ISSN 2077–9879

Eurasian Mathematical Journal

2017, Volume 8, Number 3

Founded in 2010 by
the L.N. Gumilyov Eurasian National University

in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia
the University of Padua

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

EURASIAN MATHEMATICAL JOURNAL

Aims and Scope

The Eurasian Mathematical Journal (EMJ) publishes carefully selected original research papers in all areas of mathematics written by mathematicians, principally from Europe and Asia. However papers by mathematicians from other continents are also welcome.

From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

The contents of EMJ are indexed in Scopus, Web of Science (ESCI), Mathematical Reviews, MathSciNet, Zentralblatt Math (ZMATH), Referativnyi Zhurnal – Matematika, Math-Net.Ru.

The EMJ is included in the list of journals recommended by the Committee for Control of Education and Science (Ministry of Education and Science of the Republic of Kazakhstan) and in the list of journals recommended by the Higher Attestation Commission (Ministry of Education and Science of the Russian Federation).

Information for the Authors

Submission. Manuscripts should be written in LaTeX and should be submitted electronically in DVI, PostScript or PDF format to the EMJ Editorial Office via e-mail (eurasianmj@yandex.kz).

When the paper is accepted, the authors will be asked to send the tex-file of the paper to the Editorial Office.

The author who submitted an article for publication will be considered as a corresponding author. Authors may nominate a member of the Editorial Board whom they consider appropriate for the article. However, assignment to that particular editor is not guaranteed.

Copyright. When the paper is accepted, the copyright is automatically transferred to the EMJ. Manuscripts are accepted for review on the understanding that the same work has not been already published (except in the form of an abstract), that it is not under consideration for publication elsewhere, and that it has been approved by all authors.

Title page. The title page should start with the title of the paper and authors' names (no degrees). It should contain the Keywords (no more than 10), the Subject Classification (AMS Mathematics Subject Classification (2010) with primary (and secondary) subject classification codes), and the Abstract (no more than 150 words with minimal use of mathematical symbols).

Figures. Figures should be prepared in a digital form which is suitable for direct reproduction.

References. Bibliographical references should be listed alphabetically at the end of the article. The authors should consult the Mathematical Reviews for the standard abbreviations of journals' names.

Authors' data. The authors' affiliations, addresses and e-mail addresses should be placed after the References.

Proofs. The authors will receive proofs only once. The late return of proofs may result in the paper being published in a later issue.

Offprints. The authors will receive offprints in electronic form.

Publication Ethics and Publication Malpractice

For information on Ethics in publishing and Ethical guidelines for journal publication see http://www.elsevier.com/publishingethics and http://www.elsevier.com/journal-authors/ethics.

Submission of an article to the EMJ implies that the work described has not been published previously (except in the form of an abstract or as part of a published lecture or academic thesis or as an electronic preprint, see http://www.elsevier.com/postingpolicy), that it is not under consideration for publication elsewhere, that its publication is approved by all authors and tacitly or explicitly by the responsible authorities where the work was carried out, and that, if accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright-holder. In particular, translations into English of papers already published in another language are not accepted.

No other forms of scientific misconduct are allowed, such as plagiarism, falsification, fraudulent data, incorrect interpretation of other works, incorrect citations, etc. The EMJ follows the Code of Conduct of the Committee on Publication Ethics (COPE), and follows the COPE Flowcharts for Resolving Cases of Suspected Misconduct ($http://publicationethics.org/files/u2/New_Code.pdf$). To verify originality, your article may be checked by the originality detection service CrossCheck http://www.elsevier.com/editors/plagdetect.

The authors are obliged to participate in peer review process and be ready to provide corrections, clarifications, retractions and apologies when needed. All authors of a paper should have significantly contributed to the research.

The reviewers should provide objective judgments and should point out relevant published works which are not yet cited. Reviewed articles should be treated confidentially. The reviewers will be chosen in such a way that there is no conflict of interests with respect to the research, the authors and/or the research funders.

The editors have complete responsibility and authority to reject or accept a paper, and they will only accept a paper when reasonably certain. They will preserve anonymity of reviewers and promote publication of corrections, clarifications, retractions and apologies when needed. The acceptance of a paper automatically implies the copyright transfer to the EMJ.

The Editorial Board of the EMJ will monitor and safeguard publishing ethics.

The procedure of reviewing a manuscript, established by the Editorial Board of the Eurasian Mathematical Journal

1. **Reviewing procedure**

1.1. All research papers received by the Eurasian Mathematical Journal (EMJ) are subject to mandatory reviewing.

1.2. The Managing Editor of the journal determines whether a paper fits to the scope of the EMJ and satisfies the rules of writing papers for the EMJ, and directs it for a preliminary review to one of the Editors-in-chief who checks the scientific content of the manuscript and assigns a specialist for reviewing the manuscript.

1.3. Reviewers of manuscripts are selected from highly qualified scientists and specialists of the L.N. Gumilyov Eurasian National University (doctors of sciences, professors), other universities of the Republic of Kazakhstan and foreign countries. An author of a paper cannot be its reviewer.

1.4. Duration of reviewing in each case is determined by the Managing Editor aiming at creating conditions for the most rapid publication of the paper.

1.5. Reviewing is confidential. Information about a reviewer is anonymous to the authors and is available only for the Editorial Board and the Control Committee in the Field of Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan (CCFES). The author has the right to read the text of the review.

1.6. If required, the review is sent to the author by e-mail.

1.7. A positive review is not a sufficient basis for publication of the paper.

1.8. If a reviewer overall approves the paper, but has observations, the review is confidentially sent to the author. A revised version of the paper in which the comments of the reviewer are taken into account is sent to the same reviewer for additional reviewing.

1.9. In the case of a negative review the text of the review is confidentially sent to the author.

1.10. If the author sends a well reasoned response to the comments of the reviewer, the paper should be considered by a commission, consisting of three members of the Editorial Board.

1.11. The final decision on publication of the paper is made by the Editorial Board and is recorded in the minutes of the meeting of the Editorial Board.

1.12. After the paper is accepted for publication by the Editorial Board the Managing Editor informs the author about this and about the date of publication.

1.13. Originals reviews are stored in the Editorial Office for three years from the date of publication and are provided on request of the CCFES.

1.14. No fee for reviewing papers will be charged.

2. **Requirements for the content of a review**

2.1. In the title of a review there should be indicated the author(s) and the title of a paper.

2.2. A review should include a qualified analysis of the material of a paper, objective assessment and reasoned recommendations.

2.3. A review should cover the following topics:

- compliance of the paper with the scope of the EMJ;
- compliance of the title of the paper to its content;
- compliance of the paper to the rules of writing papers for the EMJ (abstract, key words and phrases, bibliography etc.);
- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);
- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

2.4. The final part of the review should contain an overall opinion of a reviewer on the paper and a clear recommendation on whether the paper can be published in the Eurasian Mathematical Journal, should be sent back to the author for revision or cannot be published.

Web-page

The web-page of EMJ is www.emj.enu.kz. One can enter the web-page by typing Eurasian Mathematical Journal in any search engine (Google, Yandex, etc.). The archive of the web-page contains all papers published in EMJ (free access).

Subscription

For Institutions

- US$ 200 (or equivalent) for one volume (4 issues)
- US$ 60 (or equivalent) for one issue

For Individuals

- US$ 160 (or equivalent) for one volume (4 issues)
- US$ 50 (or equivalent) for one issue.

The price includes handling and postage.

The Subscription Form for subscribers can be obtained by e-mail:

eurasianmj@yandex.kz

The Eurasian Mathematical Journal (EMJ)
The Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana
Kazakhstan

ERLAN DAUTBEKOVICH NURSULTANOV

(to the 60th birthday)

On May 25, 2017 was the 60th birthday of Yerlan Dautbekovich Nursultanov, Doctor of Physical and Mathematical Sciences (1999), Professor (2001), Head of the Department of Mathematics and Informatics of the Kazakhstan branch of the M.V. Lomonosov Moscow State University (since 2001), member of the Editorial Board of the Eurasian Mathematical Journal.

E.D. Nursultanov was born in the city of Karaganda. He graduated from the Karaganda State University (1979) and then completed his postgraduate studies at the M.V. Lomonosov Moscow State University.

Professor Nursultanov's scientific interests are related to various areas of the theory of functions and functional analysis.

He introduced the concept of multi-parameter Lorentz spaces, network spaces and anisotropic Lorentz spaces, for which appropriate interpolation methods were developed. On the basis of the apparatus introduced by him, the questions of reiteration in the off-diagonal case for the real Lyons-Petre interpolation method, the multiplier problem for trigonometric Fourier series, the lower and upper bounds complementary to the Hardy-Littlewood inequalities for various orthonormal systems were solved. The convergence of series and Fourier transforms were studied with sufficiently general monotonicity conditions. The lower bounds for the norm of the convolution operator are obtained, and its upper bounds are improved (a stronger result than the O'Neil inequality). An exact cubature formula with explicit nodes and weights for functions belonging to spaces with a dominated mixed derivative is constructed, and a number of other problems in this area are solved.

He has published more than 50 scientific papers in high rating international journals included in the lists of Thomson Reuters and Scopus. 2 doctor of sciences, 9 candidate of sciences and 4 PhD dissertations have been defended under his supervision.

His merits and achievements are marked with badges of the Ministry of Education and Science of the Republic of Kazakhstan "For Contribution to the Development of Science" (2007), "Honored Worker of Education" (2011), "Y. Altynsarin" (2017). He is a laureate of the award named after K. Satpaev in the field of natural sciences for 2005, the grant holder "The best teacher of the university" for 2006 and 2011, the grant holder of the state scientific scholarship for outstanding contribution to the development of science and technology of the Republic of Kazakhstan for years 2007-2008, 2008 -2009. In 2017 he got the Top Springer Author award, established by Springer Nature together with JSC "National Center for Scientific and Technical Information".

The Editorial Board of the Eurasian Mathematical Journal congratulates Erlan Dautbekovich Nursultanov on the occasion of his 60th birthday and wishes him good health and successful work in mathematics and mathematical education.

JAMALBEK TUSSUPOV

(to the 60th birthday)

On April 10, 2017 was the 60th birthday of Jamalbek Tussupov, Doctor of Physical and Mathematical Sciences, Professor, Head of the Information Systems Department of the L.N. Gumilyov Eurasian National University, member of the Kazakhstan and American Mathematical Societies, member of the Association of Symbolic Logic, member of the Editorial Board of the Eurasian Mathematical Journal.

J. Tussupov was born in Taraz (Jambyl region of the Kazakh SSR). He graduated from the Karaganda State University (Kazakhstan) in 1979 and later on completed his postgraduate studies at S.L. Sobolev Institute of Mathematics of the Academy of Sciences of Russia (Novosibirsk).

Professor Tussupov's research interests are in mathematical logic, computability, computable structures, abstract data types, ontology, formal semantics. He solved the following problems of computable structures:

- the problems of S.S. Goncharov and M.S. Manasse: the problem of characterizing relative categoricity in the hyperarithmetical hierarchy given levels of complexity of Scott families, and the problem on the relationship between categoricity and relative categoricity of computable structures in the arithmetical and hyperarithmetical hierarchies;
- the problem of Yu.L. Ershov: the problem of finite algorithmic dimension in the arithmetical and hyperarithmetical hierarchies;
- the problem of C.J. Ash and A. Nerode: the problem of the interplay of relations of bounded arithmetical and hyperarithmetical complexity in computable presentations and the definability of relations by formulas of given complexity;
- the problem of S. Lempp: the problem of structures having presentations in just the degrees of all sets X such that for algebraic classes as symmetric irreflexive graphs, nilpotent groups, rings, integral domains, commutative semigroups, lattices, structure with two equivalences, bipartite graphs.

Professor Tussupov has published about 100 scientific papers, five textbooks for students and one monograph. Three PhD dissertations have been defended under his supervision.

Professor Tussupov is a fellow of "Bolashak" Scholarship, 2011 (Notre Dame University, USA), "Erasmus+", 2016 (Poitiers University, France). He was awarded the title "The Best Professor of 2012" (Kazakhstan). In 2015 Jamalbek Tussupov was also awarded for the contribution to science in the Republic of Kazakhstan.

The Editorial Board of the Eurasian Mathematical Journal congratulates Dr. Professor Jamalbek Tussupov on the occasion of his 60th aniversary and wishes him strong health, new achievements in science, inspiration for new ideas and fruitfull results.

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 8, Number 3 (2017), 85 – 108

USE OF BUNDLES OF LOCALLY CONVEX SPACES IN PROBLEMS OF CONVERGENCE OF SEMIGROUPS OF OPERATORS. III

B. Silvestri

Communicated by V.I. Burenkov

Key words: bundles of locally convex spaces, one-parameter semigroups, spectrum and resolvent.

AMS Mathematics Subject Classification: 55R25, 46E40, 46E10; 47A10, 47D06.

1 Introduction

In this final part of the work we accomplish in Theorem 4.2 the claim of extending the classical stability problem to the framework of bundles of $\Omega-$spaces and consequently to obtain stability results for operators acting in different Banach spaces. Let us describe the principal steps required for this result.

$(\Theta, \mathcal{E})-$structures are central in constructing in [21, Theorem 2.1] the section of C_0-semigroups $\mathcal{U}$ continuous at x_∞. However the presence in their definition of the uniform convergence over compact subsets of a topological space Y rather than the *pointwise* convergence, drastically restricts in general the fulfillment of the property (1.2) (with $\Gamma(\xi)$ replaced by $\Gamma(\rho)$) characterizing invariant structures. Possible exceptions are those where the base space X is compact and under suitable hypothesis the above property is used to determine $\Gamma(\rho)$, see [20, Remark 11].

Now first of all the property (1.2) is basic to establish in Corollary 4.2 the essential step (1.3) toward the main result (1.6). Here $\mathcal{P}$ is a suitable section of spectral projectors of the infinitesimal generators of the semigroups $\{\mathcal{U}(x)\}_{x\in X}$. For instance for obtaining (4.37) we apply Lemma 4.2. Secondly in order to prove (1.3) we need the concept of $\mu-$relatedness provided in Definition 9 requiring *$\mu-$integrable Hlcs valued maps*. Indeed see the technical Corollary 4.1 resulting by the bundle type generalization of the Lebesgue Theorem we establish in Theorem 4.1 and by Lemma 4.1, remarkable results by themself. Finally for defining $\mathcal{P}(x)$ we apply to $\mathcal{U}(x)$ the well-known integral formula (3.1) for any $x \in X$.

Therefore it appears natural in the present context to replace $(\Theta, \mathcal{E})-$structures based on function spaces of continuous maps provided with the topology of compact convergence, with those based on function spaces, provided with the topology of pointwise convergence, of Hausdorff locally convex space valued integrable maps defined on a locally compact space provided with a Radon measure.

