$

Under some additional assumptions on the sequence $\boldsymbol{a}$ monotonically tending to zero, he proved the existence of positive constants $C_1(\boldsymbol{a})$, $C_2(\boldsymbol{a})$, and $x_0 > 0$ such that the following estimates hold:

$$C_2(\boldsymbol{a}) v(\boldsymbol{a}, x) \leqslant g(\boldsymbol{a}, x) \leqslant C_1(\boldsymbol{a}) v(\boldsymbol{a}, x), \quad 0 < x \leqslant x_0. \tag{2.1}$$

S.A. Telyakovskii has improved this result by deriving estimate (2.1) with absolute constants C_1 and C_2, freeing the sequence $\boldsymbol{a}$ from additional requirements and showing that the upper bound holds for any monotone sequence $\boldsymbol{a}$, and the lower bound — for any convex sequence $\boldsymbol{a}$ (i.e. $a_n - 2a_{n+1} + a_{n+2} \geqslant 0$, $n \in \mathbb{N}$).

Theorem 2.1 ([28], [29]). *There exists a constant* $C_1 > 0$ *such that for any nonincreasing null sequence* $\boldsymbol{a}$

$$g(\boldsymbol{a}, x) \leqslant C_1 v(\boldsymbol{a}, x), \quad 0 < x \leqslant \pi/11.$$

There exists a constant $C_2 > 0$ *such that for any convex null sequence* $\boldsymbol{a}$

$$g(\boldsymbol{a}, x) \geqslant C_2(\boldsymbol{a}) v(\boldsymbol{a}, x), \quad 0 < x \leqslant \pi/11.$$

A.Yu. Popov calculated the sharp values of the constants in the estimates of Telyakovskii. He proved the following results.

Theorem 2.2 ([18]). *For any nonincreasing null sequence* $\boldsymbol{a}$,

$$g(\boldsymbol{a}, x) \leqslant v(\boldsymbol{a}, x), \quad 0 < x \leqslant \pi. \tag{2.2}$$

Theorem 2.3 ([18]). *For any convex null sequence* $\boldsymbol{a}$,

$$g(\boldsymbol{a},x) \geqslant \frac{2}{\pi^2}\, v(\boldsymbol{a},x) - 0.46\, a_{m(x)}, \quad 0 < x \leqslant \frac{\pi}{2}. \tag{2.3}$$

The estimate (2.3), in general, does not hold if there is no second negative term in its right-hand side. The question arises: is it possible to modify the Salem function $v(\boldsymbol{a},x)$ in such way so that the two-sided estimate with constants $C_1 = 1$ and $C_2 = 2\pi^{-2}$ still holds in some right half-neighbourhood of zero? The answer to this question is positive.

In [24] was shown that the estimate (2.2) can be strengthened. As a new majorant, consider the function

$$u(\boldsymbol{a},x) = x \sum_{n=1}^{[(m(x)+1)/2]} n a_n + x \sum_{n=[(m(x)+3)/2]}^{m(x)} \big(m(x)+1-n\big) a_n.$$

The following refinement of Theorem 2.2 is valid.

Theorem 2.4 ([24]). *For any nonincreasing null sequence* $\boldsymbol{a}$,

$$g(\boldsymbol{a},x) \leqslant u(\boldsymbol{a},x), \quad 0 < x \leqslant \pi.$$

Under the additional condition of convexity of the sequence $\boldsymbol{a}$, the function $2\pi^{-2}u(\boldsymbol{a},x)$ turns out to be a minorant of the sum of the sine series not only in a certain neighbourhood of zero, but practically over the entire interval $(0,\pi/2]$.

Theorem 2.5 ([24]). *For any convex null sequence* $\boldsymbol{a}$,

$$g(\boldsymbol{a},x) \geqslant \frac{2}{\pi^2}\, u(\boldsymbol{a},x), \quad 0 < x \leqslant \frac{9\pi}{20}.$$

In [21], the asymptotic behavior of sums of the particular sine series (1.2) as $x \to 0+$ was studied. Their coefficient sequences not only monotonically tend to zero, but also belong to the following two special classes. First class — let us denote it as $\mathcal{B}\downarrow$ — consists of all sequences $\boldsymbol{a}$ monotonically tending to zero such that the sequence $\{na_n\}_{n=1}^{\infty}$ does not increase, that is $(n+1)a_{n+1} \leqslant na_n$, $n \in \mathbb{N}$. Second class — let us denote it as $\mathcal{B}\uparrow$ — consists of all sequences $\boldsymbol{a}$ monotonically tending to zero such that the sequence $\{na_n\}_{n=1}^{\infty}$ does not decrease, that is $na_n \leqslant (n+1)a_{n+1}$, $n \in \mathbb{N}$.

