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CONTENTS

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SIMILAR TRANSFORMATION OF ONE CLASS OF WELL-DEFINED RESTRICTIONS

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Key words: maximal (minimal) operator, well-defined restriction, well-defined extension, similar operators, singular coefficients, Riesz basis with brackets.

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Abstract. In this paper there is considered the description of all well-defined restrictions of a maximal operator in a Hilbert space. A class of well-defined restrictions is found for which a similar transformation has the domain of the fixed well-defined restriction. The resulting theorem is applied to the study of n-order differentiation operator and the Laplace operator.

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1 Introduction

Let a closed linear operator L be given in a Hilbert space H. The linear equation

$$Lu = f \tag{1.1}$$

is said to be *well-definedly solvable* on $R(L)$ if $\|u\| \leq C\|Lu\|$ for all $u \in D(L)$ (where $C > 0$ does not depend on u) and *everywhere solvable* if $R(L) = H$. If (1.1) is simultaneously well-defined and solvable everywhere, then we say that L is a *well-defined operator*. A well-definedly solvable operator L_0 is said to be *minimal* if $R(L_0) \neq H$. A closed operator $\widehat{L}$ is called a *maximal operator* if $R(\widehat{L}) = H$ and $\operatorname{Ker}\widehat{L} \neq \{0\}$. An operator A is called a *restriction* of an operator B and B is said to be an *extension* of A if $D(A) \subset D(B)$ and $Au = Bu$ for all $u \in D(A)$.

Note that if one of the well-defined restriction L of a maximal operator $\widehat{L}$ is known, then the inverses of all well-defined restrictions of $\widehat{L}$ have in the form [9]

$$L_K^{-1}f = L^{-1}f + Kf, \tag{1.2}$$

where K is an arbitrary bounded linear operator in H such that $R(K) \subset \operatorname{Ker}\widehat{L}$.

Let L_0 be a minimal operator, and let M_0 be another minimal operator related to L_0 by the equation $(L_0u, v) = (u, M_0v)$ for all $u \in D(L_0)$ and $v \in D(M_0)$. Then $\widehat{L} = M_0^*$ and $\widehat{M} = L_0^*$ are maximal operators such that $L_0 \subset \widehat{L}$ and $M_0 \subset \widehat{M}$. A well-defined restriction L of a maximal operator $\widehat{L}$ such that L is simultaneously a well-defined extension of the minimal operator L_0 is called a *boundary well-defined extension*. The existence of at least one boundary well-defined extension L was proved by Vishik in [14], that is, $L_0 \subset L \subset \widehat{L}$.

The inverse operators to all possible well-defined restrictions L_K of the maximal operator $\widehat{L}$ have form (1.2), moreover

$$D(L_K) = \left\{u \in D(\widehat{L}) : (I - K\widehat{L})u \in D(L)\right\}$$

is dense in H if and only if $\operatorname{Ker}(I + K^*L^*) = \{0\}$. All possible well-defined extensions M_K of M_0 have inverses of the form

$$M_K^{-1}f = (L_K^*)^{-1}f = (L^*)^{-1}f + K^*f,$$

where K is an arbitrary bounded linear operator in H with $R(K) \subset \operatorname{Ker}\widehat{L}$ such that

$$\operatorname{Ker}(I + K^*L^*) = \{0\}.$$

The main result of this work is the following.

Theorem 1.1. *Let L be a boundary well-defined extension of L_0, that is, $L_0 \subset L \subset \widehat{L}$. If L_K is densely defined in H and*

$$R(K^*) \subset D(L^*) \cap D(L_K^*),$$

where K and L are the operators in representation (1.2), then the operator $\overline{KL}_K$, where the bar denotes the closure of an operator in H, is bounded in H and a well-defined restriction L_K of the maximal operator $\widehat{L}$ is similar to the well-defined extension

$$A_K = L - \overline{KL}_K L \quad \text{on} \quad D(A_K) = D(L),$$

of the minimal operator A_0, where $D(A_0) = D(L) \cap Ker(\overline{KL}_K L)$ and $A_0u = Lu$ on $D(A_0)$ (hence, $A_0 \subset L$).

The theory of well-defined restrictions and extensions is intended for the study of unbounded operators in a Hilbert space. A well-defined restriction of a certain maximal operator $\widehat{L}$ is obtained by the domain restriction of the maximal operator. All possible well-defined restrictions L_K are described using one fixed boundary well-defined restriction L in terms of the inverse operator (1.2). Then the direct operator L_K acts as a maximal operator, and its domain is given as a perturbation of the domain of a fixed boundary well-defined restriction L.

The main result of this work is the description of all well-defined restrictions L_K, which are similar to the well-defined extension A_K, of some minimal operator A_0. The domain of A_K coincides with the domain of L, and the action is defined as a perturbation of $\widehat{L}$.

It is clear that the spectra of these similar operators L_K and A_K coincide. Their eigenvectors are different. Further in the work, examples of the application of this abstract theorem to some differential equations are given. We note that a weak perturbation of the boundary condition L_K is equivalent to a singular perturbation of the action of the differential operator $\widehat{L}$.

The study of the properties of singular perturbations of some differential operators and well-defined restrictions is devoted to the works [5], [8].

2 Preliminaries

In this section, we present some results for the well-defined restrictions and extensions [3] which are used in Section 3.

Let A and B be bounded operators in a Hilbert space H. Operators A and B are said to be *similar* if there exist an invertible operator P such that $P^{-1}AP = B$. Similar operators have the same spectrum. If at least one of two operators A and B is invertible, then the operators AB and BA are similar.