To this end we introduce for any Radon measure μ on a locally compact space Y the concept of $(\Theta, \mathcal{E}, \mu)-$structure Definition 10. Roughly a $(\Theta, \mathcal{E}, \mu)-$structure $\langle \mathfrak{V}, \mathfrak{Q}, X, Y\rangle$ where $\mathfrak{Q} = \langle\langle \mathfrak{H}, \gamma\rangle, \xi, X, \mathfrak{Y}\rangle$ and $\mathfrak{V} = \langle\langle \mathfrak{E}, \tau\rangle, \pi, X, \mathfrak{N}\rangle$, is defined in the same way as a $(\Theta, \mathcal{E})-$structure except that

$$\begin{cases} \mathfrak{H}_x \subseteq \mathfrak{L}_1\left(Y, \mathcal{L}_{S_x}(\mathfrak{E}_x), \mu\right), \\ \mathfrak{Y}_x \text{ induces the topology on } \mathfrak{H}_x \text{ of pointwise convergence on } Y. \end{cases} \tag{1.1}$$

Here for any $x \in X$ we recall that $\mathcal{L}_{S_x}(\mathfrak{E}_x)$ is the Hlcs of continuous linear maps on $\mathfrak{E}_x$ provided with the topology of uniform convergence over the sets belonging to S_x. $\langle \mathfrak{V}, \mathfrak{Q}, X, Y \rangle$ is *invariant* if in addition

$$\left\{ F \in \prod_{z \in X}^{b} \mathfrak{H}_z \mid (\forall t \in Y)(F_t \bullet \mathcal{E}(\Theta) \subseteq \Gamma(\pi)) \right\} = \Gamma(\xi). \tag{1.2}$$

The main reason behind the intoduction of this concept is represented by the following result. Let $\mathfrak{V}$ be a Banach bundle, T be a suitable section of closed densely defined linear operators satisfying the property of separation of the spectrum, Γ be a curve associated with T and (K, A, ϕ) be a triplet associated with Γ, a straight extension to the bundle case of the separation of the spectrum introduced by Kato (Definition 4). If $\langle \mathfrak{V}, \mathfrak{Q}, X, \mathbb{R}^+ \rangle$ is an invariant $(\Theta, \mathcal{E}, \mathfrak{q})$ −structure for all $\mathfrak{q} \in \{\nu^\phi, \eta_s^\phi \mid s \in K\}$ (Definition 5) and $\langle \mathfrak{V}, \mathfrak{Z}, \mathsf{H} \rangle$ is $\mathfrak{q}$−related such that $\mathsf{H}_z = \mathfrak{H}_z$ for all $z \in X$, then under additional hypothesis Corollary 4.2

$$\mathcal{W}_T \in \Gamma^{x_\infty}(\xi) \Rightarrow \mathcal{P} \bullet \Gamma^{x_\infty}_{\mathcal{E}(\Theta)}(\pi) \subseteq \Gamma^{x_\infty}(\pi). \tag{1.3}$$

As we describe below, by letting $\mathfrak{n}$ be the Lebesgue measure on $\mathbb{R}^+$ we shall apply this result to a $\mathfrak{n}$−related set and to the $(\Theta, \mathcal{E}, \mathfrak{n})$ −structure underlying the $(\Theta, \mathcal{E})$ −structure used in [21, Theorem 2.1].

Now central in proving (1.3) is the fact that the global relation (1.2) implies the pointwise one. More exactly for any invariant $(\Theta, \mathcal{E}, \mu)$ −structure $\langle \mathfrak{V}, \mathfrak{D}, X, Y \rangle$ with a suitable Θ, by denoting $\mathfrak{D} = \langle \langle \mathfrak{B}, \gamma_3 \rangle, \eta, X, \mathfrak{L} \rangle$, we have Lemma 4.2

$$\left\{ H \in \left[\left(\prod_{z \in X}^{b} \mathfrak{B}_z \right)_{\diamond}^{x_\infty} \right]_{peq} \mid (\forall t \in Y)(H_t \bullet \mathcal{E}(\Theta) \subseteq \Gamma^{x_\infty}(\pi)) \right\} \subseteq \Gamma^{x_\infty}_{\diamond}(\eta). \tag{1.4}$$

To derive (1.3) we use this inclusion for the case $Y = \mathbb{R}^+$ and multiple times Corollary 4.1. (1.4) can be extended to invariant $(\Theta, \mathcal{E})$ −structures whenever $Comp(Y) = \mathcal{P}_\omega(Y)$, we shall use this remark in the case $Y = \{pt\}$.

Now if $\mathcal{E}(\Theta) \subseteq \Gamma(\pi)$ and if we show that

$$\mathcal{P} \bullet \Gamma^{x_\infty}_{\mathcal{E}(\Theta)}(\pi) \subseteq \Gamma^{x_\infty}(\pi), \tag{1.5}$$

then

$$\mathcal{P} \in \Gamma^{x_\infty}(\eta), \tag{1.6}$$

follows by (1.4) for the special case $Y = \{pt\}$. It is worthwhile remarking that (1.5) represents a satisfactory result for all practical purposes concerning the stability of $\mathcal{P}$. However in order to interpret $\mathcal{P}$ satisfying (1.5) as a bounded section continuous at x_∞ we need to employ (1.4).

Next let $Y = \mathbb{R}^+$. To prove (1.5) we apply (1.3) to the section of contractions $\mathcal{U}$ continuous at x_∞ obtained in [21, Theorem 2.1] and to the $(\Theta, \mathcal{E}, \mathfrak{n})$ −structure underlying the $(\Theta, \mathcal{E})$ −structure used in [21, Theorem 2.1]. More exactly given a suitable $(\Theta, \mathcal{E})$ −structure $\langle \mathfrak{V}, \mathfrak{W}, X, Y \rangle$ which in general is not a $(\Theta, \mathcal{E}, \mu)$ − structure and by letting $\mathfrak{W} = \langle \langle \mathfrak{M}, \gamma \rangle, \rho, X, \mathfrak{R} \rangle$, [21, Theorem 2.1] states the existence of a section T of closed operators such that $\mathcal{U} = \mathcal{W}_T \in \Gamma^{x_\infty}(\rho)$. So under the additional hypothesis that there exists an $F \in \Gamma(\rho)$ such that $F(x_\infty) = \mathcal{U}(x_\infty)$ we obtain

$$\mathcal{U} \in \Gamma^{x_\infty}_{\diamond}(\rho). \tag{1.7}$$

In order to apply (1.3) we need a procedure to extract from the initial $(\Theta, \mathcal{E})$ −structure a structure $\langle \mathfrak{V}, \mathfrak{Q}, X, Y \rangle$ which is a $(\Theta, \mathcal{E}, \mathfrak{q})$ −structure for all $\mathfrak{q} \in \{\nu^\phi, \eta_s^\phi \mid s \in K\}$ and such that

$$\Gamma^{x_\infty}_{\diamond}(\rho) \subseteq \Gamma^{x_\infty}(\xi). \tag{1.8}$$

This is performed by applying a general construction called the $(\Theta, \mathcal{E}, \mu)$ $-$structure $\langle \mathfrak{V}, \mathfrak{V}(\mathsf{M}^{\mu}, \Gamma(\rho)), X, Y\rangle$ underlying $\langle \mathfrak{V}, \mathfrak{W}, X, Y\rangle$ and defined in Definition 12.

In view of the property (1.8) we have to maintain the vicinity of the initial and the underlying structure. This is performed by applying the by now usual general result [12, Theorem 5.8] for constructing bundles with a given subspace of bounded continuous sections. Thus the choice we perform in Definition 12 for the stalks M_{x}^{μ} is essentially the weakest one in order to satisfy (1.1) and to allow the space $\Gamma(\rho)$ of bounded continuous sections of $\mathfrak{W}$ to be a subspace of the space $\Gamma(\pi_{\mathsf{M}^{\mu}})$ of bounded continuous sections of the underlying bundle $\mathfrak{V}(\mathsf{M}^{\mu}, \Gamma(\rho))$.

Proposition 4.2 shows that our choice is the right one indeed

$$\Gamma_{\diamond}^{x_{\infty}}(\rho) \subseteq \Gamma^{x_{\infty}}(\pi_{\mathsf{M}^{\mu}}). \tag{1.9}$$

It remains only to select the right measure μ in order to satisfy the hypothesis of Corollary 4.2 in particular such that the $(\Theta, \mathcal{E}, \mu)$ $-$structure underlying $\langle \mathfrak{V}, \mathfrak{W}, X, Y\rangle$ is a $(\Theta, \mathcal{E}, \mathfrak{q})$ $-$structure for all $\mathfrak{q} \in \{\nu^{\phi}, \eta_{s}^{\phi} \mid s \in K\}$. To this end let us say that $(\mathfrak{W}, \mathfrak{Z})$ satisfies the $\mathfrak{n}-$hypothesis, if the $(\Theta, \mathcal{E}, \mathfrak{n})$ $-$structure underlying $\langle \mathfrak{V}, \mathfrak{W}, X, Y\rangle$ is invariant, $\langle \mathfrak{V}, \mathfrak{Z}, \mathsf{M}^{\mathfrak{n}}\rangle$ is $\mathfrak{n}-$related and $\mathfrak{L}^{\infty}(Y, \mathfrak{n}) \blacktriangleright \Gamma(\zeta) \subseteq \Gamma(\zeta)$ (where $\Gamma(\zeta)$ is the space of bounded continuous sections of $\mathfrak{Z}$ and $\blacktriangleright$ is defined in Definition 7). Now it is possible to show that if $(\mathfrak{W}, \mathfrak{Z})$ satisfies the $\mathfrak{n}-$hypothesis, then $\langle \mathfrak{V}, \mathfrak{V}(\mathsf{M}^{\mathfrak{n}}, \Gamma(\rho)), X, Y\rangle$ is an invariant $(\Theta, \mathcal{E}, \mathfrak{q})$ $-$structure and $\langle \mathfrak{V}, \mathfrak{Z}, \mathsf{M}^{\mathfrak{n}}\rangle$ is $\mathfrak{q}-$related for all $\mathfrak{q} \in \{\nu^{\phi}, \eta_{s}^{\phi} \mid s \in K\}$. In other words the hypothesis in Corollary 4.2 for obtaining (1.3) holds true with the position $\mathfrak{Q} = \mathfrak{V}(\mathsf{M}^{\mathfrak{n}}, \Gamma(\rho))$.

Finally provided that $(\mathfrak{W}, \mathfrak{Z})$ satisfies the $\mathfrak{n}-$hypothesis we conclude that (1.5) and consequently (1.6), follow by (1.7), (1.9) with the position $\mu = \mathfrak{n}$ and (1.3).

Except when explicitly stated, we assume all the notations set in [20, 21] in particular all the vector spaces are over $\mathbb{C}$.

2 $\langle \nu, \eta, E, Z, T\rangle$ invariant set with respect to $\mathcal{F}$

In the present Section 2 let us fix a consistent class of data $\mathfrak{O}$ ([21, Definition 5]) and let $\mathfrak{G}$ denote the locally convex space relative to $\mathfrak{O}$ ([21, Definition 9]).

Definition 1. Let Z, T be two locally compact spaces, $E \in Hlcs$, $\nu \in Radon(Z)$ and $\eta \in Radon(T)^{Z}$. Set

$$\mathfrak{L}_{(1,1)}(T, E, \eta, \nu) \doteqdot \left\{\overline{F} \in \bigcap_{\lambda \in Z} \mathfrak{L}_{1}(T, E, \eta_{\lambda}) \mid \left(Z \ni \lambda \mapsto \int \overline{F}(s) d\eta_{\lambda}(s) \in E\right) \in \mathfrak{L}_{1}(Z, E, \nu)\right\}$$

Corollary 2.1. *Let Z be a locally compact space, $\nu \in Radon(Z)$, $\eta \in Radon(Y)^{Z}$, $\mathfrak{D} \in \prod_{x \in X} \mathcal{P}(\mathfrak{E}_{x})$, $D = \prod_{x \in X} \mathfrak{D}_{x}$ and assume (A) of [21, Lemma 4.4]. Thus $(\forall \overline{F} \in \mathfrak{L}_{(1,1)}(Y, \mathfrak{G}, \eta, \nu))(\forall x \in X)(\forall v \in D)$*

$$\Pr_{x} \circ \left[\int \left(\int \overline{F}(s)\, d\eta_{\lambda}(s)\right) d\nu(\lambda)\right](v) = \left[\int \left(\int \Pr_{x}(\Psi(\overline{F}))(s) v(x)\, d\eta_{\lambda}(s)\right) d\nu(\lambda)\right].$$

Proof. Let $\overline{F} \in \mathfrak{L}_{(1,1)}(Y, \mathfrak{G}, \eta, \nu)$, $x \in X$ and $v \in D$. By [21, Theorem 4.2]

$$\Pr_{x} \circ \left[\int \left(\int \overline{F}(s)\, d\eta_{\lambda}(s)\right) d\nu(\lambda)\right](v) = \int \Pr_{x} \circ \Psi\left(\int \overline{F}(s)\, d\eta_{(\cdot)}(s)\right)(\lambda)(v(x))\, d\nu(\lambda).$$

Moreover $\forall\lambda\in Z$

$$\begin{aligned}\Pr_x\circ\Psi\left(\int\overline{F}(s)\,d\eta_{(\cdot)}(s)\right)(\lambda)(v(x))&=\Pr_x\circ\left(\int\overline{F}(s)\,d\eta_\lambda(s)\right)\circ\imath_x(v(x))\\&=\int\Pr_x(\Psi(\overline{F}))(s)\circ\Pr_x\circ\imath_x(v(x))\,d\eta_\lambda(s)\\&=\int\Pr_x(\Psi(\overline{F}))(s)v(x)\,d\eta_\lambda(s),\end{aligned}$$

where in the first equality we used [21, Proposition 4.7], while in the second one [21, Theorem 4.2]. Then the statement follows. □

Definition 2. V is a $\langle\nu,\eta,E,Z,T\rangle$ invariant set with respect to $\mathcal{F}$ if

1. T,Z are two locally compact spaces;
2. $E\in Hlcs$
3. there exists $M\in Hlcs$ and $\mathfrak{b}:E\times M\to M$ bilinear;
4. $V\subseteq M$ linear subspace;
5. $\nu\in Radon(Z)$ and $\eta\in Radon(T)^Z$;
6. $\mathcal{F}\subseteq\mathfrak{L}_{(1,1)}(T,E,\eta,\nu)$
7. $\forall\overline{F}\in\mathcal{F}$
$$\left[\int\left(\int\overline{F}(s)\,d\eta_\lambda(s)\right)\,d\nu(\lambda)\right]V\subseteq V,$$
where we denote $\mathfrak{b}(e,m)$ by em, for any $e\in E$ and $m\in M$.

Proposition 2.1. *Let us assume the hypotheses of Corollary 2.1 and V be a $\langle\nu,\eta,\mathfrak{G},Z,Y\rangle$ invariant set with respect to $\mathcal{F}$ such that $V\cap D\neq\emptyset$. Then $\forall v\in V\cap D$ and $\forall\overline{F}\in\mathcal{F}$*

$$\left(X\ni x\mapsto\left[\int\left(\int\Pr_x(\Psi(\overline{F}))(s)v(x)\,d\eta_\lambda(s)\right)\,d\nu(\lambda)\right]\in\mathfrak{E}_x\right)\in V.$$

Proof. By Corollary 2.1. □

3 Construction of sets in $\Delta_\Theta\langle\mathfrak{V},\mathfrak{D},\mathfrak{W},\mathcal{E},X,\mathbb{R}^+\rangle$ through invariant sets

In the present Section 3 let us fix an entire consistent class of data $\mathfrak{O}$ ([21, Definition 5]) such that $\mathbb{R}^+$ is its locally compact space, ν defined in Definition 5 is its Radon measure, the primary family $\mathsf{E}\doteqdot\{\mathfrak{E}_x\}_{x\in X}$ underlying $\mathfrak{O}$ is a family of Banach spaces and such that for any $x\in X$ the corresponding element of the secondary family underlying $\mathfrak{O}$ is the strong operator topology on $\mathcal{L}(\mathfrak{E}_x)$. Let $\mathfrak{G}$ denote the locally convex space relative to $\mathfrak{O}$ ([21, Definition 9]). For any Banach space C let $Cld(C)$ denote, the set of all closed linear operators densely defined in C and at values in C. For any $T\in Cld(C)$ let $P(T)$ denote the resolvent set of T and $\Sigma(T)$ be the spectrum of T. Let $R(T;\cdot):P(T)\ni\zeta\mapsto(T-\zeta)^{-1}\in B(C)$ be the resolvent map of T. In the next definition we adapt to our framework a definition provided in [14, Chapter 9, §1, $n°4$].

Definition 3. Let $M > 1$ and $\beta \in \mathbb{R}$. Let $\mathcal{G}(M, \beta, \mathsf{E})$ be the set of all $T \in \prod_{x \in X} Cld(\mathfrak{E}_x)$ such that $]\beta, \infty[\subseteq P(-T(x))$ and $(\forall \xi > \beta)(\forall k \in \mathbb{N})(\forall x \in X)$

$$\|(T(x) + \xi)^{-k}\|_{B(\mathfrak{E}_x)} \leq M(\xi - \beta)^{-k}.$$

Moreover let us denote by $\{e^{-tT(x)}\}_{t \in \mathbb{R}^+}$ the strongly continuous semigroup generated by $-T(x)$.