Theorem 2.6 ([21]). *If* $\boldsymbol{a} \in \mathcal{B}\downarrow$, *then, for any* $x \in (0,\pi/3]$, *the following lower estimate holds:*

$$g(\boldsymbol{a},x) \geqslant \left(\underline{I} - \frac{1}{m(x)}\right) v(\boldsymbol{a},x) - \frac{3}{2}\, a_{m(x)+1} \sin\frac{x}{2},$$

where

$$\underline{I} = \frac{1}{\pi}\int_0^{2\pi} \frac{\sin t}{t}\, dt = 0.451\ldots.$$

Moreover, there exist sequences $\underline{\boldsymbol{a}} \in \mathcal{B}\downarrow$ *and* $\{x_k\}_{k=1}^{\infty}$ *such that*

$$x_k > 0 \quad (\forall k \in \mathbb{N}), \qquad \lim_{k\to\infty} x_k = 0, \qquad g(\underline{\boldsymbol{a}}, x_k) \sim \underline{I}\, v(\underline{\boldsymbol{a}}, x_k), \quad k \to \infty.$$

Theorem 2.7 ([21]). *If* $\boldsymbol{a} \in \mathcal{B}\uparrow$, *then, for any* $x \in (0,\pi)$, *the following upper estimate holds:*

$$g(\boldsymbol{a},x) \leqslant \overline{I}\left(1 + \frac{1}{m(x)}\right) v(\boldsymbol{a},x) + \frac{1}{2}\, a_{m(x)+1} \tan\frac{x}{4},$$

where

$$\overline{I} = \frac{1}{\pi}\int_0^\pi \frac{\sin t}{t}\,dt = 0.589\ldots.$$

Moreover, there exist sequences $\overline{\boldsymbol{a}} \in \mathcal{B}\uparrow$ *and* $\{x_k\}_{k=1}^\infty$ *such that*

$$x_k > 0 \quad (\forall k \in \mathbb{N}), \qquad \lim_{k\to\infty} x_k = 0, \qquad g(\overline{\boldsymbol{a}}, x_k) \sim \overline{I}\, v(\overline{\boldsymbol{a}}, x_k), \quad k \to \infty.$$

In [24], a lower bound for the sums of sine series with convex coefficients was studied. The following result of Popov was refined.

Theorem 2.8 ([18])**.** *For any convex null sequence* $\boldsymbol{a}$*,*

$$g(\boldsymbol{a}, x) \geqslant \frac{2}{\pi^2}\, v(\boldsymbol{a}, x) - \frac{1}{\pi}\, a_{m(x)} - a_{m(x)}\left(\frac{1}{x} - \frac{1}{2}\cot\frac{x}{2}\right), \quad 0 < x \leqslant \frac{\pi}{2}.$$

It has been established that the Salem function with a sharp constant $2\pi^{-2}$ is not, in general, a minorant for the sum of a sine series for the class of all convex sequences $\boldsymbol{a}$.

A sequence $\{\beta_k\}_{k=1}^\infty$ is called *slowly varying* if $\lim_{k\to\infty} \beta_{[\delta k]}/\beta_k = 1$ for any $\delta > 0$.

Theorem 2.9 ([24])**.** *There exists a convex slowly varying null sequence* $\boldsymbol{a}$ *such that*

$$g(\boldsymbol{a}, x_k) < \frac{2}{\pi^2}\, v(\boldsymbol{a}, x_k)$$

for a sequence of points $\{x_k\}_{k=1}^\infty$ *with* $x_k \to +0$.

