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ERLAN DAUTBEKOVICH NURSULTANOV

(to the 60th birthday)

On May 25, 2017 was the 60th birthday of Yerlan Dautbekovich Nursultanov, Doctor of Physical and Mathematical Sciences (1999), Professor (2001), Head of the Department of Mathematics and Informatics of the Kazakhstan branch of the M.V. Lomonosov Moscow State University (since 2001), member of the Editorial Board of the Eurasian Mathematical Journal.

E.D. Nursultanov was born in the city of Karaganda. He graduated from the Karaganda State University (1979) and then completed his post-graduate studies at the M.V. Lomonosov Moscow State University.

Professor Nursultanov's scientific interests are related to various areas of the theory of functions and functional analysis.

He introduced the concept of multi-parameter Lorentz spaces, network spaces and anisotropic Lorentz spaces, for which appropriate interpolation methods were developed. On the basis of the apparatus introduced by him, the questions of reiteration in the off-diagonal case for the real Lyons-Petre interpolation method, the multiplier problem for trigonometric Fourier series, the lower and upper bounds complementary to the Hardy-Littlewood inequalities for various orthonormal systems were solved. The convergence of series and Fourier transforms were studied with sufficiently general monotonicity conditions. The lower bounds for the norm of the convolution operator are obtained, and its upper bounds are improved (a stronger result than the O'Neil inequality). An exact cubature formula with explicit nodes and weights for functions belonging to spaces with a dominated mixed derivative is constructed, and a number of other problems in this area are solved.

He has published more than 50 scientific papers in high rating international journals included in the lists of Thomson Reuters and Scopus. 2 doctor of sciences, 9 candidate of sciences and 4 PhD dissertations have been defended under his supervision.

His merits and achievements are marked with badges of the Ministry of Education and Science of the Republic of Kazakhstan "For Contribution to the Development of Science" (2007), "Honored Worker of Education" (2011), "Y. Altynsarin" (2017). He is a laureate of the award named after K. Satpaev in the field of natural sciences for 2005, the grant holder "The best teacher of the university" for 2006 and 2011, the grant holder of the state scientific scholarship for outstanding contribution to the development of science and technology of the Republic of Kazakhstan for years 2007-2008, 2008 -2009. In 2017 he got the Top Springer Author award, established by Springer Nature together with JSC "National Center for Scientific and Technical Information".

The Editorial Board of the Eurasian Mathematical Journal congratulates Erlan Dautbekovich Nursultanov on the occasion of his 60th birthday and wishes him good health and successful work in mathematics and mathematical education.

JAMALBEK TUSSUPOV

(to the 60th birthday)

On April 10, 2017 was the 60th birthday of Jamalbek Tussupov, Doctor of Physical and Mathematical Sciences, Professor, Head of the Information Systems Department of the L.N. Gumilyov Eurasian National University, member of the Kazakhstan and American Mathematical Societies, member of the Association of Symbolic Logic, member of the Editorial Board of the Eurasian Mathematical Journal.

J. Tussupov was born in Taraz (Jambyl region of the Kazakh SSR). He graduated from the Karaganda State University (Kazakhstan) in 1979 and later on completed his postgraduate studies at S.L. Sobolev Institute of Mathematics of the Academy of Sciences of Russia (Novosibirsk).

Professor Tussupov's research interests are in mathematical logic, computability, computable structures, abstract data types, ontology, formal semantics. He solved the following problems of computable structures:

- the problems of S.S. Goncharov and M.S. Manasse: the problem of characterizing relative categoricity in the hyperarithmetical hierarchy given levels of complexity of Scott families, and the problem on the relationship between categoricity and relative categoricity of computable structures in the arithmetical and hyperarithmetical hierarchies;
- the problem of Yu.L. Ershov: the problem of finite algorithmic dimension in the arithmetical and hyperarithmetical hierarchies;
- the problem of C.J. Ash and A. Nerode: the problem of the interplay of relations of bounded arithmetical and hyperarithmetical complexity in computable presentations and the definability of relations by formulas of given complexity;
- the problem of S. Lempp: the problem of structures having presentations in just the degrees of all sets X such that for algebraic classes as symmetric irreflexive graphs, nilpotent groups, rings, integral domains, commutative semigroups, lattices, structure with two equivalences, bipartite graphs.