In the following definition we adapt to our framework the definition of separation of the spectrum for a closed operator provided in [14, $n°4$,§6, Chapter 3].

Definition 4. Let $M > 1$ and $\beta \in \mathbb{R}$. We say that $T \in \mathcal{G}(M, \beta, \mathsf{E})$ satisfies the property of separation of the spectrum if $(\exists\, \Gamma)(\forall x \in X)(\exists\, \Sigma'_{T(x)} \subseteq \Sigma(T(x)))(\exists\, A_{T(x)} \in Op(\mathbb{C}))$ such that Γ is a regular closed curve in $\mathbb{C}$, $\Sigma'_{T(x)}$ is bounded and

$$\Sigma'_{T(x)} \subset A_{T(x)} \subset \mathcal{O}_i(\Gamma),\ \Sigma''_{T(x)} \subset \mathcal{O}_e(\Gamma).$$

Here $\mathcal{O}_i(\Gamma)$ is the interior of Γ, namely the compact set of $\mathbb{C}$ whose frontier is Γ, $\mathcal{O}_e(\Gamma) \doteqdot \complement \mathcal{O}_i(\Gamma)$ is the exterior of Γ, finally $\Sigma''_{T(x)} \doteqdot \Sigma(T(x)) \cap \complement \Sigma'_{T(x)}$. We call any curve Γ with the above property a curve associated with T, while we call (K, A, ϕ) a triplet associated with Γ iff $K \subset \mathbb{R}^+$ is compact, A is an open neighbourhood of K and $\phi : A \to \mathbb{C}$ is such that $\phi \in C_1(A, \mathbb{R}^2)$[3] and $\phi(K) = \Gamma$.

Let $T \in \mathcal{G}(M, \beta, \mathsf{E})$ satisfy the property of separation of the spectrum and Γ a curve associated with T, then $\forall x \in X$ we set

$$P(x) \doteqdot -\frac{1}{2\pi i} \int_\Gamma R(T(x); \zeta)\, d\zeta \in B(\mathfrak{E}_x), \tag{3.1}$$

where the integration is with respect to the norm topology on $B(\mathfrak{E}_x)$. Moreover for any triplet (K, A, ϕ) associated with Γ set $R_T^\phi \in \prod_{x \in X} \mathcal{L}(\mathfrak{E}_x)^{\mathbb{R}^+}$ such that $R_T^\phi(x)(s) \doteqdot R(T(x); \phi(s))$, for all $x \in X$ and $s \in K$, while $R_T^\phi(x)(s) \doteqdot \mathbf{0}$, if $s \in \mathbb{R}^+ - K$.

Remark 1. Let $M > 1$, $\beta \in \mathbb{R}$, $T \in \mathcal{G}(M, \beta, \mathsf{E})$ satisfy the property of separation of the spectrum and Γ a curve associated with T. Then for all $x \in X$ by [14, Theorem 6.17, Chapter 3], $P(x) \in \Pr(\mathfrak{E}_x)$ and $\mathfrak{E}_x = M'_x \oplus M''_x$ direct sum of two closed subspaces of $\mathfrak{E}_x$, where $M'_x = P(x)\mathfrak{E}_x$ and $M''_x = (\mathbf{1}_x - P(x))\mathfrak{E}_x$. Moreover $T(x)$ decomposes according the previous decomposition, namely $T(x) \upharpoonright M'_x \in B(M'_x)$ such that $\Sigma(T_x \upharpoonright M'_x) = \Sigma'_{T_x}$ and $T_x \upharpoonright M''_x$ is a closed operator in M''_x such that $\Sigma(T_x \upharpoonright M''_x) = \Sigma''_{T_x}$.

Definition 5. Let $K \subset \mathbb{R}^+$ be a compact set, A an open neighbourhood of K and $\phi : A \to \mathbb{C}$ be such that $\phi \in C_1(A, \mathbb{R}^2)$ and $\phi(K) = \Gamma$. For any $s \in K$ define $\eta_s^\phi \in Radon(\mathbb{R}^+)$ such that

$$\eta_s^\phi : \mathcal{C}_{cs}\left(\mathbb{R}^+\right) \ni f \mapsto \int_{\mathbb{R}^+} e^{\phi(s)t} f(t)\, dt.$$

Moreover let $\nu^\phi \in Radon(\mathbb{R}^+)$ be the $\mathbf{0}-$extension of $\nu_0^\phi \in Radon(K)$ such that

$$\nu_0^\phi : \mathcal{C}_{cs}\left(K\right) \ni g \mapsto \int_K \frac{-g(s)}{2\pi i} \frac{d\phi}{ds}(s)\, ds.$$

Finally let $M > 1$, $\beta \in \mathbb{R}$ and $T \in \mathcal{G}(M, \beta, \mathsf{E})$, then we set $\mathcal{W}_T \in \prod_{x \in X} \mathsf{U}(B_s(\mathfrak{E}_x))$ such that $(\forall x \in X)(\forall t \in \mathbb{R}^+)$

$$\begin{cases} \mathcal{W}_T(x)(t) \doteqdot e^{-T(x)t}, \\ \overline{F}_T \doteqdot \Lambda(\mathcal{W}_T), \end{cases}$$

where Λ has been defined in [21, Definition 11].

[3]By identifying $\mathbb{C}$ with $\mathbb{R}^2$, so ϕ is derivable with contiuous derivative

Lemma 3.1. *Let $M > 1$, $\beta \in \mathbb{R}$ and $T \in \mathcal{G}(M, \beta, \mathsf{E})$ satisfy the property of separation of the spectrum. Assume that there exists a curve Γ associated with T such that*

$$Re(\Gamma) \subseteq \mathbb{R}^-. \tag{3.2}$$

Then for any triple (K, A, ϕ) associated with Γ we have that $\forall x \in X$ and $\forall v_x \in \mathfrak{E}_x$

$$P(x)v_x = -\frac{1}{2\pi i}\int_K \frac{d\phi}{ds}(s)R(T(x);\phi(s))v_x\, ds, \tag{3.3}$$

and $\forall s \in K$,

$$R(T(x),\phi(s))v_x = \int_0^{\infty} e^{\phi(s)t}e^{-tT(x)}v_x\, dt = \int_{\mathbb{R}^+} \mathcal{W}_T(x)(t)v_x\, d\eta_s^{\phi}(t). \tag{3.4}$$

Here the integration is with respect to the norm topology on $\mathfrak{E}_x$. If $\overline{F}_T \in \mathfrak{L}_{(1,1)}(\mathbb{R}^+, \mathfrak{G}, \eta^{\phi}, \nu^{\phi})$ and V is a $\langle \nu^{\phi}, \eta^{\phi}, \mathfrak{G}, K, \mathbb{R}^+\rangle$ invariant set with respect to $\{\overline{F}_T\}$, then

$$P \bullet V \subseteq V. \tag{3.5}$$

Proof. By (3.1), [4, *IV*.35, Theorem 1], and by the norm continuity of the map $B(\mathfrak{E}_x) \ni A \mapsto Aw \in \mathfrak{E}_x$ for any $w \in \mathfrak{E}_x$, we have (3.3). Moreover by (3.2) we can apply [14, equation (1.28), $n°3$, §1, Chapter 9] and (3.4) follows by Definition 5. Fix $v \in V$ so $\forall x \in X$

$$\begin{aligned}
P(x)v(x) &= -\frac{1}{2\pi i}\int_K \frac{d\phi}{ds}(s)R(T(x);\phi(s))v(x)\, ds \\
&= -\frac{1}{2\pi i}\int_K \frac{d\phi}{ds}(s)\left(\int_{\mathbb{R}^+} \mathcal{W}_T(x)(t)v(x)\, d\eta_s^{\phi}(t)\right) ds \\
&= \int_K \left(\int_{\mathbb{R}^+} \Pr_x\left(\Psi(\overline{F}_T)\right)(t)v(x)\, d\eta_s^{\phi}(t)\right) d\nu^{\phi}(s).
\end{aligned} \tag{3.6}$$

Here the first equality comes by (3.3), the second one by (3.4) and the third one by [21, Proposition 4.7] and Definition 5. Next with the notation in [21, Corollary 4.1] we choose $(\forall x \in X)(S_x = \mathcal{P}_{\omega}(\mathfrak{E}_x))$, and since $\mathfrak{D}$ is entire we can select $\mathcal{D}(x) = \mathfrak{E}_x$, for all $x \in X$, in other words $p^x_{l_x,j_x}$ is the strong operator topology on $\mathcal{L}(\mathfrak{E}_x)$. Thus (A) of [21, Lemma 4.4] is satisfied since [21, Corollary 4.1], so the statement follows by (3.6) and Proposition 2.1. □

Corollary 3.1. *Under the hypothesis of [21, Theorem 2.1] let the primary family E of $\mathfrak{D}$ be the family of stalks of the Banach Bundle $\mathfrak{V}$ of which in [21, Theorem 2.1]. Assume that the hypothesis of Lemma 3.1 are satisfied, where T is such that $-T(x)$ is the infinitesimal generator of $\mathcal{U}(x)$ for all $x \in X$, where $\mathcal{U}$ is the section of semigroups construcuted in [21, Theorem 2.1]. Moreover let $\langle \mathfrak{V}, \mathfrak{D}, X, \{pt\}\rangle$ be an invariant $(\Theta, \mathcal{E})-$structure such that $\mathcal{E}(\Theta) \subset \Gamma(\pi)$ and let (K, A, ϕ) be a triplet associated with a curve associated with T. If $\mathcal{E}(\Theta)$ is a $\langle \nu^{\phi}, \eta^{\phi}, \mathfrak{G}, K, \mathbb{R}^+\rangle$ invariant set with respect to $\{\overline{F}_T\}$ and $\overline{F}_T \in \mathfrak{L}_{(1,1)}(\mathbb{R}^+, \mathfrak{G}, \eta^{\phi}, \nu^{\phi})$, then $\{\mathcal{U}\} \in \Delta_{\Theta}\langle \mathfrak{V}, \mathfrak{D}, \mathfrak{W}, \mathcal{E}, X, \mathbb{R}^+\rangle$.*

Proof. Since (3.5) and the definition of invariant $(\Theta, \mathcal{E})-$structures. □

4 Construction of sets in $\Delta\langle \mathfrak{V}, \mathfrak{D}, \Theta, \mathcal{E}\rangle$

Assumptions 1. In this section X is a topological space, Y is a locally compact space μ is a Radon measure on Y. Let $\mathfrak{L}^{\infty}(Y, \mu)$ denote the linear space of all complex valued maps defined on Y which are $\mu-$measurable and bounded in measure for the measure μ. Let $\mathfrak{V} = \langle\langle \mathfrak{E}, \tau\rangle, \pi, X, \mathfrak{N}\rangle$ be a bundle of $\Omega-$spaces, we indicate with $\mathfrak{N} \doteqdot \{\nu_j \,|\, j \in J\}$ the directed set of seminorms on $\mathfrak{E}$.

Definition 6. Let $Z \in Hlcs$ and $\{\psi_i \,|\, i \in I\}$ a fundamental set of seminorms on Z. We denote by

$$\left(Z^Y\right)_s$$

the $Hlcs$ whose underlying linear space is Z^Y and whose locally convex topology is generated by the following set of seminorms

$$\begin{cases} \{q_s^i \,|\, s \in Y, i \in I\}, \\ q_s^i : Z^Y \ni f \mapsto \psi_i(f(s)). \end{cases} \tag{4.1}$$

Moreover for any $B \subseteq Z^Y$ we shall denote by B_s the Hlc subspace of $\left(Z^Y\right)_s$. Notice that this definition is well-set being independent by the choice of the fundamental set of seminorms, indeed the topology is that of uniform convergence over the finite subsets of Y.

Definition 7. Set

$$\begin{cases} \overset{\mu}{\blacklozenge} : \prod_{x\in X} \mathfrak{L}_1(Y, \mathfrak{E}_x, \mu) \to \prod_{x\in X} \mathfrak{E}_x, \\ \overset{\mu}{\blacklozenge}(H)(x) \doteqdot \int H(x)(s)\, d\mu(s) \in \mathfrak{E}_x, \end{cases}$$

for all $H \in \prod_{x\in X} \mathfrak{L}_1(Y, \mathfrak{E}_x, \mu)$ and for all $x \in X$. Moreover define

$$\begin{aligned} \blacktriangleright : \mathbb{C}^Y \times \prod_{x\in X} (\mathfrak{E}_x)^Y &\to \prod_{x\in X} (\mathfrak{E}_x)^Y, \\ (f \blacktriangleright H)(x)(s) &\doteqdot f(s)H(x)(s), \\ \forall f \in \mathbb{C}^Y, H &\in \prod_{x\in X} (\mathfrak{E}_x)^Y, x \in X, s \in Y. \end{aligned}$$

Notice that

$$\mathfrak{L}^\infty(Y, \mu) \blacktriangleright \prod_{x\in X} \mathfrak{L}_1(Y, \mathfrak{E}_x, \mu) \subseteq \prod_{x\in X} \mathfrak{L}_1(Y, \mathfrak{E}_x, \mu).$$

Definition 8. Set

$$\begin{cases} \bigstar : \prod_{x\in X} \mathcal{L}(\mathfrak{E}_x)^Y \times \prod_{x\in X} \mathfrak{E}_x \to \prod_{x\in X} \mathfrak{E}_x^Y, \\ (\forall x \in X)(\forall s \in Y)(F \bigstar v)(x)(s) \doteqdot F(x)(s)(v(x)). \end{cases}$$

Definition 9. $\langle \mathfrak{V}, \mathfrak{Z} \rangle$ *are $\mu-$related* if

1. $\mathfrak{Z} \doteqdot \langle \langle \mathfrak{T}, \gamma \rangle, \zeta, X, \mathfrak{K} \rangle$ ia a bundle of $\Omega-$spaces;

2. for all $x \in X$ [4]

$$\begin{cases} \mathfrak{T}_x \subseteq Meas(Y, \mathfrak{E}_x, \mu) \bigcap \mathfrak{L}_1(Y, \mathfrak{E}_x, \mu), \\ \mathfrak{K}_x = \left\{ \sup_{(s,j)\in O} q_{(s,j)}^x \,|\, O \in \mathcal{P}_\omega(Y \times J) \right\}, \\ q_{(s,j)}^x : \mathfrak{T}_x \ni f_x \mapsto \nu_j(f_x(s)), \forall s \in Y, j \in J; \end{cases} \tag{4.2}$$

3. $\Gamma(\zeta) \subset \left[\prod_{x\in X}^b \mathfrak{T}_x\right]_{ui}$;

4. $\overset{\mu}{\blacklozenge}(\Gamma(\zeta)) \subseteq \Gamma(\pi)$.

[4] In case $\mathfrak{E}_x$ is a Banach space $Meas(Y, \mathfrak{E}_x, \mu) \bigcap \mathfrak{L}_1(Y, \mathfrak{E}_x, \mu) = \mathfrak{L}_1(Y, \mathfrak{E}_x, \mu)$.