Lemma 2.1. *Let L be a densely defined well-defined restriction of a maximal operator $\widehat{L}$ in a Hilbert space H, and K be a bounded linear operator in H. Then the operator KL is bounded on $D(L)$ (hence, $\overline{KL}$ is bounded in H) if and only if*

$$R(K^*) \subset D(L^*).$$

Proof. Let $R(K^*) \subset D(L^*)$. Then, by virtue of the equality $(KL)^* = L^*K^*$, we have that $\overline{KL}$ is bounded in H. Here we have used the boundedness of the operator L^*K^*. Then the operator KL is bounded on $D(L)$. Conversely, let KL be bounded on $D(L)$. Then $\overline{KL}$ is bounded on H, by virtue of the equality $(KL)^* = (\overline{KL})^*$ and, hence, the operator $(KL)^*$ is defined on the whole space H. Moreover, the operator K^* transfers any element f in H to $D(L^*)$. Indeed, for any element g of $D(L)$ we have

$$(Lg, K^*f) = (KLg, f) = (g, (KL)^*f).$$

Therefore, K^*f belongs to the domain $D(L^*)$. □

Lemma 2.2. *Let L_K be a densely defined well-defined restriction of a maximal operator $\widehat{L}$ in a Hilbert space H. Then $D(L^*) = D(L_K^*)$ if and only if $R(K^*) \subset D(L^*) \cap D(L_K^*)$, where L and K are the operators entering representation (1.2).*

Proof. If $D(L^*) = D(L_K^*)$, then by representation (1.2) we easily get

$$R(K^*) \subset D(L^*) \cap D(L_K^*) = D(L^*) = D(L_K^*).$$

Let us prove the converse. If

$$R(K^*) \subset D(L^*) \cap D(L_K^*),$$

then we obtain

$$(L_K^*)^{-1}f = (L^*)^{-1}f + K^*f = (L^*)^{-1}(I + L^*K^*)f, \tag{2.1}$$

$$(L^*)^{-1}f = (L_K^*)^{-1}f - K^*f = (L_K^*)^{-1}(I - L_K^*K^*)f, \tag{2.2}$$

for all f in H. It follows from (2.1) that $D(L_K^*) \subset D(L^*)$, and taking into account (2.2) this implies that $D(L^*) \subset D(L_K^*)$. Thus $D(L^*) = D(L_K^*)$. □

Corollary 2.1. *Let L_K be any densely defined well-defined restriction of a maximal operator $\widehat{L}$ in a Hilbert space H. If $R(K^*) \subset D(L^*)$ and $\overline{KL}$ is a compact operator in H, then*

$$D(L^*) = D(L_K^*).$$

Proof. Compactness of $\overline{KL}$ implies compactness of L^*K^*. Then $R(I + L^*K^*)$ is a closed subspace in H. It follows from the dense definiteness of L_K that $R(I + L^*K^*)$ is a dense set in H. Hence $R(I + L^*K^*) = H$. Then from the equality (2.1) we get $D(L^*) = D(L_K^*)$. □

Lemma 2.3. *If $R(K^*) \subset D(L^*) \cap D(L_K^*)$, then the bounded operators $I + L^*K^*$ and $I - L_K^*K^*$ from (2.1) and (2.2), respectively, have bounded inverses defined on H.*

Proof. By virtue of the density of the domains of the operators L_K^* and L^* it follows that the operators $I + L^*K^*$ and $I - L_K^*K^*$ are invertible. By (2.1) and (2.2) we have $\operatorname{Ker}(I + L^*K^*) = \{0\}$ and $\operatorname{Ker}(I - L_K^*K^*) = \{0\}$, respectively. By representations (2.1) and (2.2) it also follows that

$$R(I + L^*K^*) = H \quad \text{and} \quad R(I - L_K^*K^*) = H,$$

since $D(L^*) = D(L_K^*)$. The inverse operators $(I + L^*K^*)^{-1}$ and $(I - L_K^*K^*)^{-1}$ of the closed operators $I - L_K^*K^*$ and $I + L^*K^*$, respectively, are closed. Then the closed operators $(I + L^*K^*)^{-1}$ and $(I - L_K^*K^*)^{-1}$, defined on the whole of H, are bounded. □

Under the assumptions of Lemma 2.3 the operators KL and KL_K will be (see [2]) restrictions of the bounded operators $\overline{KL}$ and $\overline{KL_K}$, respectively. Thus,

$$(I - L_K^*K^*)^{-1} = I + L^*K^* \quad \text{and} \quad (I - \overline{KL}_K)^{-1} = I + \overline{KL}.$$

In what follows, we need the following theorem.

Theorem 2.1 (Theorem 1.1 [6, p. 307])**.** *A sequence $\{\psi_j\}_{j=1}^{\infty}$ biorthogonal to a basis $\{\phi_j\}_{j=1}^{\infty}$ of a Hilbert space H is also a basis of H.*

3 Proof of Theorem 1.1

In this section we prove our main result Theorem 1.1.