It is shown that, as an alternative, one can take the modified Salem function

$$v_0(\boldsymbol{a}, x) = x\Bigg(\sum_{n=1}^{m(x)-1} n a_n + \frac{m(x)}{2}\, a_{m(x)}\Bigg).$$

Theorem 2.10 ([24])**.** *Let* $\boldsymbol{a}$ *be a positive convex null sequence. Then for some* $x_0 > 0$

$$g(\boldsymbol{a}, x) > \frac{2}{\pi^2}\, v_0(\boldsymbol{a}, x), \quad 0 < x < x_0.$$

For any $\varepsilon > 0$ *there exists a convex slowly varying null sequence* $\boldsymbol{a}$ *for which there exists a sequence of points* $\{x_k\}_{k=1}^\infty$ *with* $x_k \to +0$ *such that*

$$g(\boldsymbol{a}, x_k) < \frac{2}{\pi^2}\, x_k\Bigg(\sum_{n=1}^{m(x_k)-1} n a_n + \left(\frac{1}{2} + \varepsilon\right) m(x_k) a_{m(x_k)}\Bigg).$$

In other words, the coefficient $1/2$ multiplying the term $m(x)a_{m(x)}$ in the modified Salem majorant is sharp. This shows that in some sense the function $v_0(\boldsymbol{b}, x)$ is optimal for estimating the sum of a sine series with convex coefficients from below.

In [26], the sharp constants were found in the two-sided Telyakovskii estimate for the sum of a sine series with a monotone sequence of coefficients $\boldsymbol{a}$ under the additional condition of convexity.

S.A. Telyakovskii showed that it is convenient to compare the difference between the sum of series (1.2) and the main term of its asymptotic expansion, i.e.

$$g(\boldsymbol{a}, x) - \frac{a_{m(x)}}{2}\cot\frac{x}{2},$$

with the function

$$\sigma(\boldsymbol{a}, x) = \frac{1}{m(x)} \sum_{n=1}^{m(x)-1} n^2 \Delta^1 a_n, \qquad \Delta^1 a_n = a_n - a_{n+1} > 0.$$

Theorem 2.11 ([28], [29]). *There exist positive absolute constants C_1 and C_2 such that*

$$C_1\,\sigma(\boldsymbol{a},x)\leqslant g(\boldsymbol{a},x)-\frac{a_{m(x)}}{2}\cot\frac{x}{2}\leqslant C_2\,\sigma(\boldsymbol{a},x),\quad 0<x\leqslant\frac{\pi}{11},$$

for any convex null sequence $\boldsymbol{a}$.

In the following theorem, the sharp values of the constants C_1 and C_2 are obtained.

Theorem 2.12 ([26]). *The following equalities hold:*

$$\sup_{\boldsymbol{a}}\varlimsup_{x\to+0}\frac{g(\boldsymbol{a},x)-(a_{m(x)}/2)\cot(x/2)}{\sigma(\boldsymbol{a},x)}=\frac{\pi}{2},\tag{2.4}$$

$$\inf_{\boldsymbol{a}}\varliminf_{x\to+0}\frac{g(\boldsymbol{a},x)-(a_{m(x)}/2)\cot(x/2)}{\sigma(\boldsymbol{a},x)}=\frac{3(\pi-1)}{\pi^2},\tag{2.5}$$

moreover, the supremum in (2.4) and the infimum in (2.5) are attained for slowly varying sequences.

The following theorems answer the question how large is the deviation between the sum of sine series (1.2) and its asymptotically sharp majorant and minorant for the class of all convex sequences of coefficients.

Theorem 2.13 ([24]). *There exists a convex slowly varying null sequence $\boldsymbol{a}$ such that*

$$0<g(\boldsymbol{a},x_k)-\frac{2}{\pi^2}\,v_0(\boldsymbol{a},x_k)<\frac{1}{2}\sqrt[3]{\pi^2a_1x_ka^2_{m(x_k)}}+\frac{9+\pi^2}{6\pi^2}x_ka_{m(x_k)}$$

for some sequence of points $\{x_k\}_{k=1}^{\infty}$, $x_k\to+0$.

Theorem 2.14 ([25]). *For any $\varepsilon>0$, there exists a convex slowly varying null sequence $\boldsymbol{a}$ such that*

$$0>g(\boldsymbol{a},x_k)-\frac{a_{m(x_k)}}{2}\cot\frac{x_k}{2}-\sin\frac{x_k}{2}\sum_{n=1}^{m(x_k)-1}n(n+1)\Delta^1a_n>-a_1x_k^{3-\varepsilon}$$

for some sequence of points $\{x_k\}_{k=1}^{\infty}$, $x_k\to+0$.