Professor Tussupov has published about 100 scientific papers, five textbooks for students and one monograph. Three PhD dissertations have been defended under his supervision.

Professor Tussupov is a fellow of "Bolashak" Scholarship, 2011 (Notre Dame University, USA), "Erasmus+", 2016 (Poitiers University, France). He was awarded the title "The Best Professor of 2012" (Kazakhstan). In 2015 Jamalbek Tussupov was also awarded for the contribution to science in the Republic of Kazakhstan.

The Editorial Board of the Eurasian Mathematical Journal congratulates Dr. Professor Jamalbek Tussupov on the occasion of his 60th aniversary and wishes him strong health, new achievements in science, inspiration for new ideas and fruitfull results.

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ON FIXED POINTS OF CONTRACTION MAPS ACTING IN (q_1, q_2)-QUASIMETRIC SPACES AND GEOMETRIC PROPERTIES OF THESE SPACES

R. Sengupta

Communicated by V.I. Burenkov

Key words: fixed point, quasimetric space.

AMS Mathematics Subject Classification: 54H25, 47H04.

Abstract. We study geometric properties of (q_1, q_2)-quasimetric spaces and fixed point theorems in these spaces. In paper [1], a fixed point theorem was obtained for a contraction map acting in a complete (q_1, q_2)-quasimetric space. The graph of the map was assumed to be closed. In this paper, we show that this assumption is essential, i.e. we provide an example of a complete quasimetric space and a contraction map acting in it whose graph is not closed and which is fixed-point-free. We also describe some geometric properties of such spaces.

1 Introduction

The present paper is devoted to the problem of the existence of a fixed point of contraction maps in a complete (q_1, q_2)-quasimetric space and the geometry of (q_1, q_2)-quasimetric spaces. The basis for the theory of such spaces can be found in [1] where coincidence point theorems and a fixed point theorem for such spaces were obtained.

Let us recall the necessary definitions from [1]. Let positive real numbers q_1, q_2 and a set X be given.

Definition 1. *A function $\rho_X : X \times X \to \mathbb{R}_+$, such that $\rho_X(x, y) = 0 \iff x = y$, is called a (q_1, q_2)-quasimetric, if the generalized (q_1, q_2)-triangle inequality holds:*

$$\rho_X(x, z) \leq q_1\rho_X(x, y) + q_2\rho_X(y, z) \quad \forall x, y, z \in X.$$

The pair (X, ρ_X) is called a (q_1, q_2)-quasimetric space. A $(1, 1)$-quasimetric space is called a quasimetric space.

In definitions below, we assume that (X, ρ_X) is a (q_1, q_2)-quasimetric space and (Y, ρ_Y) is a $(\widetilde{q}_1, \widetilde{q}_2)$-quasimetric space.

Definition 2. *Given $q_0 > 0$, a (q_1, q_2)-quasimetric is called q_0-symmetric if*

$$\rho_X(x, y) \leq q_0\rho_X(y, x) \quad \forall x, y \in X.$$

In this case, the pair (X, ρ_X) is called a q_0-symmetric (q_1, q_2)-quasimetric space. If $q_0 = 1$, then the space is said to be symmetric.

Definition 3. *The map $\Phi : X \to X$ is called β-Lipschitz if*

$$\rho_X(\Phi(x), \Phi(y)) \leq \beta\rho_X(x, y) \quad \forall x, y \in X.$$

If $\beta < 1$ then the map Φ is said to be a contraction map.