Proof. We transform (1.2) to the form

$$L_K^{-1} = L^{-1} + K = (I + \overline{KL})L^{-1}. \tag{3.1}$$

By Lemma 2.1 and Lemma 2.3 the operators $\overline{KL}$ and $\overline{KL_K}$ are bounded, the operator $I + \overline{KL}$ is invertible and

$$(I + \overline{KL})^{-1} = I - \overline{KL_K}.$$

Then we have

$$\begin{aligned} A_K^{-1} &= (I + \overline{KL})^{-1} L_K^{-1} (I + \overline{KL}) \\ &= (I + \overline{KL})^{-1}(I + \overline{KL})L^{-1}(I + \overline{KL}) = L^{-1}(I + \overline{KL}). \end{aligned}$$

Hence, by Corollary 1 [4, p. 259] we have $D(A_K) = D(L)$ and

$$A_K = (I - \overline{KL_K})L = L - \overline{KL_K}L.$$

Note that $A_K = A_0 = L$ on $D(A_0) = D(L) \cap \operatorname{Ker}(\overline{KL_K}L)$ and A_K is a well-defined extension of the minimal operator A_0. □

Corollary 3.1. *Suppose the hypothesis of Theorem 1.1 is satisfied. Then a well-defined extension L_K^* of a minimal operator M_0 is similar to the well-defined operator*

$$A_K^* = L^*(I - L_K^* K^*)$$

on

$$D(A_K^*) = \big\{v \in H : (I - L_K^* K^*)v \in D(L^*)\big\}.$$

4 An application of Theorem 1.1 to the differentiation operator of order n

In this section, we give some applications of the main result to differential operators.

As a maximal operator $\widehat{L}$ in $L^2(0,1)$, we consider the operator

$$\widehat{L}y = y^{(n)},$$

with the domain $D(\widehat{L}) = W_2^n(0,1)$, $n \in \mathbb{N}$ ($W_2^n(0,1)$ in the Sobolev space). Then the minimal operator L_0 is the restriction of $\widehat{L}$ on $D(L_0) = \mathring{W}_2^n(0,1)$. As a fixed boundary well-defined extension L of the minimal operator L_0, we take the restriction of $\widehat{L}$ on

$$D(L) = \big\{y \in W_2^n(0,1) : y^{(\ell)}(0) + y^{(\ell)}(1) = 0, \ \ell = 1, 2, \ldots, n-1\big\}.$$

We find the inverse to all well-defined restrictions of $L_K \subset \widehat{L}$

$$L_K^{-1} = L^{-1} + K,$$

where

$$Kf=\sum_{\ell=1}^{n} w_\ell(x)\int_0^1 f(t)\overline{\sigma}_\ell(t)\,dt,\quad \sigma_\ell\in L^2(0,1),$$

and $w_\ell\in \operatorname{Ker}\widehat{L},\ \ell=1,2,\ldots,n$ are linearly independent functions with the properties

$$w_\ell^{(k-1)}(0)+w_\ell^{(k-1)}(1)=\begin{cases}1, & \ell=k,\\ 0, & \ell\neq k,\end{cases}\qquad \ell,k=1,2,\ldots,n.$$

Then the operator L_K is the restriction of $\widehat{L}$ on

$$\begin{aligned}&D(L_K)\\ &=\left\{u\in W_2^n(0,1):\, u^{(k-1)}(0)+u^{(k-1)}(1)=\int_0^1 u^{(n)}(t)\overline{\sigma}_k(t)\,dt,\ k=1,2,\ldots,n\right\}.\end{aligned}$$

We will consider restrictions of L_K with dense domains in $L^2(0,1)$, that is,

$$\overline{D(L_K)}=L^2(0,1).$$

If $R(K^*)\subset D(L^*)$, then by Corollary 3.1 the operators $\overline{KL}$ and $\overline{KL_K}$ will be bounded in $L^2(0,1)$. Since $\overline{KL}$ is a compact operator, then by Lemma 2.3 the operator $I+KL$ is invertible and $(I+KL)^{-1}=I-KL_K$. The operator $\overline{KL}$ is bounded if and only if

$$\sigma_\ell\in D(L^*)=\left\{\sigma_\ell\in W_2^n(0,1):\,\sigma_\ell^{(k-1)}(0)+\sigma_\ell^{(k-1)}(1)=0,\quad \ell,k=1,2,\ldots,n\right\}.$$

Hence, we have

$$KLy=\sum_{\ell=1}^{n} w_\ell(x)\int_0^1 y^{(n)}(t)\overline{\sigma}_\ell(t)\,dt=(-1)^n\sum_{\ell=1}^{n} w_\ell(x)\int_0^1 y(t)\overline{\sigma}_\ell^{(n)}(t)\,dt.$$

We find the operator KL_K. For this, we invert the operator

$$(I+KL)y=y+(-1)^n\sum_{\ell=1}^{n} w_\ell(x)\int_0^1 y(t)\overline{\sigma}_\ell^{(n)}(t)\,dt=u,$$

where $y\in D(L),\ u\in D(L_K)$. Then we can write

$$y=(I-KL_K)u=u-(-1)^n\sum_{\ell=1}^{n} w_\ell(x)\sum_{j=1}^{n}\beta_{\ell j}\int_0^1 u(t)\overline{\sigma}_j^{(n)}(t)\,dt,$$

where $\beta_{\ell j},\ \ell,j=1,2,\ldots,n$ are elements of the inverse matrix U^{-1} of the matrix U

$$\begin{aligned}&U\\ &=\begin{pmatrix} 1+(-1)^{n-1}\overline{\sigma}_1^{(n-1)}(0) & (-1)^{n-2}\overline{\sigma}_1^{(n-2)}(0) & \cdots & \overline{\sigma}_1^{(2)}(0) & -\overline{\sigma}_1^{(1)}(0) & \overline{\sigma}_1(0)\\ (-1)^{n-1}\overline{\sigma}_2^{(n-1)}(0) & 1+(-1)^{n-2}\overline{\sigma}_2^{(n-2)}(0) & \cdots & \overline{\sigma}_2^{(2)}(0) & -\overline{\sigma}_2^{(1)}(0) & \overline{\sigma}_2(0)\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ (-1)^{n-1}\overline{\sigma}_n^{(n-1)}(0) & (-1)^{n-2}\overline{\sigma}_n^{(n-2)}(0) & \cdots & \overline{\sigma}_n^{(2)}(0) & -\overline{\sigma}_n^{(1)}(0) & 1+\overline{\sigma}_n(0)\end{pmatrix}.\end{aligned}$$

