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## ERLAN DAUTBEKOVICH NURSULTANOV

(to the 60th birthday)

On May 25, 2017 was the 60th birthday of Yerlan Dautbekovich Nursultanov, Doctor of Physical and Mathematical Sciences (1999), Professor (2001), Head of the Department of Mathematics and Informatics of the Kazakhstan branch of the M.V. Lomonosov Moscow State University (since 2001), member of the Editorial Board of the Eurasian Mathematical Journal.

E.D. Nursultanov was born in the city of Karaganda. He graduated from the Karaganda State University (1979) and then completed his postgraduate studies at the M.V. Lomonosov Moscow State University.

Professor Nursultanov's scientific interests are related to various areas of the theory of functions and functional analysis.

He introduced the concept of multi-parameter Lorentz spaces, network spaces and anisotropic Lorentz spaces, for which appropriate interpolation methods were developed. On the basis of the apparatus introduced by him, the questions of reiteration in the off-diagonal case for the real Lyons-Petre interpolation method, the multiplier problem for trigonometric Fourier series, the lower and upper bounds complementary to the Hardy-Littlewood inequalities for various orthonormal systems were solved. The convergence of series and Fourier transforms were studied with sufficiently general monotonicity conditions. The lower bounds for the norm of the convolution operator are obtained, and its upper bounds are improved (a stronger result than the O'Neil inequality). An exact cubature formula with explicit nodes and weights for functions belonging to spaces with a dominated mixed derivative is constructed, and a number of other problems in this area are solved.

He has published more than 50 scientific papers in high rating international journals included in the lists of Thomson Reuters and Scopus. 2 doctor of sciences, 9 candidate of sciences and 4 PhD dissertations have been defended under his supervision.

His merits and achievements are marked with badges of the Ministry of Education and Science of the Republic of Kazakhstan "For Contribution to the Development of Science" (2007), "Honored Worker of Education" (2011), "Y. Altynsarin" (2017). He is a laureate of the award named after K. Satpaev in the field of natural sciences for 2005, the grant holder "The best teacher of the university" for 2006 and 2011, the grant holder of the state scientific scholarship for outstanding contribution to the development of science and technology of the Republic of Kazakhstan for years 2007-2008, 2008 -2009. In 2017 he got the Top Springer Author award, established by Springer Nature together with JSC "National Center for Scientific and Technical Information".

The Editorial Board of the Eurasian Mathematical Journal congratulates Erlan Dautbekovich Nursultanov on the occasion of his 60th birthday and wishes him good health and successful work in mathematics and mathematical education.

**JAMALBEK TUSSUPOV**

(to the 60th birthday)

On April 10, 2017 was the 60th birthday of Jamalbek Tussupov, Doctor of Physical and Mathematical Sciences, Professor, Head of the Information Systems Department of the L.N. Gumilyov Eurasian National University, member of the Kazakhstan and American Mathematical Societies, member of the Association of Symbolic Logic, member of the Editorial Board of the Eurasian Mathematical Journal.

J. Tussupov was born in Taraz (Jambyl region of the Kazakh SSR). He graduated from the Karaganda State University (Kazakhstan) in 1979 and later on completed his postgraduate studies at S.L. Sobolev Institute of Mathematics of the Academy of Sciences of Russia (Novosibirsk).

Professor Tussupov's research interests are in mathematical logic, computability, computable structures, abstract data types, ontology, formal semantics. He solved the following problems of computable structures:

- the problems of S.S. Goncharov and M.S. Manasse: the problem of characterizing relative categoricity in the hyperarithmetical hierarchy given levels of complexity of Scott families, and the problem on the relationship between categoricity and relative categoricity of computable structures in the arithmetical and hyperarithmetical hierarchies;

- the problem of Yu.L. Ershov: the problem of finite algorithmic dimension in the arithmetical and hyperarithmetical hierarchies;

- the problem of C.J. Ash and A. Nerode: the problem of the interplay of relations of bounded arithmetical and hyperarithmetical complexity in computable presentations and the definability of relations by formulas of given complexity;

- the problem of S. Lempp: the problem of structures having presentations in just the degrees of all sets X such that for algebraic classes as symmetric irreflexive graphs, nilpotent groups, rings, integral domains, commutative semigroups, lattices, structure with two equivalences, bipartite graphs.