Here we set for all $\mathcal{A} \subseteq \prod_{x\in X}^{b} \mathfrak{T}_x$

$$[\mathcal{A}]_{ui} \doteqdot \left\{ H \in \mathcal{A} \,|\, (\forall j \in J) \left(\int_{Y}^{\bullet} \sup_{x\in X} \nu_j(H(x)(s))\, d|\mu|(s) < \infty \right) \right\} \tag{4.3}$$

Finally $\langle \mathfrak{V}, \mathfrak{Z}, \mathsf{H} \rangle$ *are* $\mu-$*related* if

1. $\mathsf{H} = \{\mathsf{H}_x\}_{x\in X}$ such that $\mathsf{H}_x \subseteq \mathcal{L}(\mathfrak{E}_x)^Y$ for all $x \in X$,
2. $\langle \mathfrak{V}, \mathfrak{Z} \rangle$ are $\mu-$related
3. $$\left(\prod_{x\in X} \mathsf{H}_x \right) \bigstar \left(\prod_{x\in X} \mathfrak{E}_x \right) \subseteq \prod_{x\in X} \mathfrak{T}_x. \tag{4.4}$$

Theorem 4.1 (GLT). *Let $\langle \mathfrak{V}, \mathfrak{Z} \rangle$ be $\mu-$related and let $x \in X$ such that its filter of neighbourhoods admits a countable basis. Thus*

$$\overset{\mu}{\blacklozenge}([\Gamma_{\diamond}^{x}(\zeta)]_{ui}) \subseteq \Gamma_{\diamond}^{x}(\pi).$$

Proof. Let $x \in X$ and $F \in [\Gamma_{\diamond}^{x}(\zeta)]_{ui}$ thus by [20, Corollary 3.1] there exists $\eta \in \Gamma(\zeta)$ such that for all $j \in J, s \in Y$

$$\begin{cases} F(x) = \eta(x) \\ \lim_{z\to x} \nu_j(F(z)(s) - \eta(z)(s)) = 0. \end{cases} \tag{4.5}$$

Fix $j \in J$ thus by [4, Prop.6, $No2$, §1, Chapter 6] for all $z \in X$

$$\nu_j \left(\int (F(z)(s) - \eta(z)(s))\, d\mu(s) \right) \leq \int^{\bullet} \nu_j(F(z)(s) - \eta(z)(s))\, d|\mu|(s) \tag{4.6}$$

Moreover ν_j^z is continuous by definition of bundles of $\Omega-$spaces, while $F(z)$ and $\eta(z)$ are by construction $\mu-$measurable, hence by [4, Theorem 1; Corollary 3, $n°3$, § 5, Chapter 4] the map $Y \ni s \mapsto \nu_j(F(z)(s) - \eta(z)(s))$ is $\mu-$measurable thus $|\mu|-$measurable. Moreover by the hypothesis on F and by Definition 9 (3)

$$\int^{\bullet} \nu_j(F(z)(s) - \eta(z)(s))\, d|\mu|(s) \leq \int^{\bullet} \left(\sup_{x\in X} \nu_j(F(x)(s)) + \sup_{x\in X} \nu_j(\eta(x)(s)) \right) d|\mu|(s) < \infty. \tag{4.7}$$

Therefore by [4, Proposition 9, $n°3$, §1, Chapter 5] the map $Y \ni s \mapsto \nu_j(F(z)(s) - \eta(z)(s))$ is $|\mu|-$ essentially integrable hence by the fact that $\int_Y^{\bullet} f\, d|\mu| = \int_Y f\, d|\mu|$ for all $|\mu|-$essentially integrable map f, we have by (4.6)

$$\nu_j \left(\int (F(z)(s) - \eta(z)(s))\, d\mu(s) \right) \leq \int \nu_j(F(z)(s) - \eta(z)(s))\, d|\mu|(s). \tag{4.8}$$

Let $\{z_n\}_n \subset X$ be such that $\lim_{n\in\mathbb{N}} z_n = x$ thus by (4.5)

$$\lim_{n\in\mathbb{N}} \nu_j(F(z_n)(s) - \eta(z_n)(s)) = 0. \tag{4.9}$$

For all $s \in Y$

$$\nu_j(F(z)(s) - \eta(z)(s)) \leq \sup_{x\in X} \nu_j(F(x)(s)) + \sup_{x\in X} \nu_j(\eta(x)(s))$$

thus by the hypothesis on F, by Definition 9(3), by the fact that $\int_Y^{\bullet} \leq \int_Y^{*}$, by (4.9), and by the Lebesgue Theorem [4, Theorem 6, $n^{\circ}3$, §3, Chapter 4] we have

$$\lim_{n\in\mathbb{N}} \int \nu_j(F(z_n)(s) - \eta(z_n)(s))\, d|\mu|(s) = 0. \tag{4.10}$$

Finally by (4.8), (4.10) and the hypothesis on x we obtain

$$\lim_{z\to x} \nu_j \left(\int F(z)(s)\, d\mu(s) - \int \eta(z)(s)\, d\mu(s) \right) = 0,$$

thus the statement follows by Definition 9 (4) and [20, Corollary 3.1]. □

Definition 10 (**$(\Theta, \mathcal{E}, \mu)$ −structure**). We say that $\langle \mathfrak{V}, \mathfrak{Q}, X, Y\rangle$ is a $(\Theta, \mathcal{E}, \mu)-$ *structure* if

1. $\mathcal{E} \subseteq \Gamma(\pi)$;
2. $\Theta \subseteq \prod_{x\in X} Bounded(\mathfrak{E}_x)$;
3. $\forall B \in \Theta$
   (a) $\mathsf{D}(B, \mathcal{E}) \neq \emptyset$;
   (b) $\bigcup_{B\in\Theta} \mathcal{B}_B^x$ is total in $\mathfrak{E}_x$ for all $x \in X$;
4. $\mathfrak{Q} = \langle\langle \mathfrak{H}, \delta\rangle, \xi, X, \mathfrak{Y}\rangle$ is a bundle of $\Omega-$spaces such that for all $x \in X$

$$\boxed{\begin{cases} \mathfrak{H}_x \subseteq \mathfrak{L}_1\left(Y, \mathcal{L}_{S_x}(\mathfrak{E}_x), \mu\right), \\ \mathfrak{Y}_x = \left\{ \sup_{(t,j,B)\in O} P^x_{(t,j,B)} \upharpoonright \mathfrak{H}_x \,|\, O \in \mathcal{P}_\omega\left(Y \times J \times \Theta\right) \right\} \\ P^x_{(t,j,B)} : \mathfrak{L}_1\left(Y, \mathcal{L}_{S_x}(\mathfrak{E}_x), \mu\right) \ni F \mapsto \sup_{v\in\mathsf{D}(B,\mathcal{E})} \nu_j(F(t)v(x)),\ \forall t \in Y, B \in \Theta, j \in J. \end{cases}} \tag{4.11}$$

Here S_x, $\mathcal{B}_B^x$ and $\mathsf{D}(B, \mathcal{E})$ are defined in [20, equation (5.3)]. Moreover $\langle \mathfrak{V}, \mathfrak{Q}, X, Y\rangle$ is an *invariant* $(\Theta, \mathcal{E}, \mu)-$ *structure* if it is a $(\Theta, \mathcal{E}, \mu)-$ structure such that

$$\left\{ F \in \prod_{z\in X}^{b} \mathfrak{H}_z \,|\, (\forall t \in Y)(F_t \bullet \mathcal{E}(\Theta) \subseteq \Gamma(\pi)) \right\} = \Gamma(\xi). \tag{4.12}$$

Definition 11. Let μ_λ for all $\lambda > 0$ be defined as in [21, Definition 1], let $\langle \mathfrak{V}, \mathfrak{Q}, X, \mathbb{R}^+\rangle$ be a $(\Theta, \mathcal{E}, \mu)-$ structure and denote $\mathfrak{Q} = \langle\langle \mathfrak{H}, \delta\rangle, \xi, X, \mathfrak{S}\rangle$, moreover let $x \in X$, $\mathcal{O} \subseteq \Gamma(\xi)$ and $\mathcal{D} \subseteq \Gamma(\pi)$. Set

$$\mathsf{Lap}(\mathfrak{V})(x) \doteqdot \bigcap_{\lambda>0} \mathfrak{L}_1(\mathbb{R}^+, \mathcal{L}_{S_x}(\mathfrak{E}_x); \mu_\lambda).$$

Assume that

$$\Gamma^x_{\mathcal{O}}(\xi) \bigcap \mathsf{Lap}(\mathfrak{V})(x) \neq \emptyset \tag{4.13}$$

We say that $\langle \mathfrak{V}, \mathfrak{Q}, X, \mathbb{R}^+\rangle$ has the *weak-Laplace duality property on $\mathcal{O}$ and $\mathcal{D}$ at x*, shortly $w - \mathsf{LD}_x(\mathcal{O}, \mathcal{D})$ if $\forall \lambda > 0$

$$\blacksquare_{\mu_\lambda} \left(\Gamma^x_{\mathcal{O}}(\xi) \bigcap \mathsf{Lap}(\mathfrak{V})(x), \Gamma^x_{\mathcal{D}}(\pi) \right) \subseteq \Gamma^x(\pi).$$

Definition 12. Let $\langle \mathfrak{V}, \mathfrak{W}, X, Y\rangle$ be a $(\Theta, \mathcal{E})-$structure and denote $\mathfrak{W} = \langle\langle \mathfrak{M}, \gamma\rangle, \rho, X, \mathfrak{R}\rangle$. Assume that for all $x \in X$

$$\mathfrak{M}_x \subseteq \mathfrak{L}_1\left(Y, \mathcal{L}_{S_x}(\mathfrak{E}_x), \mu\right). \tag{4.14}$$

Set $\mathsf{M}^\mu \doteqdot \left\{\langle \mathsf{M}_x^\mu, \mathfrak{Y}_x\rangle\right\}_{x\in X}$ where for all $x \in X$

$$\begin{aligned}&\mathsf{M}_x^\mu \doteqdot \overline{\{\sigma(x) \,|\, \sigma \in \Gamma(\rho)\}} \text{ closure in the space } \mathfrak{L}_1\left(Y, \mathcal{L}_{S_x}(\mathfrak{E}_x), \mu\right)_s;\\ &\mathfrak{Y}_x \doteqdot \{\sup_{(t,j,B)\in O} P_{(t,j,B)}^x \upharpoonright \mathsf{M}_x^\mu \,|\, O \in \mathcal{P}_\omega(Y \times J \times \Theta)\}.\end{aligned}$$

Notice that M^μ is a nice family of Hlcs, and that $\Gamma(\rho)$ satisfies by construction $FM(3)$ with respect to M^μ. Moreover by [20, equation (5.7)] and the fact that $\{t\} \in Comp(Y)$ for all $t \in Y$

$$P_{(t,j,B)}^x = q_{(\{t\},j,B)}^x. \tag{4.15}$$

By [12, Corollary 1.6.(iii)] we deduce that $\Gamma(\rho)$ satisfies $FM(4)$ with respect to $\{\langle \mathfrak{M}_x, \mathfrak{R}_x\rangle\}_{x\in X}$. Therefore we obtain by [20, equation (5.6)] and (4.15) that for all $t \in Y$, $j \in J$, $B \in \Theta$ and for all $\sigma \in \Gamma(\rho)$

$$X \ni x \mapsto P_{(t,j,B)}^x(\sigma(x)) \text{ is } u.s.c.$$

Moreover the upper envelope of a finite set of $u.s.c.$ maps is an $u.s.c.$ map, see [2, Theorem 4,§6.2., Chapter 4], therefore for all $O \in \mathcal{P}_\omega(Y \times J \times \Theta)$

$$X \ni x \mapsto \sup_{(t,j,B)\in O} P_{(t,j,B)}^x(\sigma(x)) \text{ is } u.s.c. \tag{4.16}$$

Hence $\Gamma(\rho)$ satisfies $FM(4)$ with respect to M^μ. Finally by the boundedness of $\Gamma(\rho)$ by definition and by (4.15) we have also that for all $\sigma \in \Gamma(\rho)$ and $O \in \mathcal{P}_\omega(Comp(Y) \times J \times \Theta)$

$$\sup_{x\in X} \sup_{(t,j,B)\in O} P_{(t,j,B)}^x(\sigma(x)) < \infty.$$

Therefore we can construct the bundle generated by the couple $\langle \mathsf{M}^\mu, \Gamma(\rho)\rangle$ [20, Definition 15]

$$\mathfrak{V}(\mathsf{M}^\mu, \Gamma(\rho)).$$

Clearly $\langle \mathfrak{V}, \mathfrak{V}(\mathsf{M}^\mu, \Gamma(\rho)), X, Y\rangle$ is a $(\Theta, \mathcal{E}, \mu)-$structure that we call the $(\Theta, \mathcal{E}, \mu)-$*structure underlying* $\langle \mathfrak{V}, \mathfrak{W}, X, Y\rangle$.

Definition 13. Let $\langle \mathfrak{V}, \mathfrak{Q}, X, Y\rangle$ be a $(\Theta, \mathcal{E}, \mu)-$ structure and $A \subset \prod_{x\in X} \mathfrak{H}_x$. Define A_{peq} as the set of all pointwise equicontinuous elements in A, and A_{ceq} as the set of all compactly equicontinuous elements in A, see [20, Definition 7].

Remark 2. [20, Lemma 5.1] holds by replacing a $(\Theta, \mathcal{E})-$structure $\langle \mathfrak{V}, \mathfrak{W}, X, Y\rangle$ with a $(\Theta, \mathcal{E}, \mu)-$ structure $\langle \mathfrak{V}, \mathfrak{Q}, X, Y\rangle$ and $K \in Comp(Y)$ with $t \in Y$. In what follows when referring to [20, Lemma 5.1] for a $(\Theta, \mathcal{E}, \mu)-$ structure we shall mean the corresponding result with the replacements described here.

Lemma 4.1. *Let* $\langle \mathfrak{V}, \mathfrak{Q}, X, Y\rangle$ *be a* $(\Theta, \mathcal{E}, \mu)-$ *structure and* $\langle \mathfrak{V}, \mathfrak{Z}, \mathsf{H}\rangle$ *be* $\mu-$*related, where* $\mathfrak{Q} \doteqdot \langle\langle \mathfrak{H}, \gamma\rangle, \xi, X, \mathfrak{Y}\rangle$ *and* $\mathsf{H}_x \doteqdot \mathfrak{H}_x$ *for all* $x \in X$. *Thus*

$$\Gamma(\xi)\bigstar\mathcal{E}(\Theta) \subseteq \Gamma(\zeta) \Rightarrow (\forall x \in X)\left(\Gamma_\diamond^x(\xi)_{peq}\bigstar\Gamma_{\mathcal{E}(\Theta)}^x(\pi) \subseteq \Gamma_\diamond^x(\zeta)\right).$$

Proof. Let $j \in J$, $x \in X$ and $w \in \Gamma^{x}_{\mathcal{E}(\Theta)}(\pi)$, so there exists $v \in \mathcal{E}(\Theta)$ such that $v(x) = w(x)$ then by [20, Corollary 3.1]

$$\lim_{z \to x} \nu_j(w(z) - v(z)) = 0. \tag{4.17}$$

Moreover let $F \in \Gamma^{x}_{\diamond}(\xi)$, so by [20, Lemma 5.1] $\exists \sigma \in \Gamma(\xi)$ such that $F(x) = \sigma(x)$ and for all $t \in Y$

$$\lim_{z \to x} \nu_j \left(F(z)(t)v(z) - \sigma(z)(t)v(z)\right) = 0. \tag{4.18}$$

Moreover $(\forall t \in Y)(\exists\, M_{(t,j)} > 0)(\exists\, j_1 \in J)(\forall z \in X)$

$$\begin{aligned}\nu_j \left((F\bigstar w)(z)(t) - (\sigma\bigstar v)(z)(t)\right) &= \nu_j \left(F(z)(t)w(z) - \sigma(z)(t)v(z)\right)\\ &\leq \nu_j \left(F(z)(t)(w(z) - v(z))\right) + \nu_j \left(F(z)(t)v(z) - \sigma(z)(t)v(z)\right)\\ &\leq M_{(t,j)}\nu_{j_1}(w(z) - v(z)) + \nu_j \left(F(z)(t)v(z) - \sigma(z)(t)v(z)\right).\end{aligned}$$

Therefore by (4.17) and (4.18) for all $t \in Y$

$$\lim_{z \to x} \nu_j \left((F\bigstar w)(z)(t) - (\sigma\bigstar v)(z)(t)\right) = 0.$$

