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The Eurasian Mathematical Journal (EMJ)
The Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana
Kazakhstan

ERLAN DAUTBEKOVICH NURSULTANOV

(to the 60th birthday)

On May 25, 2017 was the 60th birthday of Yerlan Dautbekovich Nursultanov, Doctor of Physical and Mathematical Sciences (1999), Professor (2001), Head of the Department of Mathematics and Informatics of the Kazakhstan branch of the M.V. Lomonosov Moscow State University (since 2001), member of the Editorial Board of the Eurasian Mathematical Journal.

E.D. Nursultanov was born in the city of Karaganda. He graduated from the Karaganda State University (1979) and then completed his post-graduate studies at the M.V. Lomonosov Moscow State University.

Professor Nursultanov's scientific interests are related to various areas of the theory of functions and functional analysis.

He introduced the concept of multi-parameter Lorentz spaces, network spaces and anisotropic Lorentz spaces, for which appropriate interpolation methods were developed. On the basis of the apparatus introduced by him, the questions of reiteration in the off-diagonal case for the real Lyons-Petre interpolation method, the multiplier problem for trigonometric Fourier series, the lower and upper bounds complementary to the Hardy-Littlewood inequalities for various orthonormal systems were solved. The convergence of series and Fourier transforms were studied with sufficiently general monotonicity conditions. The lower bounds for the norm of the convolution operator are obtained, and its upper bounds are improved (a stronger result than the O'Neil inequality). An exact cubature formula with explicit nodes and weights for functions belonging to spaces with a dominated mixed derivative is constructed, and a number of other problems in this area are solved.

He has published more than 50 scientific papers in high rating international journals included in the lists of Thomson Reuters and Scopus. 2 doctor of sciences, 9 candidate of sciences and 4 PhD dissertations have been defended under his supervision.

His merits and achievements are marked with badges of the Ministry of Education and Science of the Republic of Kazakhstan "For Contribution to the Development of Science" (2007), "Honored Worker of Education" (2011), "Y. Altynsarin" (2017). He is a laureate of the award named after K. Satpaev in the field of natural sciences for 2005, the grant holder "The best teacher of the university" for 2006 and 2011, the grant holder of the state scientific scholarship for outstanding contribution to the development of science and technology of the Republic of Kazakhstan for years 2007-2008, 2008 -2009. In 2017 he got the Top Springer Author award, established by Springer Nature together with JSC "National Center for Scientific and Technical Information".

The Editorial Board of the Eurasian Mathematical Journal congratulates Erlan Dautbekovich Nursultanov on the occasion of his 60th birthday and wishes him good health and successful work in mathematics and mathematical education.

JAMALBEK TUSSUPOV

(to the 60th birthday)

On April 10, 2017 was the 60th birthday of Jamalbek Tussupov, Doctor of Physical and Mathematical Sciences, Professor, Head of the Information Systems Department of the L.N. Gumilyov Eurasian National University, member of the Kazakhstan and American Mathematical Societies, member of the Association of Symbolic Logic, member of the Editorial Board of the Eurasian Mathematical Journal.

J. Tussupov was born in Taraz (Jambyl region of the Kazakh SSR). He graduated from the Karaganda State University (Kazakhstan) in 1979 and later on completed his postgraduate studies at S.L. Sobolev Institute of Mathematics of the Academy of Sciences of Russia (Novosibirsk).

Professor Tussupov's research interests are in mathematical logic, computability, computable structures, abstract data types, ontology, formal semantics. He solved the following problems of computable structures:

- the problems of S.S. Goncharov and M.S. Manasse: the problem of characterizing relative categoricity in the hyperarithmetical hierarchy given levels of complexity of Scott families, and the problem on the relationship between categoricity and relative categoricity of computable structures in the arithmetical and hyperarithmetical hierarchies;

- the problem of Yu.L. Ershov: the problem of finite algorithmic dimension in the arithmetical and hyperarithmetical hierarchies;

- the problem of C.J. Ash and A. Nerode: the problem of the interplay of relations of bounded arithmetical and hyperarithmetical complexity in computable presentations and the definability of relations by formulas of given complexity;

- the problem of S. Lempp: the problem of structures having presentations in just the degrees of all sets X such that for algebraic classes as symmetric irreflexive graphs, nilpotent groups, rings, integral domains, commutative semigroups, lattices, structure with two equivalences, bipartite graphs.