Professor Tussupov has published about 100 scientific papers, five textbooks for students and one monograph. Three PhD dissertations have been defended under his supervision.

Professor Tussupov is a fellow of "Bolashak" Scholarship, 2011 (Notre Dame University, USA), "Erasmus+", 2016 (Poitiers University, France). He was awarded the title "The Best Professor of 2012" (Kazakhstan). In 2015 Jamalbek Tussupov was also awarded for the contribution to science in the Republic of Kazakhstan.

The Editorial Board of the Eurasian Mathematical Journal congratulates Dr. Professor Jamalbek Tussupov on the occasion of his 60th aniversary and wishes him strong health, new achievements in science, inspiration for new ideas and fruitfull results.

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# ON THE NUMBER OF NON-REAL EIGENVALUES OF THE STURM-LIOUVILLE PROBLEM

**A.Sh. Shukurov**

Communicated by P.D. Lamberti

**Key words:** Sturm-Liouville problem, spectral parameter in boundary condition, non-real eigenvalues, Pontryagin space, J-metric.

**AMS Mathematics Subject Classification:** 34B05, 34B08, 34B09, 34B24 , 47A75, 35P05.

**Abstract.** In this paper we consider a spectral problem for the Sturm-Liouville equation with a spectral parameter in a boundary conditions. It is shown that under certain assumptions on the coefficients of boundary conditions, problems of this type cannot have more than two non-real eigenvalues. Note that, in some special cases of boundary conditions, this kind of results have usually been obtained by using the results of the theory of Pontryagin spaces. The aim of this paper is to prove this result in a more general setting. Since the result was fairly predictable and could also be proved by using Pontryagin space methods, the author does not claim the absolute novelty of the obtained result but aims to provide an elementary proof, using only some facts of mathematical analysis and theory of ordinary differential equations, which, probably, will make the proof more accessible to a wide audience, especially to students.

## 1 Introduction

In this paper we consider the following spectral problem for the Sturm-Liouville differential equation

$$-y''(x) + q(x) \cdot y(x) = \lambda \cdot y(x), 0 < x < 1 \tag{1.1}$$

with the spectral parameter dependent boundary conditions

$$(a_0\lambda + b_0)y(0) = (c_0\lambda + d_0)y'(0), \tag{1.2}$$

$$(a_1\lambda + b_1)y(1) = (c_1\lambda + d_1)y'(1), \tag{1.3}$$

where $q$ is a real-valued continuous function on the segment $[0, 1]$; $a_i, b_i,\ c_i, d_i (i = 0, 1)$ are real numbers and

$$a_0d_0 - b_0c_0 > 0,\ a_1d_1 - b_1c_1 > 0.$$

Boundary value problems for Sturm-Liouville equation with a spectral parameter in the boundary conditions have been studied extensively (see, for example, [1]-[7], [9]-[25], [28]-[30] and the bibliography therein). Problems of such type arise upon separation of variables in one dimensional wave and heat equation and for a various type of physical problems. Most of the known papers in this direction consider the case $a_0d_0 - b_0c_0 \leq 0, a_1d_1 - b_1c_1 \geq 0$. But in this case (as can also be seen from Lemma 2.2 below) all eigenvalues are real numbers.

It is known that if $a_0d_0 - b_0c_0 > 0, a_1d_1 - b_1c_1 > 0$ then not all eigenvalues of the considered problem are necessarily real. Some special cases of boundary conditions of this kind have been investigated in [2]-[7], [9]-[11], [14]. Note that in all known papers, information on the number of non-real eigenvalues of the considered problem are obtained by associating it with self-adjoint operators acting in Pontryagin spaces (spaces with indefinite metric) and applying the results of the theory of Pontryagin spaces (from [8, 26, 27]).

In this paper the boundary conditions of this kind are treated in a more general setting. It is proved in this paper that under the mentioned assumptions on the coefficients of boundary conditions, the number of non-real eigenvalues of the considered problem cannot be more than two. Since the result was fairly predictable and could also be proved by using Pontryagin space methods, the author does not claim the absolute novelty of the obtained result but aims to provide an elementary proof, using only some facts from mathematical analysis and theory of ordinary differential equations, which, probably, will make the proof more accessible to a wide audience, especially to students.