Moreover $(\forall t \in Y)(\exists\, M_{(t,j)} > 0)(\exists\, j_1 \in J)$

$$\sup_{z \in X} \nu_j((F\bigstar w)(z)(t)) \leq M_{(t,j)} \sup_{z \in X} \nu_{j_1}(w(z)) < \infty. \tag{4.19}$$

By the antecedent of the implication of the statement we deduce that $\sigma\bigstar v \in \Gamma(\zeta)$ hence the statement follows by [20, Corollary 3.1], (4.2), by the fact that by (4.4) $F\bigstar w \in \prod_{x \in X} \mathfrak{T}_x$ and by (4.19). □

Proposition 4.1. *Let $\langle \mathfrak{V}, \mathfrak{W}, X, Y \rangle$ be a compatible $(\Theta, \mathcal{E})-$structure. Then for all $x \in X$*

$$(\Gamma^{x}_{\diamond}(\rho)_{peq})_t \bullet \Gamma^{x}_{\mathcal{E}(\Theta)}(\pi) \subseteq \Gamma^{x}_{\diamond}(\pi) \tag{4.20}$$

Proof. Notice that $(F\bigstar v)(t) = F_t \bullet v$, thus if we set $Y = \{pt\}$ the statement follows by Lemma 4.1. □

Corollary 4.1. *Let $\langle \mathfrak{V}, \mathfrak{Q}, X, Y \rangle$ be a $(\Theta, \mathcal{E}, \mu)-$ structure and $\langle \mathfrak{V}, \mathfrak{Z}, \mathsf{H} \rangle$ be $\mu-$related. If $x \in X$ is such that its filter of neighbourhoods admits a countable basis, then*

$$\Gamma(\xi)\bigstar\mathcal{E}(\Theta) \subseteq \Gamma(\zeta) \Rightarrow \overset{\mu}{\blacklozenge} \left(\left[\Gamma^{x}_{\diamond}(\xi)_{peq}\bigstar\Gamma^{x}_{\mathcal{E}(\Theta)}(\pi)\right]_{ui}\right) \subseteq \Gamma^{x}_{\diamond}(\pi).$$

Here $\mathfrak{Q} \doteqdot \langle \langle \mathfrak{H}, \gamma \rangle, \xi, X, \mathfrak{Y} \rangle$, $\mathfrak{Z} \doteqdot \langle \langle \mathfrak{T}, \delta \rangle, \zeta, X, \mathfrak{K} \rangle$ and $\mathsf{H}_x \doteqdot \mathfrak{H}_x$ for all $x \in X$.

Proof. By Theorem 4.1 and Lemma 4.1. □

Lemma 4.2. *Let $\langle \mathfrak{V}, \mathfrak{Q}, X, Y \rangle$ be an invariant $(\Theta, \mathcal{E}, \mu)-$ structure where Θ defined in [21, equation (2.16)]. Then for all $x \in X$*

$$\left\{ H \in \left[\left(\prod_{z \in X}^{b} \mathfrak{H}_z\right)^{x}_{\diamond}\right]_{peq} \mid (\forall t \in Y)(H_t \bullet \mathcal{E}(\Theta) \subseteq \Gamma^{x}(\pi)) \right\} \subseteq \Gamma^{x}_{\diamond}(\xi). \tag{4.21}$$

Proof. Let $v \in \mathcal{E}(\Theta)$, $t \in Y$ and H belong to the set in the left side of (4.21). Thus by (4.12) $\exists\, F \in \Gamma(\xi)$ such that $F_t \bullet v \in \Gamma(\pi)$, $F(x) = H(x)$ and $H_t \bullet v \in \Gamma^{x}(\pi)$ by construction. Then by [20, Corollary 3.1] we obtain for all $j \in J$

$$\lim_{z \to x} \nu_j(H(z)(t)v(z) - F(z)(t)v(z)) = 0.$$

Therefore the statement follows by [20, Lemma 5.1] and [21, equation (2.17)]. □

Remark 3. Notice that Lemma 4.2 holds if we replace invariant $(\Theta, \mathcal{E}, \mu)-$ structure with invariant $(\Theta, \mathcal{E})-$ structure, see [20, Definition 6] and assume that $Comp(Y) = \mathcal{P}_\omega(Y)$.

By recalling Definition 5 and Definition 4 we can state the following significant

Corollary 4.2. *Let $\mathfrak{V}$ be a Banach bundle and set $\mathsf{E} \doteqdot \{\mathfrak{E}_z\}_{z\in X}$. Let $M > 1$, $\beta \in \mathbb{R}$ and let $T \in \mathcal{G}(M, \beta, \mathsf{E})$ satisfy the property of separation of the spectrum. Let $x \in X$ admitting a countable basis of its filter of neighbourhoods. Let Γ be a curve associated with T and (K, A, ϕ) be a triplet associated with Γ (Definition 4) such that*

$$\begin{cases} Re(\Gamma) \subseteq \mathbb{R}^-, \\ \beta > 0 \Rightarrow -\beta \notin Re(\Gamma). \end{cases} \tag{4.22}$$

Moreover let $\mathfrak{Q} = \langle\langle \mathfrak{H}, \gamma_1\rangle, \xi, X, \mathfrak{R}\rangle$ and $\mathfrak{Z} = \langle\langle \mathfrak{T}, \gamma_2\rangle, \zeta, X, \mathfrak{K}\rangle$ be such that

1. *$\langle \mathfrak{V}, \mathfrak{Q}, X, \mathbb{R}^+\rangle$ is an invariant $(\Theta, \mathcal{E}, \mu)-$structure for all $\mu \in \{\nu^\phi, \eta_s^\phi \mid s \in K\}$, with Θ determined by $\mathcal{E}$ by [21, equation (2.16)];*
2. *$\langle \mathfrak{V}, \mathfrak{Z}, \mathsf{H}\rangle$ is $\mu-$related for all $\mu \in \{\nu^\phi, \eta_s^\phi \mid s \in K\}$, moreover $\mathsf{H}_z \doteqdot \mathfrak{H}_z$, for all $z \in X$;*
3. *there exist $F, G \in \Gamma(\xi)$ such that $F(x) = \mathcal{W}_T(x)$ and $G(x) = R^\phi(x)$;*
4. *for all $z \in X$*
$$\mathcal{C}_{cs}\left(\mathbb{R}^+, \mathcal{L}_{S_z}(\mathfrak{E}_z)\right) \subseteq \mathfrak{H}_z; \tag{4.23}$$
5. *$\Gamma(\xi) \bigstar \mathcal{E}(\Theta) \subseteq \Gamma(\zeta)$.*

Thus
$$\mathcal{W}_T \in \Gamma^x(\xi) \Rightarrow P \bullet \Gamma^x_{\mathcal{E}(\Theta)}(\pi) \subseteq \Gamma^x(\pi). \tag{4.24}$$
Moreover let $\mathfrak{D} = \langle\langle \mathfrak{B}, \gamma_3\rangle, \eta, X, \mathfrak{L}\rangle$ be such that $\langle \mathfrak{V}, \mathfrak{D}, X, \{pt\}\rangle$ is an invariant $(\Theta, \mathcal{E})-$structure. If $\Pr(\mathfrak{E}_z) \subset \mathfrak{B}_z$ for all $z \in X$ and there exists $N \in \Gamma(\eta)$ such that $N(x) = P(x)$, then
$$\mathcal{W}_T \in \Gamma^x(\xi) \Rightarrow P \in \Gamma^x(\eta). \tag{4.25}$$

Proof. In this proof we denote R_T^ϕ simply by R^ϕ. R^ϕ is $K-$supported by construction, moreover the resolvent map $R^\phi(z)$ being analytic is $\|\cdot\|_{B(\mathfrak{E}_z)}-$ continuous hence continuous as a map valued in $\mathcal{L}_{S_z}(\mathfrak{E}_z)$ for any $z \in X$. So

$$R^\phi \in \prod_{z\in X} \mathcal{C}_{cs}\left(\mathbb{R}^+, \mathcal{L}_{S_z}(\mathfrak{E}_z)\right),$$

hence by (4.23) follows
$$R^\phi \in \prod_{z\in X} \mathfrak{H}_z. \tag{4.26}$$

By (3.4) for all $s \in K$, $z \in X$ and $\forall v_z \in \mathfrak{E}_z$

$$\begin{aligned} \|R(T(z), \phi(s))v_z\| &\leq \int^*_{\mathbb{R}^+} e^{-|Re(\phi(s))|t} \|e^{-tT(z)} v_z\| dt \\ &\leq M\|v_z\| \int^*_{\mathbb{R}^+} e^{(\beta - |Re(\phi(s))|)t} dt = \frac{M\|v_z\|}{\beta - |Re(\phi(s))|}, \end{aligned} \tag{4.27}$$

where $\int^*_{\mathbb{R}^+}$ is the upper integral on $\mathbb{R}^+$ with respect to the Lebesgue measure. We considered in the first inequality [4, Proposition 6, §1, Chapter 6], in the second one the inequality [14, (1.37),

$n^\circ 4$, §1, Chapter 9], finally in the equality the Laplace transform of the map $\exp(\beta t)$. Therefore by (4.26) and (4.27)

$$R^\phi \in \left(\prod_{z\in X} \mathfrak{H}_z\right)_{peq}.$$

Thus by (4.27), [21, equation (2.16)] and (4.11)

$$R^\phi \in \left(\prod_{z\in X}^b \mathfrak{H}_z\right)_{peq}. \tag{4.28}$$

By (4.26) and (4.4) we have that $R^\phi \bigstar v \in \prod_{z\in X} \mathfrak{T}_z$ for all $v \in \prod_{z\in X}^b \mathfrak{E}_z$. By hypothesis (4.22) we deduce that $\frac{1}{\beta - |Re(\phi(s))|}$ is defined on K, hence continuous and integrable in it, thus by (4.27)

$$R^\phi \bigstar v \in \left[\prod_{z\in X} \mathfrak{T}_z\right]_{ui}. \tag{4.29}$$

By the continuity of $\frac{1}{\beta - |Re(\phi(s))|}$ on K we deduce that the map $\frac{|\frac{d\phi}{ds}(s)|}{\beta - |Re(\phi(s))|}$ is integrable in K. Hence by (4.27) and (3.3)

$$\sup_{z\in X} \|P(z)\|_{B(\mathfrak{E}_z)} \leq D \doteqdot \frac{1}{2\pi i} \int_K \frac{M\left|\frac{d\phi}{ds}(s)\right|}{\beta - |Re(\phi(s))|}\, ds. \tag{4.30}$$

Therefore for all $v \in \mathcal{E}$ by considering that $\mathcal{E} \subset \prod_{z\in X}^b \mathfrak{E}_z$

$$\sup_{z\in X} \|P(z)v(z)\|_{B(\mathfrak{E}_z)} \leq D \sup_{z\in X} \|v(z)\|_{\mathfrak{E}_z} < \infty.$$

Thus

$$P \in \prod_{z\in X}^b \mathfrak{B}_z. \tag{4.31}$$

Let $z \in X$ and $v \in \Gamma^z_{\mathcal{E}(\Theta)}(\pi)$. By (3.4) for all $s \in K$

$$(R^\phi \bigstar v)(z)(s) = \overset{\eta_s^\phi}{\blacklozenge} (\mathcal{W}_T \bigstar v)(z). \tag{4.32}$$

Moreover by (3.3)

$$P \bullet v = \overset{\nu^\phi}{\blacklozenge} (R^\phi \bigstar v). \tag{4.33}$$

Notice that $(R^\phi \bigstar v)(z)(s) = \big(R^\phi(\cdot)(s) \bullet v\big)(z)$ so by (4.32) for all $s \in K$

$$R^\phi(\cdot)(s) \bullet v = \overset{\eta_s^\phi}{\blacklozenge} (\mathcal{W}_T \bigstar v). \tag{4.34}$$

If $\mathcal{W}_T \in \Gamma^x(\xi)$ then by [14, Chapter 9, §1, $n^\circ 4$, (1.37)] and hypothesis (3) follows that $\mathcal{W}_T \in \Gamma^x_\diamond(\xi)_{peq}$. Therefore for all $w \in \Gamma^x_{\mathcal{E}(\Theta)}(\pi)$ by using [14, Chapter 9, §1, $n^\circ 4$, (1.37)] we can apply Corollary 4.1 to $\mathcal{W}_T \bigstar w$ and then since (4.34) we obtain for all $s \in K$

$$R^\phi(\cdot)(s) \bullet w \in \Gamma^x(\pi). \tag{4.35}$$

By construction $\mathcal{E}(\Theta) \subseteq \Gamma(\pi)$ so $\mathcal{E}(\Theta) \subseteq \Gamma^x_{\mathcal{E}(\Theta)}(\pi)$, hence

$$R^\phi(\cdot)(s) \bullet \mathcal{E}(\Theta) \subseteq \Gamma^x(\pi). \tag{4.36}$$

Moreover by (4.28) and hypothesis (3) follows that

$$R^{\phi} \in \left[\left(\prod_{z\in X}^{b} \mathfrak{H}_z\right)_{\diamond}^{x}\right]_{peq}.$$

Hence by Lemma 4.2 and (4.36)

$$R^{\phi} \in \Gamma_{\diamond}^{x}(\xi)_{peq}. \tag{4.37}$$

Finally (4.24) follows by (4.37), (4.33), (4.29) and Corollary 4.1. Next since (4.31), (4.30) and the hypothesis that there exists $N \in \Gamma(\eta)$ such that $N(x) = P(x)$, we obtain that

$$P \in \left[\left(\prod_{z\in X}^{b} \mathfrak{B}_z\right)_{\diamond}^{x}\right]_{peq}.$$

Thus (4.25) follows by (4.24), Remark 3 and by $\mathcal{E}(\Theta) \subseteq \Gamma_{\mathcal{E}(\Theta)}^{x}(\pi)$. □

Remark 4. [21, Proposition 4.2] can be used in order to verify part of hypothesis (1) of Corollary 4.2.

The following general result shall permit to apply Corollary 4.2 to [21, Theorem 2.1].

Proposition 4.2. *Let $\langle \mathfrak{V}, \mathfrak{W}, X, Y\rangle$ be a $(\Theta, \mathcal{E})-$structure and let us denote $\mathfrak{W} = \langle\langle \mathfrak{M}, \delta\rangle, \rho, X, \mathfrak{R}\rangle$. Assume that (4.14) holds for all $x \in X$ and let $\langle \mathfrak{V}, \mathfrak{V}(\mathsf{M}^{\mu}, \Gamma(\rho)), X, Y\rangle$ be the $(\Theta, \mathcal{E}, \mu)-$ structure underlying $\langle \mathfrak{V}, \mathfrak{W}, X, Y\rangle$. Then for all $x \in X$*

$$\Gamma_{\diamond}^{x}(\rho) \subseteq \Gamma_{\Gamma(\rho)}^{x}(\pi_{\mathsf{M}^{\mu}}), \tag{4.38}$$

inclusion to be considered modulo canonical isomorphism.