Note that the conditions $R(K^*)\subset D(L^*)$ and $\overline{D(L_K)}=L^2(0,1)$ imply that $\det U\neq 0$. Thereby, the operator

$$KL_Ku=(-1)^n\sum_{\ell=1}^{n} w_\ell(x)\sum_{j=1}^{n}\beta_{\ell j}\int_0^1 u(t)\overline{\sigma}_j^{(n)}(t)\,dt,$$

is a bounded operator in $L^2(0,1)$. Then the operator A_K has the form

$$A_K v = Lv - \overline{KL}_K Lv = v^{(n)} - (-1)^n \sum_{\ell=1}^{n} w_\ell(x) \sum_{j=1}^{n} \beta_{\ell j} \int_0^1 v^{(n)}(t)\overline{\sigma}_j^{(n)}(t)\,dt,$$

on

$$D(A_K) = D(L) = \{v \in W_2^n(0,1) : v^{(k-1)}(0) + v^{(k-1)}(1) = 0, \quad k = 1,2,\ldots,n\}.$$

The operator A_K can be written as

$$A_K v = v^{(n)} - (-1)^n \sum_{\ell=1}^{n} w_\ell(x) \sum_{j=1}^{n} \beta_{\ell j} F_j(v),$$

where

$$F_j(v) =< F_j, v >= \int_0^1 v^{(n)}(t)\overline{\sigma}_j^{(n)}(t)\,dt, \quad j = 1,2,\ldots,n.$$

It can be seen that $F_j \in W_2^{-n}(0,1)$ in the sense of Lions-Magenes (see [10]).

In this case

$$D(A_0) = \{v \in D(L) : F_j(v) = 0, \quad j = 1,2,\ldots,n\},$$

$A_0 \subset L$ and $A_0 \subset A_K$.

We transform the boundary conditions of L_K to the form

$$U\begin{pmatrix} u(0)+u(1) \\ u^{(1)}(0)+u^{(1)}(1) \\ \vdots \\ u^{(n-1)}(0)+u^{(n-1)}(1) \end{pmatrix} = \begin{pmatrix} \int_0^1 u(t)\overline{\sigma}_1^{(n)}(t)\,dt \\ \int_0^1 u(t)\overline{\sigma}_2^{(n)}(t)\,dt \\ \vdots \\ \int_0^1 u(t)\overline{\sigma}_n^{(n)}(t)\,dt \end{pmatrix}.$$

Then we get

$$u^{(\ell-1)}(0) + u^{(\ell-1)}(1) = \sum_{j=1}^{n} \beta_{\ell j} \int_0^1 u(t)\overline{\sigma}_j^{(n)}(t)\,dt, \quad \ell = 1,2,\ldots,n, \tag{4.1}$$

where $u \in D(L_K)$, $\sigma_j^{(n)} \in L^2(0,1)$, $j = 1,2,\ldots,n$. Boundary condition (4.1) is regular in the Shkalikov sense (see [12]). Then, by virtue of [12], the operator L_K has a system of root vectors forming a Riesz basis with brackets in $L^2(0,1)$. Thereby the operator A_K, being similar to the operator L_K, also has a basis with brackets property. The eigenvalues of these operators coincide. If $\{u_k\}_1^\infty$ are eigenfunctions of the operator L_K, then the eigenfunctions v_k of A_K are related to them by the relations

$$u_k = (I + KL)v_k = v_k + (-1)^n \sum_{\ell=1}^{n} w_\ell(x) \int_0^1 v_k(t)\overline{\sigma}_\ell^{(n)}(t)\,dt, \quad k = 1,2,\ldots,n.$$

If, in particular, we take

$$\sigma_\ell^{(n)}(x) = \mathrm{sign}(x - x_\ell), \quad 0 < x_\ell < 1, \quad \ell = 1,2,\ldots,n,$$

then we get

$$F_\ell(v) = -2v^{(n-1)}(x_\ell), \quad \ell = 1,2,\ldots,n.$$

By Corollary 3.1, Theorem 2.1, and [13, p. 928], we can assert that the system of root vectors of the adjoint operator

$$A_K^* v = (-1)^n \frac{d^n}{dx^n}\left[v(x) - (-1)^n \sum_{\ell,j=1}^{n} \overline{\beta}_{\ell j}\sigma_j^{(n)}(x) \int_0^1 v(t)w_\ell(t)\,dt\right],$$

on

$$D(A_K^*) = \left\{v \in L^2(0,1) :\, v(x) - (-1)^n \sum_{\ell,j=1}^{n} \overline{\beta}_{\ell j}\sigma_j^{(n)}(x) \int_0^1 v(t)w_\ell(t)\,dt \in D(L)\right\},$$

forms a Riesz basis with brackets in $L^2(0,1)$.