Professor Tussupov has published about 100 scientific papers, five textbooks for students and one monograph. Three PhD dissertations have been defended under his supervision.

Professor Tussupov is a fellow of "Bolashak" Scholarship, 2011 (Notre Dame University, USA), "Erasmus+", 2016 (Poitiers University, France). He was awarded the title "The Best Professor of 2012" (Kazakhstan). In 2015 Jamalbek Tussupov was also awarded for the contribution to science in the Republic of Kazakhstan.

The Editorial Board of the Eurasian Mathematical Journal congratulates Dr. Professor Jamalbek Tussupov on the occasion of his 60th aniversary and wishes him strong health, new achievements in science, inspiration for new ideas and fruitfull results.

Short communications

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ON PRECOMPACTNESS OF A SET IN GENERAL LOCAL AND GLOBAL MORREY-TYPE SPACES

N.A. Bokayev, V.I. Burenkov, D.T. Matin

Communicated by M.L. Goldman

Key words: Morrey spaces, precompactness, local and global Morrey-type spaces.

AMS Mathematics Subject Classification: 35Q79, 35K05, 35K20.

Abstract. Necessary and sufficient conditions for the precompactness of a set in general local Morrey-type spaces and sufficient conditions for the precompactness of a set in general global Morrey-type spaces are obtained.

1 Introduction

The classical Morrey spaces were introduced in the works of Charles Morrey in 1938 in connection with the investigation of the properties of solutions of quasilinear elliptic differential equations. In recent decades problems of the boundedness and compactness of various operators in Morrey-type spaces have been actively studied ([3], [4]).

Morrey spaces M_p^λ are defined as the sets of all functions $f \in L_p^{loc}(\mathbb{R}^n)$, for which , $0 \leq \lambda \leq \frac{n}{p}$, $0 < p \leq \infty$, with finite quasinorm

$$\|f\|_{M_p^\lambda(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\ r>0} r^{-\lambda} \left(\int\limits_{B(x,r)} |f(y)|^p dy \right)^{\frac{1}{p}} < \infty,$$

where $B(x,r)$ is the open ball in $\mathbb{R}^n$ centered at the point x of radius $r > 0$.

Note that

$$\|f\|_{M_p^0(\mathbb{R}^n)} = \|f\|_{L_p(\mathbb{R}^n)}, \|f\|_{M_p^{\frac{n}{p}}(\mathbb{R}^n)} = \|f\|_{L_\infty(\mathbb{R}^n)}.$$

If $\lambda < 0$ or $\lambda > \frac{n}{p}$, the space $M_p^\lambda(\mathbb{R}^n)$ is trivial, i.e. consists only of functions equivalent to zero on $\mathbb{R}^n$.

According to the well-known Freche-Kolmogorov theorem, a set $S \subset L_p(\mathbb{R}^n)$, where $1 \leq p < \infty$, is precompact if and only if

$$\sup_{f\in S} \|f\|_{L_p(\mathbb{R}^n)} < \infty, \tag{1.1}$$

$$\lim_{\delta\to 0^+} \sup_{|h|\leq\delta} \|f(\cdot + h) - f(\cdot)\|_{L_p(\mathbb{R}^n)} = 0 \tag{1.2}$$

and

$$\lim_{R\to\infty} \sup_{f\in S} \|f\|_{L_p({}^cB(0,R))} = 0, \tag{1.3}$$

where ${}^{c}B(0,R)$ is the complement of a ball $B(0,R)$.

Conditions (1.1)-(1.3) are equivalent to the union of conditions (1.1), (1.3) and

$$\lim_{\delta\to 0^+}\sup_{f\in S}\|A_\delta f-f\|_{L_p(\mathbb{R}^n)}=0, \tag{1.4}$$

where for any $\delta>0$ and $f\in L_1^{loc}(\mathbb{R}^n)$

$$(A_\delta f)(x)=\frac{1}{|B(x,\delta)|}\int\limits_{B(x,\delta)} f(y)dy=\int\limits_{B(0,\delta)}\omega_\delta(y)f(x-y)dy=(\omega_\delta * f)(x),\ x\in\mathbb{R}^n, \tag{1.5}$$

$\omega_\delta(x)=\frac{\chi_{B(0,\delta)}(x)}{|B(0,\delta)|}$, χ_A is the characteristic function of a set $A\subset\mathbb{R}^n$. $|A|$ is the Lebesgue measure of a set A. Recall that condition (1.4) follows from condition (1.2), and condition (1.2) follows from conditions (1.4) and (1.3).