Proof. By construction it results that $\Gamma(\rho) \subseteq \Gamma(\pi_{\mathsf{M}^{\mu}})$ modulo the canonical isomorphism, thus the statement follows since [20, Lemma 5.1] and (4.15). □

Definition 14. We call $\mathfrak{X} = \langle \mathfrak{V}, x_{\infty}, \mathcal{U}_0\rangle$ a quasi-appropriate set of contractions (isometries) if the following holds. $\mathfrak{V} = \langle\langle \mathfrak{E}, \tau\rangle, \pi, X, \|\cdot\|\rangle$ is a Banach bundle where X is a completely regular space, $x_{\infty} \in X$ such that its filter of neighbourhoods admits a countable basis and $\mathcal{U}_0 \in \prod_{x\in X_0} \mathcal{C}(\mathbb{R}^{+}, B_s(\mathfrak{E}_x))$ such that $\mathcal{U}_0(x)$ is a C_0-semigroup of contractions (isometries) on $\mathfrak{E}_x$ for all $x \in X_0 \doteqdot X - \{x_{\infty}\}$. Moreover let T_x be the infinitesimal generator of the semigroup $\mathcal{U}_0(x)$ for any $x \in X_0$, let $\langle\langle \mathfrak{E}(\mathsf{E}^{\oplus}), \tau(\mathsf{E}^{\oplus}, \mathcal{E}^{\oplus})\rangle, \pi_{\mathsf{E}^{\oplus}}, X, \mathfrak{n}^{\oplus}\rangle$ be the bundle direct sum of the family $\{\mathfrak{V}, \mathfrak{V}\}$[5] and set

$$\begin{cases} \mathcal{T}_0 \text{ is the map on } X_0 \text{ such that} \\ \mathcal{T}_0(x) \doteqdot Graph(T_x),\ x \in X_0, \\ \Phi \doteqdot \{\phi \in \Gamma^{x_{\infty}}(\pi_{\mathsf{E}^{\oplus}}) \,|\, (\forall x \in X_0)(\phi(x) \in \mathcal{T}_0(x))\}, \\ \mathcal{E} \doteqdot \{v \in \Gamma(\pi) \,|\, (\exists\, \phi \in \Phi)(v(x_{\infty}) = \phi_1(x_{\infty}))\}, \\ B_v : X \ni x \mapsto \{v(x)\},\ \forall v \in \prod_{x\in X} \mathfrak{E}_x, \\ \Theta \doteqdot \{B_w \,|\, w \in \mathcal{E}\}, \\ \mathcal{T} \text{ is the map on } X \text{ extending } \mathcal{T}_0 \text{ and such that} \\ \mathcal{T}(x_{\infty}) \doteqdot \{\phi(x_{\infty}) \,|\, \phi \in \Phi\}, \\ D(T_{x_{\infty}}) \doteqdot \mathrm{Pr}_1^{x_{\infty}}(\mathcal{T}(x_{\infty})) = \{\phi_1(x_{\infty}) \,|\, \phi \in \Phi\}. \end{cases} \tag{4.39}$$

We call $\mathfrak{X} = \langle \mathfrak{V}, x_{\infty}, \mathcal{U}_0\rangle$ an appropriate set of contractions (isometries) if it is a quasi-appropriate set of contractions (isometries) and all the following holds.

[5] well-set since $\mathfrak{B}$ is full by the Dupre' theorem

1. $D(T_{x_\infty})$ is dense in $\mathfrak{E}_{x_\infty}$,

2. $\{v(x) \mid v \in \mathcal{E}\}$ is dense in $\mathfrak{E}_x$ for all $x \in X_0$;

hence according to [21, Theorem 2.1] what follows

$$T_{x_\infty} : D(T_{x_\infty}) \ni \phi_1(x_\infty) \mapsto \phi_2(x_\infty),$$

defines a linear operator. We require that $\exists \lambda_0 > 0$ ($\exists \lambda_0 > 0, \lambda_1 < 0$) such that the range $\mathcal{R}(\lambda_0 - T_{x_\infty})$ is dense in $\mathfrak{E}_{x_\infty}$, (the ranges $\mathcal{R}(\lambda_0 - T_{x_\infty})$ and $\mathcal{R}(\lambda_1 - T_{x_\infty})$ are dense in $\mathfrak{E}_{x_\infty}$). Thus according to [21, Theorem 2.1] T_{x_∞} is the infinitesimal generator of a C_0-semigroup of contractions (isometries) on $\mathfrak{E}_{x_\infty}$. Therefore we can define *section of semigroups associated with* $\mathfrak{X}$ the following map

$$\mathcal{U} \in \prod_{x \in X} \mathsf{U}_{\|\cdot\|_{B(\mathfrak{E}_x)}}(\mathcal{L}_{S_x}(\mathfrak{E}_x)),$$

($\mathcal{U} \in \prod_{x \in X} \mathsf{U}_{is}(\mathcal{L}_{S_x}(\mathfrak{E}_x))$) such that $\mathcal{U}(x)$ is the C_0-semigroup of contractions (isometries) on $\mathfrak{E}_x$ whose infinitesimal generator is T_x for all $x \in X$. Finally set

$$\mathsf{T} : X \ni x \mapsto -T_x \in Cld(\mathfrak{E}_x).$$

We require that T satisfies the property of separation of the spectrum and that there exists a curve Γ associated with T such that

$$Re(\Gamma) \subseteq \mathbb{R}^-.$$

We call T *section of generators associated with* $\mathfrak{X}$. Finally for any curve Γ associated with T such that $Re(\Gamma) \subseteq \mathbb{R}^-$ we define *section of projectors associated with* $\mathfrak{X}$ *and* Γ the map $\mathcal{P} \in \prod_{x \in X} \Pr(\mathfrak{E}_x)$ such that for all $x \in X$

$$\mathcal{P}(x) \doteqdot -\frac{1}{2\pi i} \int_\Gamma R(-T_x; \zeta) \, d\zeta \in B(\mathfrak{E}_x).$$

Here we recall that $R(-T_x; \cdot) : P(-T_x) \ni \zeta \mapsto (-T_x - \zeta)^{-1} \in B(\mathfrak{E}_x)$ is the resolvent map of $-T_x$ and $P(-T_x)$ is its resolvent set, while the integration is with respect to the norm topology on $B(\mathfrak{E}_x)$.

Theorem 4.2 (MAIN2). *Let $\mathfrak{X} = \langle \mathfrak{V}, x_\infty, \mathcal{U}_0 \rangle$ be a quasi-appropriate set of contractions (isometries), let us denote $\mathfrak{V} = \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \| \cdot \| \rangle$ and use the notation in* (4.39). *Assume that $\{v(x_\infty) \mid v \in \mathcal{E}\}$ is dense in $\mathfrak{E}_{x_\infty}$. Then $Dom(T_{x_\infty})$ is dense in $\mathfrak{E}_{x_\infty}$. Next assume that $\mathfrak{X}$ satisfies all the remaining requests in order to be an appropriate set of contractions (isometries). Let $\mathcal{U}$ be the section of semigroups associated with $\mathfrak{X}$, Γ be a curve associated with the section of generators associated with $\mathfrak{X}$ such that $Re(\Gamma) \subseteq \mathbb{R}^-$, (K, A, ϕ) be a triplet associated with Γ and $\mathcal{P}$ be the section of projectors associated with $\mathfrak{X}$ and Γ. Let $\mathfrak{n}$ denote the Lebesgue measure on $\mathbb{R}^+$. We assume that there exist $\mathfrak{W} = \langle \langle \mathfrak{M}, \delta \rangle, \rho, X, \mathfrak{R} \rangle$ and $\mathfrak{Z} = \langle \langle \mathfrak{T}, \gamma_2 \rangle, \zeta, X, \mathfrak{K} \rangle$ with the following properties.*

1. *$\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ is a $(\Theta, \mathcal{E})-$structure;*

2. *for all $x \in X$*
$$\mathcal{C}_{cs}\left(\mathbb{R}^+, \mathcal{L}_{S_x}(\mathfrak{E}_x)\right) \subseteq \mathfrak{M}_x \subseteq \mathfrak{L}_1(\mathbb{R}^+, \mathcal{L}_{S_x}(\mathfrak{E}_x), \mathfrak{n});$$

3. *$\langle \mathfrak{V}, \mathfrak{V}(\mathsf{M}^{\mathfrak{n}}, \Gamma(\rho)), X, \mathbb{R}^+ \rangle$*[6] *is invariant and $\langle \mathfrak{V}, \mathfrak{Z}, \mathsf{M}^{\mathfrak{n}} \rangle$ is $\mathfrak{n}-$related such that $\mathfrak{L}^\infty(\mathbb{R}^+, \mathfrak{n}) \blacktriangleright \Gamma(\zeta) \subseteq \Gamma(\zeta)$;*

[6] Definition 12

4. $\Gamma(\rho)\bigstar\mathcal{E}(\Theta) \subseteq \Gamma(\zeta)$;

5. $\mathsf{U}_{\|\cdot\|_{B(\mathfrak{E}_x)}}(\mathcal{L}_{S_x}(\mathfrak{E}_x)) \subseteq \mathfrak{M}_x$ $(\mathsf{U}_{is}(\mathcal{L}_{S_x}(\mathfrak{E}_x)) \subseteq \mathfrak{M}_x)$, *for all* $x \in X$;

6. $\exists F \in \Gamma(\rho)$ *such that* $F(x_\infty) = \mathcal{U}(x_\infty)$ *and*

   i $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ *has the* $\mathsf{LD}_{x_\infty}(\{F\}, \mathcal{E})$ ***or*** *it has the* $\mathsf{LD}(\{F\}, \mathcal{E})$;

   ii *for any* $v \in \mathcal{E}$ *there exists* $\phi \in \Phi$ *such that* $v(x_\infty) = \phi_1(x_\infty)$ *and* $(\forall \{z_n\}_{n\in\mathbb{N}} \subset X \mid \lim_{n\in\mathbb{N}} z_n = x_\infty)$ *we have that* $\{\mathcal{U}(z_n)(\cdot)\phi_1(z_n) - F(z_n)(\cdot)v(z_n)\}_{n\in\mathbb{N}}$ *is a bounded equicontinuous sequence.*

Thus we can state what follows.

1. *If* $\exists G \in \Gamma(\rho)$ *such that* $G(x_\infty) = R^\phi(x_\infty)$, *then* $\mathcal{P} \bullet \Gamma^{x_\infty}_{\mathcal{E}(\Theta)}(\pi) \subseteq \Gamma^{x_\infty}(\pi)$.

2. *Let* $\mathfrak{D} = \langle \langle \mathfrak{B}, \gamma_3 \rangle, \eta, X, \mathfrak{L} \rangle$ *be such that* $\langle \mathfrak{V}, \mathfrak{D}, X, \{pt\} \rangle$ *is an invariant* $(\Theta, \mathcal{E})-$*structure. If* $\Pr(\mathfrak{E}_x) \subset \mathfrak{B}_x$ *for all* $x \in X$ *and if there exists* $N \in \Gamma(\eta)$ *such that* $N(x_\infty) = \mathcal{P}(x_\infty)$, *then*
$$\mathcal{P} \in \Gamma^{x_\infty}(\eta), \tag{4.40}$$
and
$$\boxed{\{\langle \mathcal{T}, x_\infty, \Phi \rangle\} \in \Delta \langle \mathfrak{V}, \mathfrak{D}, \Theta, \mathcal{E} \rangle .} \tag{4.41}$$

Proof. $\mathfrak{V}$ is full since the Dupre' theorem [12, Corollary 2.10]. So by [21, Proposition 2.2] and the density assumption follows that $Dom(T_{x_\infty})$ is dense in $\mathfrak{E}_{x_\infty}$. Since [12, 2.2] we deduce that the set of all bounded continuous sections of any bundle of $\Omega-$spaces over a completely regular space satisfies the property $FM(3)$. Therefore $\mathfrak{M}_x \subset \mathsf{M}^{\mathfrak{n}}_x$ for all $x \in X$, since the immersion $\langle \mathfrak{M}_x, \mathfrak{R}_x \rangle \hookrightarrow \mathfrak{L}_1(Y, \mathcal{L}_{S_x}(\mathfrak{E}_x), \mathfrak{n})_s$ is continuous. Thus
$$\mathcal{C}_{cs}\left(\mathbb{R}^+, \mathcal{L}_{S_x}(\mathfrak{E}_x)\right) \subset \mathsf{M}^{\mathfrak{n}}_x. \tag{4.42}$$
Now since $Re(\Gamma) \subseteq \mathbb{R}^-$ and since $d\phi/ds$ is continuous and then bounded on K, we deduce by hypothesis (3) that
$$\begin{aligned} &\left\langle \mathfrak{V}, \mathfrak{V}(\mathsf{M}^{\mathfrak{n}}, \Gamma(\rho)), X, \mathbb{R}^+ \right\rangle \text{ is an invariant } (\Theta, \mathcal{E}, \mu)-\text{structure and} \\ &\langle \mathfrak{V}, \mathfrak{Z}, \mathsf{M}^{\mathfrak{n}} \rangle \text{ is } \mu-\text{related}, \forall \mu \in \{\nu^\phi, \eta^\phi_s \mid s \in K\}. \end{aligned} \tag{4.43}$$
In particular [21, equation (2.14)] holds, so we can apply [21, Theorem 2.1] to obtain $\mathcal{U} \in \Gamma^{x_\infty}(\rho)$ and in virtue of hypothesis (6) that $\mathcal{U} \in \Gamma^{x_\infty}_\diamond(\rho)$. Thus by Proposition 4.2 we have
$$\mathcal{U} \in \Gamma^{x_\infty}_\diamond(\pi_{\mathsf{M}^{\mathfrak{n}}}). \tag{4.44}$$
Now for the position $\mathfrak{Q} = \mathfrak{V}(\mathsf{M}^{\mathfrak{n}}, \Gamma(\rho))$ the hypotheses (1) and (2) of Corollary 4.2 are satisfied since (4.43). Moreover $F, G \in \Gamma(\pi_{\mathsf{M}^{\mathfrak{n}}})$ indeed $\Gamma(\rho) \subseteq \Gamma(\pi_{\mathsf{M}^{\mathfrak{n}}})$ modulo the canonical isomorphism, so hypothesis (3) of Corollary 4.2 is satisfied. Hence statement (1) follows by (4.44) and (4.24), while (4.40) follows by (4.44) and (4.25). Next $\langle \mathcal{T}, x_\infty, \Phi \rangle \in \mathsf{Gr}(\mathfrak{V}, \mathfrak{V})$ since [21, Theorem 2.1], while $\mathcal{P}(x)T_x \subseteq T_x\mathcal{P}(x)$ for all $x \in X$ since the resolvent map of any operator commutes with its operator see for example [14, § 6.1. Chapter 3], thus (4.41) follows by (4.40). $\square$

Remark 5. By [20, equation (5.8)] follows that (4.40) is equivalent to say that for all $v \in \mathcal{E}$
$$\lim_{z \to x_\infty} \|(\mathcal{P}(z) - N(z))\, v(z)\| = 0.$$

5 Kurtz Bundle Construction

In this section we construct a special bundle $\mathfrak{E}$ of Banach space such that for it [21, Theorem 2.1] reduces to the [15, Theorem 2.1.] showing in this way that (a particular case) of the construction of Kurtz falls into the general setting of bundle of $\Omega-$spaces.

Notation 1. In this section we shall use the notation of [15] with the additional specification of denoting with $\|\cdot\|_n$ the norm in the Banach space L_n. Moreover we denote by X the Alexandrov (one-point) compactification of the locally compact space $\mathbb{N}$ with the discrete topology. Here we recall some definitions given in [15]. $\langle L, \|\cdot\|\rangle$ is a Banach space and $\{\langle L_n, \|\cdot\|_n\rangle\}_{n\in\mathbb{N}}$ is a sequence of Banach spaces, moreover $\{P_n \in B(L, L_n)\}_{n\in\mathbb{N}}$ is a sequence of bounded linear operators such that $\forall f \in L$

$$\lim_{n\to\infty} \|P_n f\|_n = \|f\|. \tag{5.1}$$

Given an element $f \in L$ and a sequence $\{f_n\}_{n\in\mathbb{N}}$ such that $f_n \in L_n$ for all $n \in \mathbb{N}$ we set

$$\lim_{n\to\infty} f_n \overset{K}{=} f \overset{def}{\Leftrightarrow} \lim_{n\to\infty} \|f_n - P_n f\|_n = 0. \tag{5.2}$$

In addition to the requirements of [15] we assume also that

$$(\forall n \in \mathbb{N})(\overline{P_n(L)} = L_n) \tag{5.3}$$

We shall set here $L_\infty \doteqdot L$, $\|\cdot\| \doteqdot \|\cdot\|_\infty$, where $\|\cdot\|$ is the norm on L. Finally for all Z we recall that $B_s(Z)$ is the locally convex space of all linear bounded operators on Z with the strong operator topology.