5 Example in case $n = 2$

If the maximal operator $\widehat{L}$ acts as

$$\widehat{L}y = -y''$$

on the domain $D(\widehat{L}) = W_2^2(0,1)$, then the minimal operator L_0 is a restriction of $\widehat{L}$ on $D(L_0) = \mathring{W}_2^2(0,1)$. As a fixed operator L we take the restriction of $\widehat{L}$ on

$$D(L) = \big\{y \in W_2^2(0,1) :\, y(0) = y(1) = 0\big\}.$$

Then

$$L_K^{-1} f = L^{-1} f + Kf = -\int_0^x (x-t)f(t)\,dt + x\int_0^1 (1-t)f(t)\,dt$$
$$+ (1-x)\int_0^1 f(t)\overline{\sigma}_1(t)\,dt + x\int_0^1 f(t)\overline{\sigma}_2(t)\,dt,$$

$$Kf = (1-x)\int_0^1 f(t)\overline{\sigma}_1(t)\,dt + x\int_0^1 f(t)\overline{\sigma}_2(t)\,dt.$$

KL is bounded in $L^2(0,1)$, if $R(K^*) \subset D(L^*) = D(L)$, that is,

$$\sigma_1,\, \sigma_2 \in D(L) = \big\{\sigma_1, \sigma_2 \in W_2^2(0,1) :\, \sigma_1(0) = \sigma_1(1) = \sigma_2(0) = \sigma_2(1) = 0\big\},$$

and has the form

$$KLy = -(1-x)\int_0^1 y(t)\overline{\sigma}_1''(t)\,dt - x\int_0^1 y(t)\overline{\sigma}_2''(t)\,dt.$$

The operator KL_K is also bounded in $L^2(0,1)$ and

$$KL_K u = -\frac{1-x}{\Delta}\Big[\big(1 - \overline{\sigma}_2'(1)\big)\int_0^1 u(t)\overline{\sigma}_1''(t)\,dt + \overline{\sigma}_1'(1)\int_0^1 u(t)\overline{\sigma}_2''(t)\,dt\Big]$$
$$- \frac{x}{\Delta}\Big[\big(1 + \overline{\sigma}_1'(0)\big)\int_0^1 u(t)\overline{\sigma}_2''(t)\,dt - \overline{\sigma}_2'(0)\int_0^1 u(t)\overline{\sigma}_1''(t)\,dt\Big],$$

where

$$\Delta = \big(1 + \overline{\sigma}_1'(0)\big)\big(1 - \overline{\sigma}_2'(1)\big) + \overline{\sigma}_2'(0)\,\overline{\sigma}_1'(1).$$

Then the operator A_K has the form

$$A_K v = -v'' - \frac{1}{\Delta}\Big[\big((1-x)(1-\overline{\sigma}_2'(1)) - x\overline{\sigma}_2'(0)\big)\int_0^1 v''(t)\overline{\sigma}_1''(t)\,dt + \big((1-x)\overline{\sigma}_1'(1) + x(1+\overline{\sigma}_1'(0))\big)\int_0^1 v''(t)\overline{\sigma}_2''(t)\,dt\Big],$$

on

$$D(A_K) = D(L) = \big\{v \in W_2^2(0,1):\ v(0) = v(1) = 0\big\},$$

where σ_1'', $\sigma_2'' \in L^2(0,1)$.

We rewrite the operator A_K in the form

$$A_K v = -v'' + a(x)F_1(v) + b(x)F_2(v), \tag{5.1}$$

where

$$a(x) = -\frac{1}{\Delta}\big((1-x)(1-\overline{\sigma}_2'(1)) - x\overline{\sigma}_2'(0)\big),\ \ F_1(v) = \int_0^1 v''(t)\overline{\sigma}_1''(t)\,dt,$$
$$b(x) = -\frac{1}{\Delta}\big((1-x)\overline{\sigma}_1'(1) + x(1+\overline{\sigma}_1'(0)\big),\ \ F_2(v) = \int_0^1 v''(t)\overline{\sigma}_2''(t)\,dt.$$

Note that $F_1, F_2 \in W_2^{-2}(0,1)$ in the sense of Lions-Magenes (see [10]).

Further, we see that the operator L_K acts as $\widehat{L}$ on the domain

$$D(L_K) = \left\{ u \in W_2^2(0,1): \begin{pmatrix} 1+\overline{\sigma}_1'(0) & 0 & -\overline{\sigma}_1'(1) & 0 \\ \overline{\sigma}_2'(0) & 0 & 1-\overline{\sigma}_2'(1) & 0 \end{pmatrix} \begin{pmatrix} u(0) \\ u'(0) \\ u(1) \\ u'(1) \end{pmatrix} = \begin{pmatrix} -\int_0^1 u(t)\overline{\sigma}_1''(t)\,dt \\ -\int_0^1 u(t)\overline{\sigma}_2''(t)\,dt \end{pmatrix} \right\},$$

and the determinant of the matrix composed of the first and third columns of the boundary conditions matrix is

$$J_{13} = \big(1+\overline{\sigma}_1'(0)\big)\big(1-\overline{\sigma}_2'(1)\big) + \overline{\sigma}_2'(0)\,\overline{\sigma}_1'(1) = \Delta \neq 0,$$

since $R(K^*) \subset D(L^*)$ and $\overline{D(L_K)} = L^2(0,1)$. Then the left-hand side of this boundary condition is non-degenerate according to Marchenko [11], hence regular according to Birkhoff (see [12]). By virtue of Theorem (see [12, p. 15]), the system of root vectors of the operator L_K forms a Riesz basis with brackets in $L^2(0,1)$. Thus, by virtue of Theorem (1.1) the system of root vectors of A_K also forms a Riesz basis with brackets and the eigenvalues of L_K and A_K coincide, and the eigenfunctions are related to each other as follows

$$u_k = v_k - (1-x)\int_0^1 v_k(t)\overline{\sigma}_1''(t)\,dt - x\int_0^1 v_k(t)\overline{\sigma}_2''(t)\,dt,\ \ k \in \mathbb{N}.$$