At the end of the section, we present a result that refines the asymptotics of the sum of a sine series (1.1) with a convex slowly varying sequence of coefficients, obtained by S. Aljančić, R. Bojanić and M. Tomić, in the case when the sequence of coefficients satisfies the additional regularity condition.

Theorem 2.15 ([27]). *Let $\boldsymbol{a}$ be a non-negative convex null sequence, and let $\{n\Delta^1a_n\}_{n=1}^{\infty}$ be a convex slowly varying sequence. Then*

$$g(\boldsymbol{a},x)-\frac{a_{m(x)}}{x}\sim(\gamma+\ln\pi)\frac{m(x)\Delta^1a_{m(x)}}{x},\quad x\to+0,$$

where γ — is the Euler constant.

3 The fractional monotonicity

Let us give the corresponding definitions.

Definition 1. *Let $\alpha \in (-\infty, \infty)$. The Cesàro numbers $\{A_n^\alpha\}_{n=0}^\infty$ are defined as the coefficients in the expansion*

$$(1-x)^{-\alpha-1} = \sum_{n=0}^{\infty} A_n^\alpha x^n \quad \textit{for } x \in (0,1).$$

The following properties of the Cesàro numbers are known (see [31]):

(1) $A_n^0 = 1$ for $n = 0, 1, \ldots$ and $A_0^\alpha = 1$ for any α.

(2) If $\alpha \neq -1, -2, \ldots$, then there are constants $C_1(\alpha) > 0$ and $C_2(\alpha) > 0$ depending only on α such that

$$C_2(\alpha)\, n^\alpha \leqslant |A_n^\alpha| \leqslant C_1(\alpha)\, n^\alpha \quad \text{for all } n > 0.$$

(3) For $\alpha > -1$ and any n, $A_n^\alpha > 0$; for $\alpha > 0$, $A_n^\alpha \uparrow \infty$ as $n \to \infty$; and, for $-1 < \alpha < 0$, $A_n^\alpha \downarrow 0$ as $n \to \infty$.

(4) For all α, β and $n = 0, 1, \ldots$

$$\sum_{k=0}^{n} A_{n-k}^\alpha A_k^\beta = A_n^{\alpha+\beta+1}.$$

In particular, $A_n^\alpha - A_{n-1}^\alpha = A_n^{\alpha-1}$.

(5) For $\alpha > -1$ and $n = 0, 1, \ldots$ we have

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\ldots(\alpha+n)}{n!}.$$

Given a number sequence $\boldsymbol{a} = \{a_n\}_{n=0}^\infty$ and a real number α, we set

$$\Delta^\alpha a_n = \sum_{k=0}^{\infty} A_k^{-\alpha-1} a_{n+k}$$

for $n = 0, 1, \ldots$ if this series converges (this is so, for example, if $\alpha > 0$ and the sequence $\boldsymbol{a}$ is bounded).

Definition 2. *Let $\alpha > 0$, and let $\boldsymbol{a} = \{a_n\}_{n=0}^\infty$ be a sequence of real numbers. We say that $\boldsymbol{a} \in M_\alpha$ if $\lim_{n\to\infty} a_n = 0$ and $\Delta^\alpha a_n \geqslant 0$ for $n = 0, 1, \ldots$.*

It follows from Definition 2 that the class M_0 coincides with the class of null sequences of nonnegative numbers, M_1 is the class of monotone nonincreasing null sequences, M_2 is the class of convex null sequences, etc. In addition, in [8, Lemma 1, assertion b)] it was shown that $M_\alpha \subset M_\beta$ for $\alpha > \beta \geqslant 0$.

Definition 3. *Let $\gamma \in (0,1)$. We say that a sequence $\boldsymbol{a} \in P_\gamma$ if $\boldsymbol{a} \in M_0$ and*

$$\sum_{n=1}^{\infty} n^{-\gamma} a_n < \infty.$$

In [8], M.I. Dyachenko proved the following statements. They were proved for cosine series, but the analogous statements remain valid for sine series.