Lemma 5.1. *Let $f, g \in L$ and $\{f_n\}_{n\in\mathbb{N}}$ such that $f_n \in L_n$ for all $n \in \mathbb{N}$. Then* $(\lim_{n\to\infty} f_n \overset{K}{=} f) \wedge (\lim_{n\to\infty} f_n \overset{K}{=} g) \Rightarrow f = g$

Proof. Let $(\lim_{n\to\infty} f_n \overset{K}{=} f)$ and $(\lim_{n\to\infty} f_n \overset{K}{=} g)$ thus

$$\lim_{n\in\mathbb{N}} \|P_n(f-g)\| \leq \lim_{n\in\mathbb{N}} \|P_n f - f_n\| + \lim_{n\in\mathbb{N}} \|P_n g - f_n\| = 0,$$

so the statement follows by (5.1). □

Definition 15. Set

$$\begin{cases} \mathsf{L} \doteqdot \{\langle L_x, \|\cdot\|_x\rangle\}_{x\in X}, \\ \mathcal{E}(L) \doteqdot \{\sigma^f \mid f \in L\}, \end{cases}$$

where $\sigma^f \in \prod_{x\in X} L_x$ such that $\sigma^f(n) \doteqdot P_n f$ for all $n \in \mathbb{N}$ and $\sigma^f(\infty) \doteqdot f$.

Definition 16. By (5.1) the sequence $\{\|P_n f\|_n\}_{n\in\mathbb{N}}$ is bounded for all $f \in L$ so $\sigma^f \in \prod_{x\in X}^b L_x$. Moreover by (5.1) $\mathcal{E}(L)$ satisfies $FM(4)$, finally by the request (5.3) it satisfies also $FM(3)$. Therefore we can define the *Kurtz bundle* the following bundle

$$\mathfrak{V}(\mathsf{L}, \mathcal{E}(L))$$

generated by the couple $\langle \mathsf{L}, \mathcal{E}(L)\rangle$, see in [20, Definition 15].

Remark 6. By [20, Remark 11] we have that

$$\mathcal{E}(L) \subseteq \Gamma(\pi_{\mathsf{L}}) \text{ modulo the canonical isomorphism.} \tag{5.4}$$

Finally by applying the principle of uniform boundedness, [14, Theorem 1.29, $No3$, Chapter 3], we deduce that the sequence $\{\|P_n\|_{B(L,L_n)}\}_{n\in\mathbb{N}}$ is bounded.

Definition 17. Fix $\mathcal{U}_0 \in \prod_{n\in\mathbb{N}} \mathcal{C}\left(\mathbb{R}^+, B_s(L_n)\right)$ such that $\mathcal{U}_0(x)$ is a $(C_0)-$semigroup of isometries on L_n for all $n \in \mathbb{N}$. Denote by T_n the infinitesimal generator of the semigroup $\mathcal{U}_0(n)$ for any $n \in \mathbb{N}$. Let us take the positions [21, equation (2.18)], where $\langle\langle \mathfrak{E}(\mathsf{E}^{\oplus}), \tau(\mathsf{E}^{\oplus}, \mathcal{E}^{\oplus})\rangle, \pi_{\mathsf{E}^{\oplus}}, X, \mathfrak{n}^{\oplus}\rangle$ is the bundle direct sum of the family $\{\mathfrak{V}(\mathsf{L}, \mathcal{E}(L)), \mathfrak{V}(\mathsf{L}, \mathcal{E}(L))\}$. In addition we maintain the [21, Notation 1] where $\mathfrak{V}$ has to be considered the Kurtz bundle and $x_\infty \doteqdot \infty$, thus $\mathcal{T} \in \prod_{x\in X} Graph(L_x \times L_x)$ so that $\mathcal{T} \upharpoonright X - \{\infty\} \doteqdot \mathcal{T}_0$ and

$$\mathcal{T}(\infty) \doteqdot \{\phi(\infty) \mid \phi \in \Phi\},$$

and $D(T_\infty) \doteqdot \mathrm{Pr}_1^\infty(\mathcal{T}(\infty)) = \{\phi_1(\infty) \mid \phi \in \Phi\}$. Finally $\mathcal{S} \doteqdot \{S_x\}_{x\in X}$ where $(\forall B \in \Theta)(\forall x \in X)$

$$\begin{cases} \mathsf{D}(B, \mathcal{E}) \doteqdot \mathcal{E} \cap \left(\prod_{x\in X} B_x\right) \\ \mathcal{B}_B^x \doteqdot \{v(x) \mid v \in \mathsf{D}(B, \mathcal{E})\}\} \\ S_x \doteqdot \{\mathcal{B}_B^x \mid B \in \Theta\}. \end{cases} \tag{5.5}$$

Proposition 5.1. *Let $\overline{f} \in \prod_{x\in X} L_x$ Thus*

$$\lim_{n\to\infty} \overline{f}(n) \stackrel{K}{=} \overline{f}(\infty) \Leftrightarrow \overline{f} \in \Gamma^\infty(\pi_{\mathsf{L}}).$$

Proof. By (5.4) and implication (3) $\Rightarrow$ (1) of [20, Corollary 3.1] we have that $\lim_{n\to\infty} \overline{f}(n) \stackrel{K}{=} \overline{f}(\infty)$ implies that

$$\overline{f} \text{ is continuous at } \infty,$$

indeed $\sigma^{\overline{f}(\infty)} \in \Gamma(\pi_{\mathsf{L}})$ modulo isomorphism. By the upper semicontinuity of $\|\cdot\| : \mathfrak{E} \to \mathbb{R}^+$, due to the construction of the bundle $\mathfrak{V}(\mathsf{L}, \mathcal{E}(L))$ and to [15, 1.6.(ii)], and by the fact that the composition of any u.s.c. map with any continuous one at a point is an u.s.c. map in the same point, we deduce that $\|\cdot\| \circ \overline{f}$ is u.s.c. at ∞. Thus $\sup_{x\in X} \|\overline{f}(x)\|_x < \infty$, indeed we applied to the u.s.c. map $\|\cdot\| \circ \overline{f}$ the fact that X is compact (so quasi compact), $-\|\cdot\| \circ \overline{f}$ is l.s.c, the [2, Theorem 3, §6.2., Chapter 4] and [2, formula (2), §5.4, Chapter 4]. Therefore

$$\overline{f} \in \prod_{x\in X}^{b} L_x.$$

Then $\overline{f} \in \Gamma^\infty(\pi_{\mathsf{L}})$. The remaining implication follows by [20, Corollary 3.1] and by the fact that $\mathfrak{V}(\mathsf{L}, \mathcal{E}(L))$ is full since X is compact so completely regular and since the Dupre' theorem see for example [12, Corollary 2.10]. $\square$

Proposition 5.2. *We have*

$$\Gamma^\infty(\pi_{\mathbf{L}^{\oplus}}) = \left\{\sigma_1 \oplus \sigma_2 \mid \sigma_i \in \prod_{x\in X} L_x, \lim_{n\to\infty} \sigma_i(n) \stackrel{K}{=} \sigma(\infty), i = 1, 2\right\}.$$

Here, we used the [20, Convention 1] and set $(\sigma_1 \oplus \sigma_2)(x) \doteqdot \sigma_1(x) \oplus \sigma_2(x)$.

Proof. By [20, Convention 1] and [20, Corollary 4.1] $\sigma_1 \oplus \sigma_2$ is continuous at ∞ if and only if σ_i is continuous at ∞ for all $i = 1, 2$. Thus the statement follows by Proposition 5.1. $\square$

Proposition 5.3. *Let $\mathcal{U}_0 \in \prod_{n\in\mathbb{N}} \mathcal{C}\left(\mathbb{R}^+, B_s(L_n)\right)$ be such that $\mathcal{U}_0(x)$ is a $(C_0)-$semigroup of contractions on L_n for all $n \in \mathbb{N}$. Moreover let us denote by T_n the infinitesimal generator of* *the semigroup* $\mathcal{U}_0(n)$ *for any* $n \in \mathbb{N}$. *Thus with the positions [21, equation (2.18)] where* $\mathfrak{V}$ *is the Kurtz bundle we have*

$$\begin{cases} \Phi = \left\{\sigma_1 \oplus \sigma_2 \,|\, (\forall i \in \{1,2\})(\sigma_i \in \prod_{x \in X} L_x)(1-2)\right\} \\ (1) \lim_{n\to\infty} \sigma_i(n) \overset{K}{=} \sigma_i(\infty) \\ (2)(\forall n \in \mathbb{N})(\sigma_1(n), \sigma_2(n)) \in Graph(T_n), \end{cases} \tag{5.6}$$

and

$$\begin{cases} \mathcal{E} = \left\{\sigma^{\sigma_1(\infty)} \,|\, \sigma_1 \in \prod_{x \in X} L_x(1-2-3)\right\} \\ (1) \lim_{n\to\infty} \sigma_1(n) \overset{K}{=} \sigma_1(\infty) \\ (2)(\forall n \in \mathbb{N})(\sigma_1(n) \in Dom(T_n)) \\ (3)(\exists\, f \in L_\infty)(\lim_{n\to\infty} T_n \sigma_1(n) \overset{K}{=} f). \end{cases} \tag{5.7}$$

Moreover, there exists a unique function f *satisfying* (3) *in* (5.7) *and* $(\forall \sigma_1 \in \mathcal{E})((\sigma_1, \sigma_2) \in \Phi)$, *where* $\sigma_2 \in \prod_{x \in X} L_x$ *such that* $(\forall n \in \mathbb{N})(\sigma_2(n) \doteqdot T_n \sigma_1(n))$ *and* $\sigma_2(\infty) \doteqdot f$.

Proof. The first statement follows by Proposition 5.2, while the second one follows by the first one and Lemma 5.1. □

Assumptions 2. We assume $\exists \{I_n \in B(L_n, L)\}_{n \in \mathbb{N}}$ such that

$$\begin{cases} \sup_{n\in\mathbb{N}} \|I_n\|_{B(L_n, L)} < \infty, \\ (\forall f \in L)(\forall n \in \mathbb{N})(I_n \circ P_n = Id). \end{cases} \tag{5.8}$$

Moreover we assume that

$$\overline{\lim_{n\to\infty}} \|P_n\| \leq 1. \tag{5.9}$$

In addition we assume that $(\forall g \in L)(\exists\, \sigma_1 \in \prod_{x \in X} L_x)$ such that

$$\begin{cases} (1) \lim_{n\to\infty} \sigma_1(n) \overset{K}{=} \sigma_1(\infty) \\ (2)(\forall n \in \mathbb{N})(\sigma_1(n) \in Dom(T_n)) \\ (3)(\exists\, f \in L_\infty)(\lim_{n\to\infty} T_n \sigma_1(n) \overset{K}{=} f) \\ (4) g = \sigma_1(\infty). \end{cases} \tag{5.10}$$

Set

$$\mathfrak{U} \doteqdot \left\{F \in \mathcal{C}\left(\mathbb{R}^+, B_s(L)\right) \,|\, (\forall s \in \mathbb{R}^+)(\forall v \in L)(\|F(s)v\| = \|v\|)\right\}. \tag{5.11}$$

In the following definition we shall give the data for constructing a bundle $\mathfrak{W}$ such that $\langle \mathfrak{V}(\mathsf{L}, \mathcal{E}(L)), \mathfrak{W}, X, \mathbb{R}^+ \rangle$ would be a $(\Theta, \mathcal{E})$ −structure.

Definition 18. Set $P_\infty \doteqdot I_\infty \doteqdot Id : L \to L$, moreover $\forall U \in \mathfrak{U}$ set $F_U \in \prod_{x \in X} \mathcal{C}_c\left(\mathbb{R}^+, \mathcal{L}_{S_x}(L_x)\right)$ such that $\forall x \in X$

$$\begin{cases} F_U(x) \doteqdot P_x \circ U(\cdot) \circ I_x, \\ P_x \circ U(\cdot) \circ I_x : \mathbb{R}^+ \ni s \mapsto P_x \circ U(s) \circ I_x \in B(L_x). \end{cases}$$

Now we can define $\forall x \in X$

$$\mathsf{M}_x \doteqdot \operatorname{span}\left\{F_U(x) \,|\, U \in \mathfrak{U}\right\}.$$

M_x has to be considered as Hlcs with the topology induced by that on $\mathcal{C}_c\left(\mathbb{R}^+, \mathcal{L}_{S_x}(L_x)\right)$. [7] Moreover set

$$\mathcal{M} \doteqdot \operatorname{span}\left\{F_U \,|\, U \in \mathfrak{U}\right\}.$$

[7] $\mathcal{C}_c\left(\mathbb{R}^+, \mathcal{L}_{S_x}(L_x)\right)$ is Hausdorff for all $x \in X$ by the fact that $\bigcup_{B \in \Theta} \mathcal{B}_B^x = L_x$, see later Proposition 5.4.

Theorem 5.1. *M_x as Hlcs is well-defined for any $x \in X$, moreover $\mathcal{M} \subset \prod_{x\in X}^{b} \mathsf{M}_x$ and $\mathsf{M}_x = \{F(x) \mid F \in \mathcal{M}\}$. Finally $\mathcal{M}$ satisfies $FM(3) - FM(4)$ with respect to M.*

Proof. By [21, Remark 4] we have that $\mathcal{C}_c(\mathbb{R}^+, B_s(L_x)) \subset \mathcal{C}_c(\mathbb{R}^+, \mathcal{L}_{S_x}(L_x))$ hence for the first sentence of the statement it is sufficient to show that $P_x \circ U(\cdot) \circ I_x \in \mathcal{C}_c(\mathbb{R}^+, B_s(L_x))$ for any $U \in \mathfrak{U}$. For $x = \infty$ is trivial so let $n \in \mathbb{N}$ and $f_n \in L_n$ thus for all $s \in \mathbb{R}^+$ and all net $\{s_\alpha\}_{\alpha\in D}$ in $\mathbb{R}^+$ converging at s we have

$$\lim_{\alpha\in D} \|P_n \circ U(s_\alpha) \circ I_n(f_n) - P_n \circ U(s) \circ I_n(f_n)\|_n = \lim_{\alpha\in D} \|P_n(U(s_\alpha) - U(s))I_n f_n\|_n = 0,$$

where we used the fact that U is strongly continuous and P_n is norm continuous by construction. Thus the first sentence of the statement follows. Let $v \in \mathcal{E}$ and $U \in \mathfrak{U}$ thus $\forall K \in Comp(\mathbb{R}^+)$

$$\begin{aligned}\sup_{n\in\mathbb{N}}\sup_{s\in K} \|P_n U(s) I_n v(n)\|_n &\leq M \sup_{n\in\mathbb{N}}\sup_{s\in K} \|U(s) I_n v(n)\|_\infty \\ &= M \sup_{n\in\mathbb{N}} \|I_n v(n)\|_\infty \\ &\leq M \sup_{n\in\mathbb{N}} \|I_n\| \sup_{n\in\mathbb{N}} \|v(n)\|_\infty < \infty.\end{aligned}$$

Here $M \doteqdot \sup_{n\in\mathbb{N}} \|P_n\|$, in the second one the hypothesis that $U(s)$ is an isometry for all $s \in \mathbb{R}^+$, in the final inequality we considered (5.8), $\mathcal{E} \subset \prod_{x\in X}^{b} L_x$ and that $M < \infty$ by Remark 6. Therefore by [21, Remark 4] $\mathcal{M} \subset \prod_{x\in X}^{b} \mathsf{M}_x$. The equality $\mathsf{M}_x = \{F(x) \mid F \in \mathcal{M}\}$ comes by construction, in particular $\mathcal{M}$ satisfies the $FM(3)$ with respect to the M. $\forall v \in \mathcal{E}$

$$\begin{aligned}\overline{\lim_{n\to\infty}} \sup_{s\in K} \|P_n U(s) I_n v(n)\|_n &\leq \overline{\lim_{n\to\infty}} \left(\|P_n\| \sup_{s\in K} \|U(s) I_n v(n)\|_n \right), && \text{[2, Proposition 11, §5.6, Chapter 4]} \\ &\leq \overline{\lim_{n\to\infty}} \|P_n\| \, \overline{\lim_{n\to\infty}} \sup_{s\in K} \|U(s) I_n v(n)\|_n, && \text{[2, Proposition 13, §5.7, Chapter 4]} \\ &\leq \overline{\lim_{n\to\infty}} \|I_n v(n)\|_\infty, && (5.9), (5.11) \\ &= \overline{\lim_{n\to\infty}} \|I_n P_n f\|_\infty, && v \in \mathcal{E} \subset \Gamma(\pi) \simeq \mathcal{E}(L) \\ &= \|f\|_\infty, && (5.8) \\ &= \|v(\infty)\|_\infty.\end{aligned}$$

Thus by considering that U is a map of isometries we have

$$\overline{\lim_{n\to\infty}} \sup_{s\in K} \|P_n U(s) I_n v(n)\|_n \leq \sup_{s\in K} \|P_\infty U(s) I_\infty v(\infty)\|_\infty.$$

Hence by [2, Proposition 3, §7.1, Chapter 4] and [2, formula (13), §5.6, Chapter 4] we deduce that

$$X \ni x \mapsto \sup_{s\in K} \|P_x U(s) I_x v(x)\|_x \text{ is } u.s.c. \text{ at } \infty,$$

therefore it is $u.s.c.$ on X because of it is continuous in each point in $\mathbb{N}$ due to the fact that the topology induced on $\mathbb{N}$ by that on X is the discrete topology. So $\mathcal{M}$ satisfies the $FM(4)$ with respect to the M. □

Definition 19. Theorem 5.1 allows us to construct a bundle of $\Omega-$space, namely the bundle $\mathfrak{V}(\mathsf{M}, \mathcal{M})$ generated by the couple $\langle \mathsf{M}, \mathcal{M} \rangle$, see [20, Definition 15].