## 2 Auxiliary facts

Let us consider the following auxiliary problem

$$-w''(x) + q(x) \cdot w(x) = \lambda \cdot w(x), 0 < x < 1, \tag{2.1}$$

$$w(0) = c_0\lambda + d_0, w'(0) = a_0\lambda + b_0, \tag{2.2}$$

$$(a_1\lambda + b_1)w(1) = (c_1\lambda + d_1)w'(1). \tag{2.3}$$

**Lemma 2.1.** *The sets of complex (non-real) eigenvalues of problems (1.1)-(1.3) and (2.1)-(2.3) is the same.*

*Proof.* Let $\lambda$ and $w$ be an eigenvalue of the problem (2.1)-(2.3) and a corresponding eigenfunction, respectively. Then it is obvious that $\lambda$ is also an eigenvalue of the problem (1.1)-(1.3) and $w$ is an eigenfunction of this problem corresponding to an eigenvalue $\lambda$.

Now, let $\lambda$ be an eigenvalue of the problem (1.1)-(1.3) and $y$ a corresponding eigenfunction. There are only two possibilities: $y(0) = 0$ and $y(0) \neq 0$.

We are going to show that the case $y(0) = 0$ is impossible. Assume the contrary: $y(0) = 0$. Then from (1.2) we find that $y'(0) = 0$ (since $\lambda$ is a nonreal number, $c_0\lambda + d_0 \neq 0$ ). But, since $y$ is an eigenfunction, according to the definition of an eigenfunction $y \not\equiv 0$. This and the previous sentence imply that there is a nontrivial solution of the differential equation which satisfies the initial conditions $y(0) = 0$ and $y'(0) = 0$. Contradiction: the solution of the Cauchy problem is unique (it is evident that the unique solution of the considered Cauchy problem is a function that is identically zero).

Consider the case $y(0) \neq 0$. It is evident that the function

$$w(x) = \frac{c_0\lambda + d_0}{y(0)} \cdot y(x)$$

is an eigenfunction of the problem (2.1)-(2.3) corresponding to an eigenvalue $\lambda$: $\lambda$ is an eigenvalue of the problem (2.1)-(2.3).

The lemma is proved. □

**Lemma 2.2.** *The following equality*

$$\int_0^1 |w(x,\lambda)|^2 dx = \frac{b_1c_1 - a_1d_1}{|c_1\lambda + d_1|^2} \cdot |w(1,\lambda)|^2 + a_0d_0 - b_0c_0$$

*holds true for all* $\lambda$ *with* $\mathrm{Im}\lambda \neq 0$.

*Proof.* Let $w(x,\lambda)$ be an eigenfunction, corresponding to an eigenvalue $\lambda$. From (2.1) we obtain the following equalities:

$$-w''(x,\lambda)\overline{w(x,\lambda)} + q(x)\cdot w(x,\lambda)\overline{w(x,\lambda)} = \lambda\cdot w(x,\lambda)\overline{w(x,\lambda)};$$

$$-\int_0^1 w''(x,\lambda)\overline{w(x,\lambda)}dx + \int_0^1 q(x)\cdot |w(x,\lambda)|^2dx = \lambda\cdot\int_0^1 |w(x,\lambda)|^2dx;$$

$$-\int_0^1 \overline{w(x,\lambda)}dw'(x,\lambda) + \int_0^1 q(x)\cdot |w(x,\lambda)|^2dx = \lambda\cdot\int_0^1 |w(x,\lambda)|^2dx;$$

$$-\,w'(x,\lambda)\overline{w(x,\lambda)}|_0^1 + \int_0^1 |w'(x,\lambda)|^2dx + \int_0^1 q(x)\cdot|w(x,\lambda)|^2dx =$$
$$= \lambda\cdot\int_0^1 |w(x,\lambda)|^2dx;$$