Theorem 3.1 ([8]). *Let $\alpha \in (0,1)$, a sequence $\boldsymbol{a} \in M_\alpha \cap P_\alpha$. Then a function $f(\boldsymbol{a},x)$ exists for $x \in (0,2\pi)$, such that $f(\boldsymbol{a},x) \in C((0,2\pi))$ and $|f(\boldsymbol{a},x)| \leqslant C(\alpha)x^{-\alpha}$ for $x \in (0,\pi)$, where $C(\alpha)$ depends only on α.*

Theorem 3.2 ([8]). *Let $\alpha \in (0,1)$, a sequence $\boldsymbol{b} \in M_1$ and $\boldsymbol{b} \notin P_\alpha$. Then there exists a sequence $\boldsymbol{a} \in M_\alpha$ such that $a_n \leqslant b_n$ for all n, but series (1.1) diverges at the point $\pi/2$.*

Theorem 3.3 ([8]). *Let $\alpha \in (1,2)$ and $\boldsymbol{a} \in M_\alpha$. Then for any $\gamma \in (0,\pi)$ we have $f(\boldsymbol{a},x+t) - f(\boldsymbol{a},x) = o(t^{\alpha-1})$ as $t \to +0$ uniformly for $x \in [\gamma, 2\pi-\gamma]$.*

Theorem 3.4 ([8]). *Let $\alpha \in (1,2)$ and a function φ be defined on $[0,1]$ and $\varphi(t) \downarrow 0$ as $t \downarrow 0$. Then there exist a sequence $\boldsymbol{a} \in M_\alpha$ and a sequence $\{t_n\}_{n=1}^\infty$ such that $t_n \downarrow 0$ as $n \to \infty$ and $|f(\boldsymbol{a},\pi/2+t_n) - f(\boldsymbol{a},\pi/2)| \geqslant Ct_n^{\alpha-1}\varphi(t_n)$ for all n where $C > 0$ does not depend on n.*

In [9], the following statements connected with Theorem 1.1 were obtained.

Theorem 3.5 ([9]). *Let $\alpha \in (0,1)$, $p \in (1/\alpha,\infty)$, a sequence $\boldsymbol{a} \in M_\alpha$ and $J_p(\boldsymbol{a}) < \infty$. Then series (1.1) converges at any $x \in (0,2\pi)$.*

Theorem 3.6 ([9]). *Let $\alpha \in (1/2,1)$, $p \in (1/\alpha,\infty)$, a sequence $\boldsymbol{a} \in M_\alpha$ and $J_p(\boldsymbol{a}) < \infty$. Then the function $f(\boldsymbol{a},x) \in L_p([0,\pi])$.*

Theorem 3.7 ([9]). *Let $\alpha \in (1/2,1)$. Then there exists a sequence $\boldsymbol{a} \in M_\alpha$ such that $J_p(\boldsymbol{a}) < \infty$ for every $p \in (1,1/\alpha)$, but (1.1) is not a Fourier–Lebesgue series.*

It is natural to suppose that the following hypothesis is true.

Hypothesis 3.1. *Let $\alpha \in (1/2,1)$, $p \in (2,1/(1-\alpha))$, a function $f \in L_p([0,\pi])$ and has the Fourier series of type (1.1) or (1.2) with $\boldsymbol{a} \in M_\alpha$. Then $J_p(\boldsymbol{a}) < \infty$.*

This conjecture is still unsolved, but M.I. Dyachenko and E. D. Nursultanov [12] proved, in particular, the following result.

Theorem 3.8 ([12]). *Let $\alpha \in (1/2,1)$ and $p > 1/(1-\alpha)$. Then there exists an even function $f \in L_p([0,\pi])$ such that its Fourier coefficients $\boldsymbol{a} \in M_\alpha$, but $J_p(\boldsymbol{a}) = \infty$.*

As for asymptotic properties of the sums of trigonometric series with fractional monotone coefficients, the results are the following. For cosine series they were established by M.I. Dyachenko [10].