Remark 7. By [20, Remark 11] we have

$$\mathcal{M} \subseteq \Gamma(\pi_{\mathsf{M}}) \text{ modulo the canonical isomorphism.} \tag{5.12}$$

Hence by $\mathsf{M}_x = \{F(x) \mid F \in \mathcal{M}\}$ we have that $\mathfrak{V}(\mathsf{M}, \mathcal{M})$ is full.

Proposition 5.4. *We have that* $\overline{\bigcup_{B\in\Theta}\mathcal{B}_B^x} = L_x$ *for all* $x \in X$. *Moreover,* $\langle\mathfrak{V}(\mathsf{L},\mathcal{E}(L)),\mathfrak{V}(\mathsf{M},\mathcal{M}),X,\mathbb{R}^+\rangle$ *is a* $(\Theta,\mathcal{E})-$*structure.*

Proof. By assumptions (5.10), (5.3), Proposition 5.3 and [21, Remark 4] we obtain that $\overline{\bigcup_{B\in\Theta}\mathcal{B}_B^x} = L_x$ for all $x \in X$. The remaining part of the statement follows by the construction of $\mathcal{M}$ and M. □

Corollary 5.1. *If* $D(T_{x_\infty})$ *is dense in* $\mathfrak{E}_{x_\infty}$, *and* $\exists\lambda_0 > 0, \lambda_1 < 0$ *such that the ranges* $\mathcal{R}(\lambda_0 - T_{x_\infty})$ *and* $\mathcal{R}(\lambda_1 - T_{x_\infty})$ *are dense in* $\mathfrak{E}_{x_\infty}$*), then* $\langle\mathcal{T},\infty,\Phi\rangle \in \mathsf{Gr}(\mathfrak{V}(\mathsf{L},\mathcal{E}(L)),\mathfrak{V}(\mathsf{L},\mathcal{E}(L)))$ *and the following*

$$T_\infty : D(T_\infty) \ni \phi_1(\infty) \mapsto \phi_2(\infty)$$

is a well-defined operator which is the generator of a C_0-*semigroup of isometries on* $\mathfrak{E}_\infty$.

Proof. By Propositions 5.4 and 5.5 we have that the first part of hypotheses of [21, Theorem 2.1] is satisfied so the statement follows by the first part of the statement of [21, Theorem 2.1]. □

Definition 20. Let us denote by $\mathcal{U}_\infty$ the C_0-semigroup of isometries on L_∞. Moreover set $\mathcal{U} \in \prod_{x\in X}\mathsf{U}_{is}(B_s(L_x))$ if $\mathcal{U}\upharpoonright\mathbb{N} = \mathcal{U}_0$ and $\mathcal{U}(\infty) = \mathcal{U}_\infty$.

Theorem 5.2. $(\exists\, F \in \Gamma(\pi_{\mathsf{M}}))(F(\infty) = \mathcal{U}(\infty))$ *such that* $(\forall v \in \mathcal{E})(\exists\,\phi \in \Phi)$ *s.t.* $\phi_1(x_\infty) = v(x_\infty)$ *and* $(\forall\{z_n\}_{n\in\mathbb{N}} \subset X \mid \lim_{n\in\mathbb{N}} z_n = x_\infty)$ *we have that* $\{\mathcal{U}(z_n)(\cdot)\phi_1(z_n) - F(z_n)(\cdot)v(z_n)\}_{n\in\mathbb{N}}$ *is a bounded equicontinuous sequence. Moreover we can choose* F *such that* $F = F_{\mathcal{U}_\infty}$.

Proof. By Proposition 5.3 and (5.12) the statement is equivalent to showing that $\forall\sigma_1 \in \prod_{x\in X} L_x$ satisfying $(1-2-3)$ of (5.7) and $(\forall\{z_n\}_{n\in\mathbb{N}} \subset X \mid \lim_{n\in\mathbb{N}} z_n = \infty)$ we have that

$$\{\mathcal{U}(z_n)(\cdot)\sigma_1(z_n) - F_{\mathcal{U}_\infty}(z_n)(\cdot)\sigma^{\sigma_1(\infty)}(z_n)\}_{n\in\mathbb{N}} \tag{5.13}$$

is a bounded equicontinuous sequence. Moreover by the second assumption (5.8) and (5.13)

$$\{\mathcal{U}(z_n)(\cdot)\sigma_1(z_n) - P_{z_n}\mathcal{U}_\infty(z_n)(\cdot)\sigma_1(\infty)\}_{n\in\mathbb{N}} \tag{5.14}$$

is a bounded equicontinuous sequence. Set $\sigma_2 \in \prod_{x\in X} L_x$ such that $\sigma_2(x) \doteqdot T_x\sigma_1(x)$, for all $x \in X$, thus

$$\sigma_i \in \Gamma^\infty(\pi_{\mathsf{L}}),$$

for all $i = 1, 2$, indeed for $i = 1$ follows by (1) of (5.7) and Proposition 5.1, while for $i = 2$ follows by construction of T_∞, the second part of Proposition 5.3, the fact that by construction $\Phi \subseteq \Gamma(\pi_{\mathsf{E}^\oplus})$, see [21, equation (2.18)], and finally by [20, Corollary 4.1]. Therefore in particular σ_i is continuous at ∞. Thus by considering that $\sigma^{\sigma_i(\infty)} \in \Gamma(\pi_{\mathsf{L}})$ modulo isomorphism by (5.4) and that $\mathfrak{V}(\mathsf{L},\mathcal{E}(L))$ is full we deduce by [20, Proposition 3.1]

$$\lim_{n\in\mathbb{N}}\|\sigma_i(z_n) - \sigma^{\sigma_i(\infty)}(\pi\circ\sigma_i(z_n))\|_{\pi\circ\sigma_i(z_n)} = 0.$$

Then by considering that $\pi\circ\sigma_i = Id$ because of σ_i is a section, we have

$$\lim_{n\in\mathbb{N}}\|\sigma_i(z_n) - P_{z_n}\sigma_i(\infty)\|_{z_n} = 0. \tag{5.15}$$

The statement now follows by (5.15), (5.14) and by using the same argumentation used in proof of [15, Theorem 1.2] for proving a similar result. □

Proposition 5.5. *With the notation of [21, Definition 1] we have that*

$$\mathsf{M}_x \subset \bigcap_{\lambda>0}\mathfrak{L}_1(\mathbb{R}^+,\mathcal{L}_{S_x}(L_x);\mu_\lambda),$$

and [21, equation (2.14)] holds.

Proof. Follows by [21, Proposition 4.2]. □

Theorem 5.3. *$\langle \mathfrak{V}(\mathsf{L}, \mathcal{E}(L)), \mathfrak{V}(\mathsf{M}, \mathcal{M}), X, \mathbb{R}^+ \rangle$ has the full Laplace duality property, moreover $\forall U \in \mathsf{U}_{1,\|\cdot\|}(B_s(L))$, $\forall \lambda > 0$ and $\forall f \in L$ we have that*

$$\mathfrak{L}(F_U)(\cdot)(\lambda)\sigma^f(\cdot) = \sigma^{(\lambda - T_U)^{-1} f}.$$

Here T_U is the infinitesimal generator of the semigroup U.

Proof. Let $f \in L$ and $U \in \mathfrak{U}$ thus for all $x \in X$ and $\lambda > 0$ we have

$$\begin{aligned} \int_0^\infty e^{-\lambda s} P_x U(s) I_x \sigma^f(x)\, ds &= \int_0^\infty e^{-\lambda s} P_x U(s) f\, ds \\ &= P_x \int_0^\infty e^{-\lambda s} U(s) f\, ds, \end{aligned} \tag{5.16}$$

where the first equality follows by the second assumption 5.8, while the second one by the linearity and continuity of P_x and by [4, Proposition 1, n°, §1, Chapter 6]. Thus the first sentence of the statement by (5.4) and (5.12). The second sentence of the statement folllows by the (5.16) and by Hille-Yosida Theorem, see [15, Theorem 1.2.]. □

Corollary 5.2. *Let us assume the hypotheses of Corollary 5.1. Then $(\forall g \in L)(\forall K \in Comp(\mathbb{R}^+))$*

$$\lim_{z \to \infty} \sup_{s \in K} \|(\mathcal{U}(z)(s) \circ P_z - P_z \circ \mathcal{U}_\infty(s))\, g\| = 0. \tag{5.17}$$

Moreover

$$\mathcal{U} \in \Gamma^\infty(\rho). \tag{5.18}$$

In particular

$$\{\langle \mathcal{T}, \infty, \Phi \rangle\} \in \Delta_\Theta \left\langle \mathfrak{V}(\mathsf{L}, \mathcal{E}), \mathfrak{V}(\mathsf{M}, \mathcal{M}), \mathcal{E}, X, \mathbb{R}^+ \right\rangle. \tag{5.19}$$

Proof. By Proposition 5.5 follows [21, equation (2.14)], hypothesis (i) of [21, Theorem 2.1] follows by Theorem 5.3, (ii) by Theorem (5.2), finally (iii) follows by [2, Corollary of Proposition 16, §2.9, Chapter 9] and by the fact that $\{\{n\} \mid n \in \mathbb{N}\}$ is a base for the topology on $\mathbb{N}$. Thus by [21, Theorem 2.1] we obtain (5.18), (5.19) and $(\forall v \in \mathcal{E})(\forall K \in Comp(\mathbb{R}^+))$

$$\lim_{z \to \infty} \sup_{s \in K} \|\mathcal{U}(z)(s) v(z) - F(z)(s) v(z)\| = 0, \tag{5.20}$$

where F is any map in Theorem 5.2. Now by Theorem 5.2 we can take in the previous equation $F = F_{\mathcal{U}_\infty}$, moreover by (5.7) and assumption (5.10) we have

$$\mathcal{E} = \{\sigma^g \mid g \in L\},$$

therefore by (5.8) $\forall s \in \mathbb{R}^+$, $\forall z \in X$ and $\forall g \in L$

$$F_{\mathcal{U}_\infty}(z)\sigma^g(z) = (P_z \circ \mathcal{U}_\infty(s))g.$$

Hence (5.17) follows by (5.20). □

References

[1] N. Bourbaki, *Algebre 1*, Diffusion C.C.L.S, 1970.

[2] N. Bourbaki, *General topology Part 1, Part 2,* Springer-Verlag, 1989.

[3] N. Bourbaki, *Topological vector spaces,* Springer-Verlag, 1989.

[4] N. Bourbaki, *Integration I, II,* Springer-Verlag, 2003.

[5] O. Bratteli, D.W. Robinson, *Operator algebras and quantum statistical mechanics I. Csp∗- and Wsp∗-algebras, symmetry groups, decomposition of states. 2° ed,* Springer-Verlag, New York Heidelberg Berlin, 1987.

[6] V.I. Burenkov, P.D. Lamberti, *Spectral stability of Dirichlet second order uniformly elliptic operators,* J. Differential Equations 244 (2008), no. 7, 1712–1740.

[7] V.I. Burenkov, P.D. Lamberti, *Spectral stability of general non-negative self-adjoint operators with applications to Neumann-type operators,* J. Differential Equations 233 (2007), no. 2, 345–379.

[8] V.I. Burenkov, P.D. Lamberti, M. Lanza de Cristoforis, *A real analyticity result for symmetric functions of the eigenvalues of a domain-dependent Neumann problem for the Laplace operator,* Mediterr. J. Math. 4 (2007), no. 4, 435–449.

[9] W. Chojnacki, *Multiplier algebras, Banach bundles, and one-parameter semigroups,* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 28 (1999), no. 2, 287–322.

[10] N. Dunford, J.T. Schwartz, *Linear operators Vol 1,2,3,* Wiley Interscience, 1988.

[11] J.M. Fell, R.S. Doran, *Representations of* sp∗*-algebras, locally compact groups, and Banach* sp∗*-algebraic bundles. Vol. 1-2,* Pure and Applied Mathematics, 126. Academic Press, Inc., Boston, MA, 1988.

[12] G. Gierz, *Bundles of topological vector spaces and their duality,* Lecture Notes in Mathematics, 955. Springer-Verlag, 1982.

[13] H. Jarchow, *Locally convex spaces,* B.G. Teubner, 1981.

[14] T. Kato, *Perturbation theory for linear operators,* Springer-Verlag, 1980.

[15] T.G. Kurtz, *Extensions of Trotter's operator semigroup approximation theorems,* J. Functional Analysis 3 1969 354–375.

[16] P.D. Lamberti, M. Lanza de Cristoforis, *A real analyticity result for symmetric functions of the eigenvalues of a domain-dependent Neumann problem for the Laplace operator,* Mediterr. J. Math. 4 (2007), no. 4, 435–449.

[17] M. Lanza de Cristoforis, *Singular perturbation problems in potential theory and applications,* Complex analysis and potential theory, 131–139, World Sci. Publ., Hackensack, NJ, 2007.

[18] M. Lanza de Cristoforis, *Asymptotic behavior of the solutions of the Dirichlet problem for the Laplace operator in a domain with a small hole. A functional analytic approach.* Analysis (Munich) 28 (2008), no. 1, 63–93.

[19] B. Silvestri, *Integral equalities for functions of unbounded spectral operators in Banach spaces*, Dissertationes Math. 464 (2009), 60 pp. preprint `http://arxiv.org/abs/0804.3069v2`

[20] B. Silvestri, *Use of bundles of locally convex spaces in problems of convergence of semigroups of operators. I,* Eurasian Math. J. 7 (2015), no. 3, 53-88.

[21] B. Silvestri, *Use of bundles of locally convex spaces in problems of convergence of semigroups of operators. II,* Eurasian Math. J. 7 (2015), no. 4, 46-78.

[22] M. Takesaki, *Theory of operator algebras II,* Springer-Verlag, 2003.

[23] K. Yosida, *Functional analysis,* Springer-Verlag, 1980.

Benedetto Silvestri
Dipartimento di Matematica Pura ed Applicata
Università degli Studi di Padova
Via Trieste, 63
35121 Padova, Italy
E-mails: 6260rstlvs@gmail.com

Received: 24.03.2016