$$-w'(1,\lambda)\overline{w(1,\lambda)} + w'(0,\lambda)\overline{w(0,\lambda)} + R_0 = \lambda\cdot\int_0^1 |w(x,\lambda)|^2dx,$$

where $R_0 = \int_0^1 |w'(x,\lambda)|^2dx + \int_0^1 q(x)\cdot|w(x,\lambda)|^2dx$;

$$-\frac{a_1\lambda+b_1}{c_1\lambda+d_1}\cdot|w(1,\lambda)|^2 + (a_0\lambda+b_0)(c_0\bar\lambda+d_0) + R_0 = \lambda\cdot\int_0^1 |w(x,\lambda)|^2dx;$$

$$-\,\frac{(a_1\lambda+b_1)(c_1\bar\lambda+d_1)}{|c_1\lambda+d_1|^2}\cdot|w(1,\lambda)|^2 + (a_0c_0|\lambda|^2 + a_0d_0\lambda+$$
$$+\,b_0c_0\bar\lambda + b_0d_0) + R_0 = \lambda\cdot\int_0^1 |w(x,\lambda)|^2dx;$$

$$-\frac{(a_1c_1|\lambda|^2+b_1d_1) + (a_1d_1\lambda+b_1c_1\bar\lambda)}{|c_1\lambda+d_1|^2}\cdot|w(1,\lambda)|^2 + \big(a_0c_0|\lambda|^2 + a_0d_0\lambda+$$
$$+b_0c_0\bar\lambda + b_0d_0) + R_0 = \lambda\cdot\int_0^1 |w(x,\lambda)|^2dx;$$

$$-\frac{a_1d_1\lambda+b_1c_1\bar\lambda}{|c_1\lambda+d_1|^2}\cdot|w(1,\lambda)|^2 + a_0d_0\lambda+$$
$$+b_0c_0\bar\lambda + R_1 = \lambda\cdot\int_0^1 |w(x,\lambda)|^2dx, \tag{2.4}$$

where $R_1 = R_0 - \frac{a_1c_1|\lambda|^2+b_1d_1}{|c_1\lambda+d_1|^2}\cdot|w(1,\lambda)|^2 + a_0c_0|\lambda|^2 + b_0d_0$ .

Let $\lambda = \lambda_1 + i\lambda_2$, where $\lambda_1, \lambda_2 \in R$ and $\lambda_2 \neq 0$. Then $\bar\lambda = \lambda_1 - i\lambda_2$ and from (2.4) we obtain that

$$-\frac{i\lambda_2 \cdot a_1 d_1 - i\lambda_2 \cdot b_1 c_1}{|c_1\lambda + d_1|^2} \cdot |w(1,\lambda)|^2 + i\lambda_2 \cdot a_0 d_0 - i\lambda_2 \cdot b_0 c_0 + R_2 =$$
$$= \lambda_1 \cdot \int_0^1 |w(x,\lambda)|^2 dx + i\lambda_2 \cdot \int_0^1 |w(x,\lambda)|^2 dx,$$

where $R_2 = R_1 + \lambda_1 \left( a_0 d_0 + b_0 c_0 - \frac{a_1 d_1 + b_1 c_1}{|c_1\lambda + d_1|^2} \cdot |w(1,\lambda)|^2 \right)$ is a real number. Comparing real and imaginary parts of both - left and right hand sides, we arrive at the validity of the statement.

The lemma is proved. □

**Lemma 2.3.** *The following equality holds:*

$$(w'(x,\lambda)w(x,\mu) - w(x,\lambda)w'(x,\mu))\,|_0^1 = (\mu - \lambda) \cdot \int_0^1 w(x,\lambda)w(x,\mu)dx.$$

*Proof.* Taking into account expressions of $w''(x,\lambda)$ and $w''(x,\mu)$ from (2.1) in the following equality

$$(w'(x,\lambda)w(x,\mu) - w(x,\lambda)w'(x,\mu))' = w''(x,\lambda)w(x,\mu) - w(x,\lambda)w''(x,\mu),$$

we find that

$$(w'(x,\lambda)w(x,\mu) - w(x,\lambda)w'(x,\mu))' = (q(x) - \lambda)w(x,\lambda)w(x,\mu) -$$
$$- (q(x) - \mu)w(x,\lambda)w(x,\mu) = (\mu - \lambda)w(x,\lambda)w(x,\mu).$$