Note also that if $A\subset\mathbb{R}^n$ is a bounded set, then for the precompactness of a set $S\subset L_p(A)$ it is necessary and sufficient that conditions (1.1) - (1.2) are satisfied, where $\mathbb{R}^n$ replaced by the set A.

Sufficient conditions ensuring the precompactness of set in the Morrey spaces M_p^λ were obtained in [12], [13] and in the generalized Morrey spaces $M_p^{w(\cdot)}$ in [1], [2].

The aim of this paper is to obtain necessary and sufficient conditions for the precompactness of a set in the general local Morrey-type spaces and sufficient conditions for the precompactness of a set in the general global Morrey-type spaces.

2 Notation and preliminaries

Definition 1. Let $0<p,\theta\le\infty$, and let w be a nonnegative measurable function on $(0,\infty)$. We denote by $LM_{p\theta,w(\cdot)}$ the general local Morrey-type space, the space of all functions $f\in L_p^{loc}(R^n)$ with finite quasi-norm

$$\|f\|_{LM_{p\theta,w(\cdot)}}=\Big\|w(r)\,\|f\|_{L_p(B(0,r))}\Big\|_{L_\theta(0,\infty)}.$$

We denote by Ω_θ the set of all functions that are nonnegative, measurable on $(0,\infty)$, not equivalent 0 and such, that for some $t>0$

$$\|w(r)\|_{L_\theta(t,\infty)}<\infty.$$

The space $LM_{p\theta,w(\cdot)}$ is non-trivial, that is, it consists not only of functions, equivalent to 0 on $\mathbb{R}^n$, if and only if $w\in\Omega_\theta$ [8].

Definition 2. Let $0<p,\theta\le\infty$ and let w be a nonnegative measurable function on $(0,\infty)$. The general global Morrey-type space $GM_{p\theta,w(\cdot)}$ is defined as the set of all functions $f\in L_p^{loc}(\mathbb{R}^n)$ with finite quasi-norm

$$\|f\|_{GM_{p\theta,w(\cdot)}}=\sup_{x\in\mathbb{R}^n}\Big\|w(r)\,\|f\|_{L_p(B(x,r))}\Big\|_{L_\theta(0,\infty)}.$$

We denote by $\Omega_{p\theta}$ the set of all functions that are nonnegative, measurable on $(0,\infty)$, not equivalent to 0 and such, that for some $t>0$ (and therefore for all $t>0$)

$$\|w(r)r^{\frac{n}{p}}\|_{L_\theta(0,t)}<\infty,\quad \|w(r)\|_{L_\theta(t,\infty)}<\infty.$$

The space $GM_{p\theta,w(\cdot)}$ is non-trivial, that is, it consists not only of functions equivalent to 0 in $\mathbb{R}^n$ if and only if $w\in\Omega_{p\theta}$ [8], [9]. If $\theta=\infty$, then $GM_{p\infty,w(\cdot)}\equiv M_p^{w(\cdot)}$.

Definition 3. Let $0 < p, \theta \le \infty$, $\lambda > 0$ if $\theta < \infty$ and $\lambda \ge 0$ if $\theta = \infty$. We denote by $LM_{p\theta}^{\lambda}$ the local Morrey-type space, the space of all functions $f \in L_p^{loc}(R^n)$ with finite quasinorm

$$\|f\|_{LM_{p\theta}^{\lambda}} = \left(\int_0^{\infty} \left(\frac{\|f\|_{L_p(B(0,r))}}{r^{\lambda}} \right)^{\theta} \frac{dr}{r} \right)^{\frac{1}{\theta}}.$$

Definition 4. Let $0 < p, \theta \le \infty$, $0 < \lambda < \frac{n}{p}$ if $\theta < \infty$ and $0 \le \lambda \le \frac{n}{p}$ if $\theta = \infty$. The global Morrey-type space $GM_{p\theta}^{\lambda}$ is defined as the set of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{GM_{p\theta}^{\lambda}} = \sup_{x \in \mathbb{R}^n} \left(\int_0^{\infty} \left(\frac{\|f\|_{L_p(B(x,r))}}{r^{\lambda}} \right)^{\theta} \frac{dr}{r} \right)^{\frac{1}{\theta}}.$$