Definition 4. *A sequence* $\{x_i\}_{i=1}^{\infty} \subset X$ *such that*

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall i, j \in \mathbb{N} : i > j > N \Rightarrow \quad \rho_X(x_j, x_i) < \varepsilon$$

is called a Cauchy sequence.

Definition 5. *A sequence* $\{x_i\}_{i=1}^{\infty} \subset X$ *is said to converge to a point* $x_0 \in X$ *in a quasimetric space* (X, ρ_X) *if* $\lim\limits_{i\to\infty} \rho_X(x_0, x_i) = 0$.

Definition 6. *A quasimetric space* (X, ρ_X) *is said to be complete if any Cauchy sequence converges in this space.*

Let a function $F : X \to Y$ be given. Denote $gphF = \{(x, y) \in X \times Y : y = F(x)\}$.

Definition 7. *A map* $F : X \to Y$ *is said to be closed if the set* $gphF$ *is closed, that is if for all sequences* $\{x_i\} \subset X$ *and* $\{y_i\} \subset Y$, *such that they converge to points* x_0 *and* y_0 *respectively and* $(x_i, y_i) \in gph(F)$ *for all* i, *we have* $(x_0, y_0) \in gph(F)$.

Let us show that if a (q_1, q_2)-quasimetric space X is q_0-symmetric then any contraction map $F : X \to X$ acting in it is closed. Consider the contrary: let there exist sequences $\{x_i\}$ and $\{y_i\}$, that converge to points x_0 and y_0 respectively and $(x_i, y_i) \in gph(F)$ for all i, however $(x_0, y_0) \notin gph(F)$. In other words, $\rho_X(y_0, F(x_0)) > 0$. While

$$\rho_X(y_0, F(x_0)) \le q_1\rho_X(y_0, y_n) + q_2\rho_X(y_n, F(x_0)) \le q_1\rho_X(y_0, y_n) + \beta q_2 q_0 \rho_X(x_0, x_n) \to 0.$$

Thus, we obtained a contradiction.

The following example shows that the identity map in a (q_1, q_2)-quasimetric space is not necessarily closed.

Example 1 (proposed by S.E. Zhukovskiy)**.** *On the standard Euclidean space* $\mathbb{R}^2$ *by* $S(r)$ *denote the circle of radius* r *with center at zero. Let* $\mathcal{X}$ *be system of sets, consisting of all circles* $S(r)$, $r > 1$ *and points* x *of the unit circle* $S(1)$. *Set*

$$d(U, V) = h_X^+(U, V), \qquad U, V \in \mathcal{X},$$

$$h_X^+(U, V) = \inf\{\varepsilon \ge 0 : U \subset N_\varepsilon(V)\}, \tag{1.1}$$

$$N_\varepsilon(V) = \bigcup_{v\in V} \{u \in \mathbb{R}^2 : |u - v| < \varepsilon\}. \tag{1.2}$$

Then $(\mathcal{X}, d)$ *is a quasimetric space.*

It is known that such a space $(\mathcal{X}, d)$ *is complete. For any point* $x \in S(1)$, *we have*

$$d\left(x, S\left(1 + \frac{1}{n}\right)\right) = h_X^+\left(x, S\left(1 + \frac{1}{n}\right)\right) = \frac{1}{n} \to 0 \ \ as \quad n \to \infty.$$

Thus, every point $x \in S(1)$ *is a limit of the sequence* $\left\{S\left(1 + \frac{1}{n}\right)\right\}$.

Let us show that the identity map F *in this space is not closed. Let us denote* $x_n = S\left(1+\frac{1}{n}\right)$. *Let us take two different points* $a, b \in S(1)$. *It was shown above that* $x_n \to a$ *and* $x_n \to b$. *It is obvious* $(x_n, x_n) \in gph(F)$ *for all* n. *However,* $(a, b) \notin gph(F)$, *since* $a \ne b$. *Thus, the identity map is not closed.*

2 Main results

Let us consider the fixed point theorem from [1].