Note that the sums of cosine series are usually estimated using the function

$$q(\boldsymbol{a},x) = \sum_{n=0}^{[\pi/x]}(n+1)(a_n - a_{n+1}).$$

Theorem 3.9 ([10]). *For any $\alpha \in (1,2)$, there exists a sequence $\boldsymbol{a} \in M_\alpha$ and a monotone null sequence $\{t_l\}_{l=1}^\infty$ such that*

$$\lim_{l\to\infty}\frac{q(\boldsymbol{a},t_l)}{f(\boldsymbol{a},t_l)} = 0.$$

Theorem 3.10 ([10]). *Let $\alpha > 2$. Then there exists a constant $C(\alpha) > 0$ such that if a sequence $\boldsymbol{a} \in M_\alpha$, then, for $x \in (0,\pi/6)$, the sum of series (1.1) satisfies the inequality $f(\boldsymbol{a},x) \geqslant C(\alpha)q(\boldsymbol{a},x)$.*

In the same paper [10], an example showing that the condition $\boldsymbol{a} \in M_2$ does not guarantee the validity of the lower bound in terms of $q(\boldsymbol{a},x)$ was given. Of course, the condition $\boldsymbol{a} \in M_2$ is sufficient for the upper bound $f(\boldsymbol{a},x) \leqslant Cq(\boldsymbol{a},x)$, $x \in (0,\pi)$, to hold. So, for cosine series, we need 2-monotonicity for the upper estimate, and $(2+\varepsilon)$-monotonicity for the lower estimate.

For the sine series the situation is quite different. This was shown by M.I. Dyachenko and A.P. Solodov in the paper [13]. They proved the following results.

Theorem 3.11 ([13]). *For any $\alpha \in (0,1)$, there exists a sequence $\boldsymbol{a} \in M_\alpha$ such that series (1.2) diverges almost everywhere.*

Theorem 3.12 ([13]). *Let $\alpha > 1$. Then there exist positive constants $C(\alpha)$ and $x(\alpha)$ such that if a sequence $\boldsymbol{a} \in M_\alpha$, then, for $x \in (0, x(\alpha))$, the sum of series (1.2) satisfies the inequality $g(\boldsymbol{a}, x) \geqslant C(\alpha) v(\boldsymbol{a}, x)$.*

Also, it was shown in [13] that there exists a sequence $\boldsymbol{a} \in M_1$ and a monotone null sequence $\{t_l\}_{l=1}^\infty$ such that

$$\lim_{l\to\infty} \frac{g(\boldsymbol{a}, t_l)}{v(\boldsymbol{a}, t_l)} = 0.$$

In [2], the following analogue of Theorem 1.2 was obtained.

Theorem 3.13 ([2]). *Let an even 2π-periodic function f be in the class $\operatorname{Lip}\beta$ with some $0 < \beta < 1$ and its cosine Fourier coefficients be in the class M_α with some $0 < \alpha < 1$. Then for some $C > 0$ we have $a_n \leqslant C/n^{\alpha+\beta}$ for $n = 1, 2, \dots$.*

This result cannot be improved as it follows from the next statement.

Theorem 3.14 ([2]). *Let $\alpha \in (0,1)$ and $\beta \in (0,1)$. Then there exists an even 2π-periodic function $f \in \operatorname{Lip}\beta$ such that its cosine Fourier coefficients are in the class M_α and also there exists a monotone increasing sequence of natural numbers $\{l_r\}_{r=1}^\infty$ such that the Fourier coefficients $a_{l_r}(f) \geqslant l_r^{-\alpha-\beta}$ for all r.*

Also in [2], the following property of α-monotone sequences was established.

Theorem 3.15 ([2]). *Let $\alpha \in (0,1)$ and $\boldsymbol{a} = \{a_n\}_{n=0}^\infty$ be an α-monotone sequence. Then for any $n \geqslant 1$ holds the inequality $a_k \geqslant a_n A_{n-k}^{\alpha-1}$ for all $0 \leqslant k \leqslant n-1$, and this inequality cannot be improved.*

In [11], M.I. Dyachenko proved the following generalization of one part of Theorem 1.3.

Theorem 3.16 ([11]). *Let $\alpha \in (0,1)$, the coefficients of series (1.2) belong to the class M_α and $na_n \to 0$ as $n \to \infty$. Then series (1.2) uniformly converges.*

As for reverse statement, the following is true.

Theorem 3.17 ([11]). *Let $\alpha \in (0,1)$, series (1.2) uniformly converge and its coefficients belong to the class M_α. Then $n^\alpha a_n \to 0$ as $n \to \infty$ and this result cannot be improved.*

Also in [11] the following generalization of Kolmogorov's theorem was obtained.

Theorem 3.18 ([11]). *Let $\alpha > 1$ and a sequence $\boldsymbol{a} \in M_\alpha$. Then the sum of series (1.1) $f(\boldsymbol{a}, x) \in L([0,\pi])$.*

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