If in the particular case we take

$$\begin{aligned} \overline{\sigma}_1''(x) &= \operatorname{sign}(x-x_1) - \operatorname{sign}(x-x_2), \\ \overline{\sigma}_2''(x) &= x\big[\operatorname{sign}(x-x_1) - \operatorname{sign}(x-x_2)\big], \end{aligned} \tag{5.2}$$

where $0 < x_1 < x_2 < 1$, then we get

$$F_1(v) = 2v'(x_2) - 2v'(x_1),$$
$$F_2(v) = 2x_2 v'(x_2) - 2x_1 v'(x_1) - 2v(x_2) + 2v(x_1),$$

in (5.1).

In this case

$$D(A_0) = \{v \in D(L) : \, v(x_1) = v(x_2), \quad v'(x_1) = v'(x_2)\},$$

$A_0 \subset L$ and $A_0 \subset A_K$.

By Corollary 3.1, Theorem 2.1, and [13, p. 928], for $n = 2$, we can assert that the system of root vectors of the operator

$$A_K^* v = (-1)^2 \frac{d^2}{dx^2}\Big[v(x) - c(x)\int_0^1 (1-t)v(t)\,dt - d(x)\int_0^1 tv(t)\,dt\Big],$$

on

$$D(A_K^*) = \Big\{v \in L^2(0,1) : \, v(x) - c(x)\int_0^1 (1-t)v(t)\,dt - d(x)\int_0^1 tv(t)\,dt \in D(L)\Big\},$$

forms a Riesz basis with brackets in $L^2(0,1)$, where

$$c(x) = -\frac{1}{\Delta}\Big[\big(1 - \sigma_2'(1)\big)\sigma_1''(x) + \sigma_1'(1)\sigma_2''(x)\Big],$$
$$d(x) = \frac{1}{\Delta}\Big[\big(1 + \sigma_1'(0)\big)\sigma_2''(x) - \sigma_2'(0)\sigma_1''(x)\Big].$$

Note that

$$\sigma_1'', \sigma_2'' \in L^2(0,1), \quad D(L) = \big\{y \in W_2^2(0,1) : \, y(0) = y(1) = 0\big\}.$$

For clarity, we consider the special case (5.2), then we have

$$c(x) = \frac{\operatorname{sign}(x - x_1) - \operatorname{sign}(x - x_2)}{\Delta}\left[1 + \frac{x_2^3 - x_1^3}{3} - \frac{x_2^2 - x_1^2}{2}x\right],$$
$$d(x) = \frac{\operatorname{sign}(x - x_1) - \operatorname{sign}(x - x_2)}{\Delta}\left[\left(1 + x_2 - x_1 - \frac{x_2^2 - x_1^2}{2}\right)x - \frac{x_2^2 - x_1^2}{2} + \frac{x_2^3 - x_1^3}{3}\right].$$

The domain of A_K^* will have the form

$$\begin{aligned} D(A_K^*) = \Big\{ & v \in L^2(0,1) \cap W_2^2(0,x_1) \cap W_2^2(x_1,x_2) \cap W_2^2(x_2,1) : \, v(0) = v(1) = 0, \\ & v(x_1 - 0) - v(x_1 + 0) = -c(x_1 + 0)\int_0^1 (1-t)v(t)\,dt - d(x_1 + 0)\int_0^1 tv(t)\,dt, \\ & v(x_2 + 0) - v(x_2 - 0) = -c(x_2 - 0)\int_0^1 (1-t)v(t)\,dt - d(x_2 - 0)\int_0^1 tv(t)\,dt, \\ & v'(x_1 - 0) - v'(x_1 + 0) = -c'(x_1 + 0)\int_0^1 (1-t)v(t)\,dt - d'(x_1 + 0)\int_0^1 tv(t)\,dt, \\ & v'(x_2 + 0) - v'(x_2 - 0) = -c'(x_2 - 0)\int_0^1 (1-t)v(t)\,dt - d'(x_2 - 0)\int_0^1 tv(t)\,dt \Big\}, \end{aligned}$$

where

$$
\begin{aligned}
c(x_1+0) &= \frac{2}{\Delta}\Big(1+\frac{x_2^3-x_1^3}{3}-\frac{x_2^2-x_1^2}{2}x_1\Big),\\
d(x_1+0) &= -\frac{2}{\Delta}\Big(\big(1+x_2-x_1-\frac{x_2^2-x_1^2}{2}\big)x_1-\frac{x_2^2-x_1^2}{2}+\frac{x_2^3-x_1^3}{3}\Big),\\
c(x_2-0) &= \frac{2}{\Delta}\Big(1+\frac{x_2^3-x_1^3}{3}-\frac{x_2^2-x_1^2}{2}x_2\Big),\\
d(x_2-0) &= -\frac{2}{\Delta}\Big(\big(1+x_2-x_1-\frac{x_2^2-x_1^2}{2}\big)x_2-\frac{x_2^2-x_1^2}{2}+\frac{x_2^3-x_1^3}{3}\Big),\\
c'(x_1+0) &= -\frac{1}{\Delta}\big(x_2^2-x_1^2\big),\\
d'(x_1+0) &= \frac{2}{\Delta}\Big(1+x_1-x_2+\frac{x_2^2-x_1^2}{2}\Big),\\
c'(x_2-0) &= c'(x_1+0), \quad d'(x_2-0)=d'(x_1+0),\\
\Delta &= 1+x_2-x_1-\frac{x_2^2-x_1^2}{2}+\frac{x_2^3-x_1^3}{3}+\frac{x_2-x_1}{12}\big((x_2-x_1)^3+6x_1x_2\big)\neq 0,
\end{aligned}
$$