Clearly the spaces $LM_{p\theta}^{\lambda}$, $GM_{p\theta}^{\lambda}$ are particular cases of the spaces $LM_{p\theta,w(\cdot)}$, $GM_{p\theta,w(\cdot)}$ respectively, with $w(r) = r^{-\lambda-\frac{1}{\theta}}$. The condition $r^{-\lambda-\frac{1}{\theta}} \in \Omega_\theta$ is equivalent to the condition $\lambda > 0$ if $\theta < \infty$ and $\lambda \ge 0$ if $\theta = \infty$. Respectively, the condition $r^{-\lambda-\frac{1}{\theta}} \in \Omega_{p\theta}$ is equivalent to the condition $0 < \lambda < \frac{n}{p}$ if $\theta < \infty$ and $0 \le \lambda \le \frac{n}{p}$ if $\theta = \infty$.

Note that for any $x \in \mathbb{R}^n$ and $\delta > 0$

$$|(A_\delta f)(x)| \le \frac{1}{|B(x,\delta)|} \int_{B(x,\delta)} |f(y)| dy \le \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy = (Mf)(x), \tag{2.1}$$

where Mf is the Hardy-Littlewood maximal function.

In the sequel we shall use the following special case of the theorem on boundedness of the operator M in the space $LM_{p\theta,w(\cdot)}$, proved in [3], [4], [7].

Theorem 2.1. *Let $1 < p < \infty$, $0 < \theta \le \infty$, $w \in \Omega_\theta$. In order that the maximal operator M be bounded in the space $LM_{p\theta,w(\cdot)}$, it is necessary and sufficient that for some $c_1 > 0$*

$$t^{-\frac{n}{p}} \left\| w(r) r^{\frac{n}{p}} \right\|_{L_\theta(0,t)} \le c_1 \|w(r)\|_{L_\theta(t,\infty)}, \tag{2.2}$$

for any $t > 0$. If $w \in \Omega_{p\theta}$, then condition (2.2) is a sufficient condition for the boundedness of the operator M in the space $GM_{p\theta,w(\cdot)}$.

Corollary 2.1. *Let $1 < p < \infty$, $0 < \theta \le \infty$. If $w \in \Omega_\theta$ and condition (2.2), is satisfied, then there exists $c_2 > 0$, depending only on n, p, θ and c_1, such that for any $\delta > 0$ and $f \in LM_{p\theta,w(\cdot)}$*

$$\|A_\delta f\|_{LM_{p\theta,w(\cdot)}} \le c_2 \|f\|_{LM_{p\theta,w(\cdot)}}. \tag{2.3}$$

Corollary 2.2. *Let $1 < p < \infty$, $0 < \theta \le \infty$. Then for any $0 < \lambda < \frac{n}{p}$ for $\theta < \infty$ and $0 \le \lambda \le \frac{n}{p}$, for $\theta = \infty$ there exists $c_3 > 0$, depending only on n, p, θ and λ, such that for any $\delta > 0$ and $f \in LM_{p\theta}^{\lambda}$*

$$\|A_\delta f\|_{LM_{p\theta}^{\lambda}} \le c_3 \|f\|_{LM_{p\theta}^{\lambda}},$$

Lemma 2.1. *Let $1 \le p \le \infty, 0 < \theta \le \infty$ and $w \in \Omega_{p\theta}$. Then for any $\delta > 0$ and $f \in GM_{p\theta,w(\cdot)}$*

$$\|A_\delta f\|_{GM_{p\theta,w(\cdot)}} \le \|f\|_{GM_{p\theta,w(\cdot)}}. \tag{2.4}$$