Integrating both sides of this equality from 0 to 1 we obtain that

$$(w'(x,\lambda)w(x,\mu) - w(x,\lambda)w'(x,\mu))\,|_0^1 = (\mu - \lambda) \int_0^1 w(x,\lambda)w(x,\mu)dx.$$

The lemma is proved. □

**Lemma 2.4.** *If $\mu \neq \lambda$ and $\mu \neq \overline{\lambda}$, then the following equalities*

$$\int_0^1 w(x,\lambda)w(x,\mu)dx = \frac{b_1 c_1 - a_1 d_1}{(c_1\lambda + d_1)(c_1\mu + d_1)} w(1,\lambda)w(1,\mu) + a_0 d_0 - b_0 c_0 \tag{2.5}$$

*and*

$$\int_0^1 \overline{w(x,\lambda)}w(x,\mu)dx = \frac{b_1 c_1 - a_1 d_1}{(c_1\overline{\lambda} + d_1)(c_1\mu + d_1)} \overline{w(1,\lambda)}w(1,\mu) + a_0 d_0 - b_0 c_0 \tag{2.6}$$

*hold.*

*Proof.* Using

$$\begin{aligned} w'(1,\lambda)w(1,\mu) - w(1,\lambda)w'(1,\mu) - w'(0,\lambda)w(0,\mu) + w(0,\lambda)w'(0,\mu) \\ = \frac{a_1\lambda + b_1}{c_1\lambda + d_1} w(1,\lambda)w(1,\mu) - \frac{a_1\mu + b_1}{c_1\mu + d_1} w(1,\lambda)w(1,\mu) \\ - (a_0\lambda + b_0)(c_0\mu + d_0) + (a_0\mu + b_0)(c_0\lambda + d_0) \\ = (\mu - \lambda)\{\frac{b_1 c_1 - a_1 d_1}{(c_1\lambda + d_1)(c_1\mu + d_1)} w(1,\lambda)w(1,\mu) + a_0 d_0 - b_0 c_0\} \end{aligned}$$

and taking into account Lemma 2.3 , we immediately obtain the validity of the equality (2.5). The equality (2.6) is proved in a similar way.

The lemma is proved □

For simplicity, let us denote

$$w(x,\xi) = u_\xi(x) + i\nu_\xi(x),$$

$$\frac{w(1,\xi)}{c_1\xi + d_1} = A_\xi + iB_\xi,$$

where $u_\xi(x), \nu_\xi(x)$ are real valued functions and $A_\xi, B_\xi$ are real numbers. Using this notation, equalities in Lemma 2.2 (written for $\lambda$ and $\mu$ separately) and Lemma 2.4 can be written in the following form:

$$\int_0^1 \left(u_\lambda^2(x) + \nu_\lambda^2(x)\right) dx = (b_1c_1 - a_1d_1)(A_\lambda^2 + B_\lambda^2) + a_0d_0 - b_0c_0, \tag{2.7}$$

$$\int_0^1 \left(u_\mu^2(x) + \nu_\mu^2(x)\right) dx = (b_1c_1 - a_1d_1)\left(A_\mu^2 + B_\mu^2\right) + a_0d_0 - b_0c_0, \tag{2.8}$$

$$\int_0^1 (u_\lambda(x) + i\nu_\lambda(x))\,(u_\mu(x) + i\nu_\mu(x))dx = (b_1c_1 - a_1d_1)\times$$

$$\times(A_\lambda + iB_\lambda)(A_\mu + iB_\mu) + a_0d_0 - b_0c_0, \tag{2.9}$$

$$\int_0^1 (u_\lambda(x) - i\nu_\lambda(x))(u_\mu(x) + i\nu_\mu(x))dx = (b_1c_1 - a_1d_1)\times$$

$$\times(A_\lambda - iB_\lambda)(A_\mu + iB_\mu) + a_0d_0 - b_0c_0. \tag{2.10}$$

**Lemma 2.5.** *Assume that equalities (2.7)-(2.10) hold for some real numbers $a_i, b_i, c_i, d_i (i = 0, 1)$, $A_\xi, B_\xi (\xi = \lambda, \mu)$ and continuous functions $u_\xi(x), \nu_\xi(x) (\xi = \lambda, \mu)$. Then $\nu_\lambda(x) \equiv \nu_\mu(x) \equiv 0$.*