since $x_1, x_2 \in (0,1)$. Moreover, the operator A_K^* acts as follows

$$A_K^* v = -v''(x) + c''(x)\int_0^1 (1-t)v(t)\,dt + d''(x)\int_0^1 tv(t)\,dt,$$

where

$$
\begin{aligned}
c''(x) = {} & \frac{2}{\Delta}\Big[1+\frac{x_2^3-x_1^3}{3}+\frac{x_2^2-x_1^2}{2}(x_2-x_1)\Big]\big(\delta'(x-x_1)-\delta'(x-x_2)\big)\\
& -\frac{1}{\Delta}\big(x_2^2-x_1^2\big)\big(\delta(x-x_1)-\delta(x-x_2)\big),
\end{aligned}
$$

$$
\begin{aligned}
d''(x) = {} & \frac{2}{\Delta}\Big[\Big(1+x_2-x_1-\frac{x_2^2-x_1^2}{2}\Big)(x_1-x_2)-\frac{x_2^2-x_1^2}{2}+\frac{x_2^3-x_1^3}{3}\Big]\\
& \times\big(\delta'(x-x_1)-\delta'(x-x_2)\big)\\
& -\frac{1}{\Delta}\Big(1+x_2-x_1+\frac{x_2^2-x_1^2}{2}\Big)\big(\delta(x-x_1)-\delta(x-x_2)\big),
\end{aligned}
$$

here δ is the Dirac delta-function.

6 An application to the Laplace operator

In the Hilbert space $L_2(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^m$ with infinitely smooth boundary $\partial\Omega$, let us consider the minimal L_0 and maximal $\widehat{L}$ operators generated by the Laplace operator

$$-\Delta u = -\Big(\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\cdots+\frac{\partial^2 u}{\partial x_m^2}\Big). \tag{6.1}$$

The closure L_0 in the space $L_2(\Omega)$ of the Laplace operator (6.1) with the domain $C_0^\infty(\Omega)$ is called the minimal operator corresponding to the Laplace operator.

The operator $\widehat{L}$, adjoint to the minimal operator L_0 corresponding to the Laplace operator is called the maximal operator (see [7]). Note that

$$D(\widehat{L}) = \{u \in L_2(\Omega) : \ \widehat{L}u = -\Delta u \in L_2(\Omega)\}.$$

Denote by L the operator corresponding to the Dirichlet problem with the domain

$$D(L) = \{u \in W_2^2(\Omega) : \ u|_{\partial\Omega} = 0\}.$$

Then the inverse operators L_K^{-1} to all possible well-defined restrictions of the maximal operator $\widehat{L}$ corresponding to the Laplace operator (6.1) have the following form:

$$u \equiv L_K^{-1} f = L^{-1} f + Kf, \tag{6.2}$$

where K is an arbitrary bounded (in $L_2(\Omega)$) linear operator with

$$R(K) \subset \operatorname{Ker} \widehat{L} = \{u \in L_2(\Omega) : \ -\Delta u = 0\}.$$

Then the direct operator L_K is determined from the following problem:

$$\widehat{L}u = f, \quad f \in L_2(\Omega), \tag{6.3}$$

$$D(L_K) = \{u \in D(\widehat{L}) : \ (I - K\widehat{L})u|_{\partial\Omega} = 0\}, \tag{6.4}$$

where I is the unit operator in $L_2(\Omega)$. There are no other linear well-defined restrictions of the operator $\widehat{L}$ (see [1]).

The operators $(L_K^*)^{-1}$, corresponding to the operators L_K^*,

$$v \equiv (L_K^*)^{-1} g = L^{-1} g + K^* g,$$

describe the inverse operators to all possible well-defined extensions of the minimal operator L_0 if and only if K satisfies the condition (see [1])

$$\operatorname{Ker}(I + K^* L) = \{0\}.$$

Note that the last condition is equivalent to the following one: $\overline{D(L_K)} = L_2(\Omega)$. If the additional condition

$$KR(L_0) = \{0\}$$

is imposed on the operator K from (6.2), then the operator L_K corresponding to problem (6.3), (6.4), will turn out to be boundary well-defined. By applying Theorem 1.1 to this particular case we have

Theorem 6.1. *Let the operator K have the form*

$$Kf(x) = \phi(x) \iint\limits_{\Omega} f(\xi)\overline{g(\xi)}d\xi, \quad x, \ \xi \in \Omega \subset \mathbb{R}^m,$$

where ϕ is a harmonic function in $L_2(\Omega)$, $g \in L_2(\Omega)$, and

$$K^* f(x) = g(x) \iint\limits_{\Omega} f(\xi)\overline{\phi(\xi)}d\xi.$$

If K satisfies the assumptions of Theorem 1.1, then $g \in W_2^2(\Omega)$, $g(x) \mid_{\partial\Omega}= 0$,

$$\iint\limits_{\Omega} \phi(\xi)(\Delta\overline{g})(\xi)d\xi \neq 1,$$

and the well-defined operator

$$A_K u(x) = -\Delta u(x) + \frac{\phi(x)}{1 + \iint\limits_{\Omega} \phi(\xi)(\Delta\overline{g})(\xi)d\xi} \iint\limits_{\Omega} (\Delta u)(\xi)(\Delta\overline{g})(\xi)d\xi,$$

,

$$D(A_K) = \Big\{u \in W_2^2(\Omega) : \big(u(x) \mid_{\partial\Omega}= 0\Big\}$$

describes a relatively bounded perturbation of L_K which has the same eigenvalues as L_K.