Corollary 2.3. *Let $1 \le p \le \infty$, $0 < \theta \le \infty$ and $0 \le \lambda \le \frac{n}{p}$ if $\theta < \infty$ and $0 \le \lambda \le \frac{n}{p}$ if $\theta = \infty$. Then*

$$\|A_\delta f\|_{GM^{\lambda}_{p\theta}} \le \|f\|_{GM^{\lambda}_{p\theta}}.$$

Lemma 2.2. *Let $1 < p < \infty$, $0 < \theta < \infty$, $w \in \Omega_\theta$ and let condition (2.2) be satisfied. Then for any function $f \in LM_{p\theta,w(\cdot)}$*

$$\lim_{\delta\to 0^+} \|A_\delta f - f\|_{LM_{p\theta,w(\cdot)}} = 0. \tag{2.5}$$

Lemma 2.3. *Let $0 < p < \infty$, $0 < \theta < \infty$, $w \in \Omega_\theta$. Then for any function $f \in LM_{p\theta,w(\cdot)}$*

$$\lim_{R_1\to 0^+} \left\| f\chi_{B(0,R_1)} \right\|_{LM_{p\theta,w(\cdot)}} = 0. \tag{2.6}$$

Lemma 2.4. *Let $0 < p \le \infty$, $0 < \theta < \infty$, $w \in \Omega_\theta$. Then for any function $f \in LM_{p\theta,w(\cdot)}$*

$$\lim_{R_2\to \infty} \left\| f\chi_{^cB(0,R_2)} \right\|_{LM_{p\theta,w(\cdot)}} = 0. \tag{2.7}$$

Remark 1. For the global spaces $GM_{p\theta,w(\cdot)}$, relations (2.5) - (2.7) with $GM_{p\theta,w(\cdot)}$ instead of $LM_{p\theta,w(\cdot)}$, in general, are not satisfied. A corresponding example is given in [2].

3 Main results

Theorem 3.1. *Let $1 < p < \infty$, $0 < \theta < \infty$, $w \in \Omega_\theta$, and let condition (2.2) be satisfied. In order that a set $S \subset LM_{p\theta,w(\cdot)}$ be precompact it is necessary and sufficient that*

$$\sup_{f\in S} \|f\|_{LM_{p\theta,w(\cdot)}} < \infty, \tag{3.1}$$

$$\lim_{R_1\to 0^+} \sup_{f\in S} \left\| f\chi_{B(0,R_1)} \right\|_{LM_{p\theta,w(\cdot)}} = 0, \tag{3.2}$$

$$\lim_{\delta\to 0^+} \sup_{f\in S} \|A_\delta f - f\|_{L_p(B(0,R_2)\setminus B(0,R_1))} = 0 \tag{3.3}$$

and

$$\lim_{R_2\to \infty} \sup_{f\in S} \left\| f\chi_{^cB(0,R_2)} \right\|_{LM_{p\theta,w(\cdot)}} = 0. \tag{3.4}$$

If $\theta = \infty$, then conditions (3.1)- (3.4) are sufficient for the precompactness of a set $S \subset LM_{p\infty,w(\cdot)}$.

Corollary 3.1. *Let $1 < p < \infty$, $0 < \theta < \infty$, $0 < \lambda < \frac{n}{p}$. In order that a set $S \subset LM^{\lambda}_{p\theta}$ be precompact it is necessary and sufficient that*

$$\sup_{f\in S} \|f\|_{LM^{\lambda}_{p\theta}} < \infty, \tag{3.5}$$

$$\lim_{R_1\to 0^+} \sup_{f\in S} \left\| f\chi_{B(0,R_1)} \right\|_{LM^{\lambda}_{p\theta}} = 0, \tag{3.6}$$

$$\lim_{\delta\to 0^+} \sup_{f\in S} \|A_\delta f - f\|_{L_p(B(0,R_2)\setminus B(0,R_1))} = 0, \tag{3.7}$$

and

$$\lim_{R_2\to \infty} \sup_{f\in S} \left\| f\chi_{^cB(0,R_2)} \right\|_{LM^{\lambda}_{p\theta}} = 0. \tag{3.8}$$

Theorem 3.2. *Let $1 \le p \le \infty$, $0 < \theta \le \infty$, $w \in \Omega_{p\theta}$, and $S \subset GM_{p\theta,w(\cdot)}$. If*

$$\sup_{f\in S} \|f\|_{GM_{p\theta,w(\cdot)}} < \infty, \tag{3.9}$$

$$\lim_{R_1\to 0^+} \sup_{f\in S} \left\|f\chi_{B(0,R_1)}\right\|_{GM_{p\theta,w(\cdot)}} = 0, \tag{3.10}$$

$$\lim_{\delta\to 0^+} \sup_{f\in S} \|A_\delta f - f\|_{L_p(B(0,R_2)\setminus B(0,R_1))} = 0 \tag{3.11}$$

and

$$\lim_{R_2\to\infty} \sup_{f\in S} \left\|f\chi_{{}^cB(0,R_2)}\right\|_{GM_{p\theta,w(\cdot)}} = 0. \tag{3.12}$$

then the set S is precompact.