Theorem 2.1. *A closed contraction map acting from a complete (q_1, q_2)-quasimetric space to itself has a unique fixed point.*

Below we present an example showing that in Theorem 2.1 the assumption of the contraction map being closed is essential.

Before constructing the example, we introduce the concept of a stabilizing sequence.

Let (X, ρ) be a (q_1, q_2)-quasimetric space.

Definition 8. *A sequence $\{x_i\} \subset X$ is said to stabilize to a point $a \in X$ if there exists $N \in \mathbb{N}$, such that $x_i = a$ for all $i \geq N$. This sequence is said to be stabilizing if there exists such a point $a \in X$.*

It is obvious that any stabilizing sequence $\{x_i\} \subset X$ is Cauchy and convergent.

Lemma 2.1. *Given a Cauchy sequence $\{x_n\}$, assume that there exists a subsequence $\{x_{n_k}\}$ that stabilizes to a certain $a \in X$. Then $\lim\limits_{n\to\infty} x_n = a$ and a is the only limit of the sequence $\{x_n\}$.*

Proof. Let us prove that the sequence $\{x_n\}$ converges to a. Assume that $\{x_n\}$ does not converge to a. Then there exists an $\varepsilon > 0$ such that for all $N \in \mathbb{N}$ there exists $i > N$, such that $\rho(a, x_i) > \varepsilon$. By virtue of the assumption, there exist arbitrarily large numbers j, such that $x_j = a$. Therefore, there exist arbitrarily large numbers i and j, such that $i > j$ and $\rho(x_j, x_i) > \varepsilon$. The statement above contradicts the fact that $\{x_n\}$ is a Cauchy sequence.

Let us prove the uniqueness of the limit. Let there exist $b \in X$, such that $a \neq b$ and $x_n \to b$. Set $\rho(b, a) = \gamma > 0$. Let us pick $\varepsilon = \frac{\gamma}{2}$. Then by virtue of the assumption made there exist arbitrarily large numbers i such that $x_i = a$ and therefore $\rho(b, x_i) > \varepsilon$. The above contradicts the assumption that $\{x_n\}$ converges to b and proves that a is the unique limit. □

Corollary 2.1. *Let us consider a sequence $\{x_n\} \subset X$ such that it has subsequences $\{x_{n_k}\}, \{x_{n_l}\}$, which stabilize to $a, b \in X$ respectively, moreover $a \neq b$. Then the sequence $\{x_n\}$ is not a Cauchy sequence.*

Corollary 2.2. *If a Cauchy sequence $\{x_n\}$ has at least two different limits then any element of the sequence appears in $\{x_n\}$ a finite number of times.*

Proof. Let a certain element c appear an infinite number of times. Then there exists a subsequence that stabilizes to c and therefore by virtue of Lemma 2.1, c is the only limit of the original sequence which contradicts the assumption. □

The example below illustrates that in Theorem 2.1 the assumption of the contraction map being closed is essential.

Example 2. Set $X = \{0, 1, 2, 3, ...\}$. Let us define the function $\rho : X \times X \to \mathbb{R}_+$ by

$$\rho(k, n) = \begin{cases} \frac{1}{2^{n-1}} - \frac{1}{2^{k-1}}, & \text{if} \quad k > n. \\ \frac{1}{2^n}, & \text{if} \quad k < n. \\ 0, & \text{if} \quad k = n. \end{cases}$$

Let us prove that ρ is a quasimetric. It is enough to verify the triangle inequality:

$$\rho(k, n) \leq \rho(k, m) + \rho(m, n) \quad \forall k, m, n \in X.$$

It is obvious that if the values of at least two out of three variables $k, n, m \in X$ coincide, then this inequality holds. Let us assume that are $k, n, m \in X$ are pairwise non-identical. Consider 6 cases:

1. Let $m > k > n$. Then

$$\rho(k,m)+\rho(m,n)=\frac{1}{2^m}+\frac{1}{2^{n-1}}-\frac{1}{2^{m-1}}=\frac{1}{2^{n-1}}-\frac{1}{2^m}\geq\frac{1}{2^{n-1}}-\frac{1}{2^{k-1}}=\rho(k,n).$$

2. Let $k > m > n$. Then

$$\rho(k,m)+\rho(m,n)=\frac{1}{2^{m-1}}-\frac{1}{2^{k-1}}+\frac{1}{2^{n-1}}-\frac{1}{2^{m-1}}=\frac{1}{2^{n-1}}-\frac{1}{2^{k-1}}=\rho(k,n).$$

3. Let $k > n > m$. Then

$$\rho(k,m)+\rho(m,n)=\frac{1}{2^{m-1}}-\frac{1}{2^{k-1}}+\frac{1}{2^n}=\frac{1}{2^{m-1}}-\frac{1}{2^n}+\frac{1}{2^{n-1}}-\frac{1}{2^{k-1}}\geq\frac{1}{2^{n-1}}-\frac{1}{2^{k-1}}=\rho(k,n).$$

4. Let $m > n > k$. Then

$$\rho(k,m)+\rho(m,n)=\frac{1}{2^m}+\frac{1}{2^{n-1}}-\frac{1}{2^{m-1}}=\frac{2}{2^n}-\frac{1}{2^m}\geq\frac{2}{2^n}-\frac{1}{2^n}=\frac{1}{2^n}=\rho(k,n).$$

5. Let $n > m > k$. Then

$$\rho(k,m)+\rho(m,n)=\rho(k,m)+\frac{1}{2^n}\geq\frac{1}{2^n}=\rho(k,n).$$

6. Let $n > k > m$. Then

$$\rho(k,m)+\rho(m,n)=\rho(k,m)+\frac{1}{2^n}\geq\frac{1}{2^n}=\rho(k,n).$$

Therefore, ρ is a quasimetric.

Proof. Let us now prove that the constructed quasimetric space (X, ρ_X) is complete.

Let $\{x_i\}$ be a Cauchy sequence. If there exists a subsequence that stabilizes to a certain element from X then by virtue of Lemma 2.1, it converges to this element, which is the only limit of the sequence. Let us consider the case, when the Cauchy sequence $\{x_i\}$ does not contain a stabilizing subsequence. Then by definition every element of the sequence appears only a finite number of times. Therefore, for an arbitrary element $a \in X$, there exists a number $N(a) \in \mathbb{N}$ such that $x_i > a$ for all $i \geq N(a)$. Then $\rho(a, x_i) = \frac{1}{2^{x_i}}$. Let us take an arbitrary $\varepsilon > 0$ and a natural $N' \geq N(a)$ such that $\rho(a, x_i) = \frac{1}{2^{x_i}} < \varepsilon$ for all $i \geq N'$. By virtue of the arbitrary choice of ε, it follows that the sequence $\{x_i\}$ converges to a.

Thus, in the given space any Cauchy sequence converges. Therefore, the constructed quasimetric space (X, ρ_X) is complete.

Consider the map $\Phi : X \to X$, $\Phi(n) = n+1$ for all $n \in X$. The map Φ is a contraction map, since

$$\rho(\Phi(n),\Phi(m))=\rho(n+1,m+1)=\frac{1}{2}\rho(n,m)\quad\forall n,m\in X.$$

It is obvious, that the map Φ does not have fixed points. Thus, the space (X, ρ) and the map Φ are desired. □

Let us directly verify that the graph of Φ is not closed. In the example above, take two different points $a, b \in X$ such that $b \neq \Phi(a)$. Let us take $x_n = n$, then $x_n \to a$, $x_n \to b$ and $(x_n, x_{n+1}) \in gph(\Phi)$ for all $n \in \mathbb{N}$. However, $(a, b) \notin gph(\Phi)$, since $\Phi(a) \neq b$. Thus, the map Φ is not closed.