The system of root vectors of A_K is complete in $L_2(\Omega)$. Morever, if $\{v_k\}$ is a system of eigenfunctions of L_K , then the system of eigenvectors $\{u_k\}$ of A_K has the form

$$u_k(x) = ((I + \overline{KL})v_k)(x) = v_k(x) + \phi(x) \iint\limits_{\Omega} v_k(\xi)(\Delta\overline{g})(\xi)d\xi, \quad k = 1, 2, \ldots$$

We can rewrite

$$A_K u(x) = -\Delta u(x) + \frac{\phi(x)}{1 + \iint\limits_{\Omega} \phi(\xi)(\Delta\overline{g})(\xi)d\xi} F(u),$$

where

$$F(u) = \iint\limits_{\Omega} (\Delta u)(\xi)(\Delta\overline{g})(\xi)d\xi.$$

Note that $F \in W_2^{-2}(\Omega)$ in the sense of Lions-Magenes (see [10]).

In this case

$$D(A_0) = \left\{ v \in D(L) : \iint\limits_{\Omega} (\Delta u)(\xi)(\Delta\overline{g})(\xi)d\xi = 0 \right\},$$

$A_0 \subset L$ and $A_0 \subset A_K$.

Consider a more visual cases when $m = 2$ and $m = 3$, that is, $\Omega \subset \mathbb{R}^2$ and $\Omega \subset \mathbb{R}^3$ respectively. To do this, we define the operator K by using the function g constructed in the following way. Let $x_0 \in \Omega$, be a point lying strictly inside the closed domain $\overline{\Omega}$. As functions $g(x)$ we take the solution to the following Dirichlet problem

$$-(\Delta g)(x) = -\ln|x - x_0|, \quad g|_{\partial\Omega} = 0, \tag{6.5}$$

for $m = 2$ and

$$-(\Delta g)(x) = |x - x_0|, \quad g|_{\partial\Omega} = 0, \tag{6.6}$$

for $m = 3$, respectively. Then we get the following:

$$A_K u(x) = -\Delta u(x) + \frac{\phi(x)}{1 + \iint\limits_{\Omega} \phi(\xi)(\ln|\xi - x_0|)d\xi} \int\limits_{\partial\Omega} \frac{\partial u}{\partial n}(\xi)(\ln|\xi - x_0|)d\xi$$
$$+ \frac{\phi(x)u(x_0)}{1 + \iint\limits_{\Omega} \phi(\xi)(\ln|\xi - x_0|)d\xi},$$
$$D(A_K) = \Big\{u \in W_2^2(\Omega) : u(x) \mid_{\partial\Omega}= 0\Big\}$$

for the case $m = 2$ and

$$A_K u(x) = -\Delta u(x) + \frac{\phi(x)}{1 + \iint\limits_{\Omega} \phi(\xi) \frac{1}{|\xi - x_0|} d\xi} \int\limits_{\partial\Omega} \frac{\partial u}{\partial n}(\xi) \frac{1}{|\xi - x_0|} d\xi$$

$$+ \frac{\phi(x) u(x_0)}{1 + \iint\limits_{\Omega} \phi(\xi) \frac{1}{|\xi - x_0|} d\xi},$$

$$D(A_K) = \left\{ u \in W_2^2(\Omega) : u(x) \,|_{\partial\Omega} = 0 \right\}$$

for the case $m = 3$. We have obtained a relatively bounded perturbation A_K of L_K which has the same eigenvalues as the operator L_K. The system of root vectors of A_K is complete in the $L_2(\Omega)$. If $\{v_k\}$ is a system of eigenfunctions of L_K, then the system of eigenfunctions $\{u_k\}$ of A_K has the form

$$u_k(x) = ((I + \overline{KL}) v_k)(x) = v_k(x) + \phi(x) \iint\limits_{\Omega} v_k(\xi) \ln |\xi - x_0| d\xi, \quad k = 1, 2, \ldots,$$

in the case $m = 2$ and

$$u_k(x) = ((I + \overline{KL}) v_k)(x) = v_k(x) + \phi(x) \iint\limits_{\Omega} v_k(\xi) \frac{1}{|\xi - x_0} | d\xi, \quad k = 1, 2, \ldots,$$

in the case $m = 3$, respectively.

Thus, we have constructed a singular perturbation A_K of the L_K with a complete system of root vectors. Indeed,

$$L_K^{-1} = L^{-1} + K = (I + KL) L^{-1},$$

where KL is compact operator, $I + KL$ is invertable operator. Selfadjoint operator L^{-1} is positive, compact and belongs to the Neumann-Schatten class. Then by Theorem 8.1 [6, p. 257]the system of root vectors L_K is complete in $L_2(\Omega)$. Hence, by Theorem 1.1 the system of root vectors of A_K is complete in $L_2(\Omega)$.

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