Moreover, conditions (3.9) and (3.11) are necessary for the precompactness of the set S.

Corollary 3.2. *Let $1 < p < \infty$, $0 < \theta \le \infty$, $0 < \lambda < \frac{n}{p}$ if $\theta < \infty$ and $0 \le \lambda \le \frac{n}{p}$ if $\theta = \infty$, and $S \subset GM^\lambda_{p\theta}$. If*

$$\sup_{f\in S} \|f\|_{GM^\lambda_{p\theta}} < \infty, \tag{3.13}$$

$$\lim_{R_1\to 0^+} \sup_{f\in S} \left\|f\chi_{B(0,R_1)}\right\|_{GM^\lambda_{p\theta}} = 0, \tag{3.14}$$

$$\lim_{\delta\to 0^+} \sup_{f\in S} \|A_\delta f - f\|_{L_p(B(0,R_2)\setminus B(0,R_1))} = 0 \tag{3.15}$$

and

$$\lim_{R_2\to\infty} \sup_{f\in S} \left\|f\chi_{{}^cB(0,R_2)}\right\|_{GM^\lambda_{p\theta}} = 0. \tag{3.16}$$

then the set S is precompact.

Remark 2. In [15], necessary and sufficient conditions for the precompactness of a set are obtained for a rather wide class of Banach spaces B of functions f defined on $\mathbb{R}$ under the assumption that the translation operator $T(t)f(s) = f(s+t), s \in \mathbb{R}$, is bounded and for all $f \in B$

$$\lim_{t\to 0} \|T(t)f(s) - f(s)\|_B = 0. \tag{3.17}$$

However, the translation operator $T(t)$ is not bounded in $LM_{p\theta,w(\cdot)}$, and, being bounded in $GM_{p\theta,w(\cdot)}$, it does not satisfy condition (3.17), as an example, given in [2] shows. For this reason the result of [15] cannot be applied to the spaces $LM_{p\theta,w(\cdot)}$ and $GM_{p\theta,w(\cdot)}$.

In [14], sufficient conditions for precompactness of a set are obtained for the so-called Banach function spaces and for $\theta \ge 1$ the sufficiency in Theorems 3.1 and 3.2 can be derived by using the results of [14]. In [13], also necessary conditions for precompactness of a set are established under additional assumptions including the assumption that the norm is absolutely continuous. However, the appropriate statement in [14] cannot be aplied to the spaces $LM_{p\theta,w(\cdot)}$ and $GM_{p\theta,w(\cdot)}$ because their norms (or quasi-norms if $0 < \theta < 1$) are not absolutely continuous. Indeed, it is not difficult to verify that, for example, in the one-dimensional case $\lim_{k\to\infty} \left\| |x|^{\lambda-\frac{1}{p}} \chi_{(0,\frac{1}{k})}(x) \right\|_{LM^\lambda_{p\theta}} \ne 0$.

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Nurzhan Adilkhanovich Bokayev
Department of Mechanics and Mathematics
L.N. Gumilyov Eurasian National University
13 Munaitpasov St
010008 Astana, Kazakhstan
E-mail: bokayev2011@yandex.ru

Victor Ivanovich Burenkov
S.M.Nikolskii Mathematical Institute
Peoples Friendship University of Russia(RUDN University)
6 Miklukho-Maklay St
117198 Moscow, Russia
and
V.A. Steklov Mathematical Institute
Russian Academy of Sciences
8 Gubkin St, Moscow, 119991 Russia
E-mail: burenkov@cf.ac.uk

Dauren Tulutaevich Matin
Department of Mechanics and Mathematics
L.N. Gumilyov Eurasian National University
13 Munaitpasov St
010008 Astana, Kazakhstan
E-mail: d.matin@mail.ru

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