Let us discuss some geometric properties of (q_1, q_2)-quasimetric spaces. Let (X, ρ) be a (q_1, q_2)-quasimetric space and the set X consists of not less than two points. Then $q_1 \geq 1, q_2 \geq 1$.

In [1] a set Q was defined that consists of points $q = (q_1, q_2) \in \mathbb{R}^2$ such that $q_1 \geq 1, q_2 \geq 1$ and the (q_1, q_2)-generalized triangle inequality holds for the quasimetric ρ . It is obvious that the set Q is convex and closed. Besides, if a (q_1, q_2)-quasimetric space is symmetric then the set Q is symmetric with respect to the bisector of the first quadrant.

In [1] the concept of a best point was also defined. A point $q = (q_1, q_2) \in Q$ is said to be the best point, if $q \leq q'$ for all $q' \in Q$ (coordinatewise inequality is implied). The best point is unique, though it does not always exist. However, if (X, ρ) is symmetric and the best point $q = (q_1, q_2)$ exists, then $q_1 = q_2$.

A natural question arises: does there exist a (q_1, q_2)-quasimetric space (that is not symmetric), for which there exists a best point $q = (q_1, q_2) \in Q$, such that $q_1 \neq q_2$?

The example below illustrates that such a point can indeed exist for a nonsymmetric (q_1, q_2)-quasimetric space.

Example 3. Set

$$X = \{1, 2, 3, ...\}.$$

Let us define the function $\rho : X \times X \to \mathbb{R}_+$ by

$$\rho(k, n) = \begin{cases} \frac{1}{2^{k-1}}, & \text{if} \quad k > n. \\ \frac{1}{2^k}, & \text{if} \quad k < n. \\ 0, & \text{if} \quad k = n. \end{cases}$$

Let us prove that ρ is a $(2, 1)$-quasimetric. For this it is enough to verify the $(2, 1)$-generalized triangle inequality:

$$\rho(k, n) \leq 2(\rho(k, m)) + \rho(m, n) \quad \forall k, m, n \in X.$$

It is obvious that if the values of at least two out of three variables $k, n, m \in X$ coincide, then this inequality holds. Let us assume that are $k, n, m \in X$ are pairwise non-identical. Consider 6 cases:

1. Let $m > k > n$. Then

$$2(\rho(k, m)) + \rho(m, n) = 2(\frac{1}{2^k}) + \frac{1}{2^{m-1}} \geq \frac{1}{2^{k-1}} = \rho(k, n).$$

2. Let $k > m > n$. Then

$$2(\rho(k, m)) + \rho(m, n) = 2(\frac{1}{2^{k-1}}) + \frac{1}{2^{m-1}} \geq \frac{1}{2^{k-1}} = \rho(k, n).$$

3. Let $k > n > m$. Then

$$2(\rho(k, m)) + \rho(m, n) = 2(\frac{1}{2^{k-1}}) + \frac{1}{2^m} \geq \frac{1}{2^{k-1}} = \rho(k, n).$$

4. Let $m > n > k$. Then

$$2(\rho(k, m)) + \rho(m, n) = 2(\frac{1}{2^k}) + \frac{1}{2^{m-1}} \geq \frac{1}{2^k} = \rho(k, n).$$

5. Let $n > m > k$. Then

$$2(\rho(k, m)) + \rho(m, n) = 2(\frac{1}{2^k}) + \frac{1}{2^m} \geq \frac{1}{2^k} = \rho(k, n).$$

6. Let $n > k > m$. Then

$$2(\rho(k, m)) + \rho(m, n) = 2(\frac{1}{2^{k-1}}) + \frac{1}{2^m} \geq \frac{1}{2^k} = \rho(k, n).$$

Therefore, ρ is a $(2, 1)$-quasimetric and $(2, 1)$ is the best point, as in the case 1, an arbitrarily large number m could be taken so that the inequality does not hold for $q_1 < 2$.

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