*Proof.* From (2.9) and (2.10) we obtain

$$2\int_0^1 u_\lambda(x)\cdot(u_\mu(x) + i\nu_\mu(x))dx = 2A_\lambda(b_1c_1 - a_1d_1)(A_\mu + iB_\mu) + 2(a_0d_0 - b_0c_0). \tag{2.11}$$

Now (2.7),(2.8) and (2.11) imply that

$$\int_0^1 \{u_\lambda^2(x) + \nu_\lambda^2(x)\}dx + \int_0^1 \{u_\mu^2(x) + \nu_\mu^2(x)\}dx - 2\int_0^1 u_\lambda(x)\cdot(u_\mu(x) + i\nu_\mu(x))dx =$$

$$= (b_1c_1 - a_1d_1)\cdot(A_\lambda^2 + B_\lambda^2) + (b_1c_1 - a_1d_1)\cdot(A_\mu^2 + B_\mu^2) + 2A_\lambda\cdot(a_1d_1 - b_1c_1)\cdot(A_\mu + iB_\mu).$$

We find from this equality (comparing real and imaginary parts of left and right hand sides) that

$$\int_0^1 \{u_\lambda(x) - u_\mu(x)\}^2 dx + \int_0^1 \nu_\lambda^2(x)dx + \int_0^1 \nu_\mu^2(x)dx =$$
$$= -(a_1d_1 - b_1c_1)\{(A_\lambda - A_\mu)^2 + B_\lambda^2 + B_\mu^2\}.$$

Since $a_1d_1 - b_1c_1 > 0$, we obtain that

$$\nu_\lambda(x) \equiv \operatorname{Im}\{w(x,\lambda)\} \equiv 0$$

and

$$\nu_\mu(x) \equiv \operatorname{Im}\{w(x,\mu)\} \equiv 0.$$

The lemma is proved. □

**Remark 1.** It is obvious that if $\lambda$ is an eigenvalue of the problem (1.1)-(1.3) and $y(x)$ is a corresponding eigenfunction, then $\bar{\lambda}$ is also an eigenvalue of the problem (1.1)-(1.3) and $\overline{y(x)}$ is an eigenfunction corresponding to $\bar{\lambda}$.

## 3 Main result

The main result of this note is the following theorem.

**Theorem 3.1.** *Nonreal eigenvalues of the problem (1.1)-(1.3) are not more than two.*

*Proof.* Note that to prove the theorem it sufficies to show that complex numbers $\lambda$ and $\mu$ can not simultaneously be eigenvalues of the problem (1.1)-(1.3) if $\mu \neq \lambda$ and $\mu \neq \bar{\lambda}$.

Assume the contrary: $\mu \neq \lambda$, $\mu \neq \bar{\lambda}$ and $\lambda, \mu$ are eigenvalues of the problem (1.1)-(1.3). Then by Lemma 2.1 , $\lambda$ and $\mu$ are also eigenvalues of the problem (2.1)-(2.3).

Since $w(x,\lambda)$ and $w(x,\mu)$ are eigenfunctions corresponding to nonreal eigenvalues $\lambda$ and $\mu$,

$$\nu_\lambda(x) \equiv \operatorname{Im}\{w(x,\lambda)\} \not\equiv 0$$

and

$$\nu_\mu(x) \equiv \operatorname{Im}\{w(x,\mu)\} \not\equiv 0.$$

These two relations and Lemma 2.5 gives us contradiction which shows that our proposition is false: if $\mu \neq \lambda$ and $\mu \neq \bar{\lambda}$, then nonreal numbers $\lambda, \mu$ cannot simultaneously be eigenvalues of the problem (1.1)-(1.3).

The theorem is proved. □

**Remark 2.** It can be easily seen that the result of this paper remains true in the case when $q(x) \in L_2(0,1)$ as well as when $q(x) \in L_1(0,1)$.

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