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ERLAN DAUTBEKOVICH NURSULTANOV

(to the 60th birthday)

On May 25, 2017 was the 60th birthday of Yerlan Dautbekovich Nursultanov, Doctor of Physical and Mathematical Sciences (1999), Professor (2001), Head of the Department of Mathematics and Informatics of the Kazakhstan branch of the M.V. Lomonosov Moscow State University (since 2001), member of the Editorial Board of the Eurasian Mathematical Journal.

E.D. Nursultanov was born in the city of Karaganda. He graduated from the Karaganda State University (1979) and then completed his post-graduate studies at the M.V. Lomonosov Moscow State University.

Professor Nursultanov's scientific interests are related to various areas of the theory of functions and functional analysis.

He introduced the concept of multi-parameter Lorentz spaces, network spaces and anisotropic Lorentz spaces, for which appropriate interpolation methods were developed. On the basis of the apparatus introduced by him, the questions of reiteration in the off-diagonal case for the real Lyons-Petre interpolation method, the multiplier problem for trigonometric Fourier series, the lower and upper bounds complementary to the Hardy-Littlewood inequalities for various orthonormal systems were solved. The convergence of series and Fourier transforms were studied with sufficiently general monotonicity conditions. The lower bounds for the norm of the convolution operator are obtained, and its upper bounds are improved (a stronger result than the O'Neil inequality). An exact cubature formula with explicit nodes and weights for functions belonging to spaces with a dominated mixed derivative is constructed, and a number of other problems in this area are solved.

He has published more than 50 scientific papers in high rating international journals included in the lists of Thomson Reuters and Scopus. 2 doctor of sciences, 9 candidate of sciences and 4 PhD dissertations have been defended under his supervision.

His merits and achievements are marked with badges of the Ministry of Education and Science of the Republic of Kazakhstan "For Contribution to the Development of Science" (2007), "Honored Worker of Education" (2011), "Y. Altynsarin" (2017). He is a laureate of the award named after K. Satpaev in the field of natural sciences for 2005, the grant holder "The best teacher of the university" for 2006 and 2011, the grant holder of the state scientific scholarship for outstanding contribution to the development of science and technology of the Republic of Kazakhstan for years 2007-2008, 2008 -2009. In 2017 he got the Top Springer Author award, established by Springer Nature together with JSC "National Center for Scientific and Technical Information".

The Editorial Board of the Eurasian Mathematical Journal congratulates Erlan Dautbekovich Nursultanov on the occasion of his 60th birthday and wishes him good health and successful work in mathematics and mathematical education.

JAMALBEK TUSSUPOV

(to the 60th birthday)

On April 10, 2017 was the 60th birthday of Jamalbek Tussupov, Doctor of Physical and Mathematical Sciences, Professor, Head of the Information Systems Department of the L.N. Gumilyov Eurasian National University, member of the Kazakhstan and American Mathematical Societies, member of the Association of Symbolic Logic, member of the Editorial Board of the Eurasian Mathematical Journal.

J. Tussupov was born in Taraz (Jambyl region of the Kazakh SSR). He graduated from the Karaganda State University (Kazakhstan) in 1979 and later on completed his postgraduate studies at S.L. Sobolev Institute of Mathematics of the Academy of Sciences of Russia (Novosibirsk).

Professor Tussupov's research interests are in mathematical logic, computability, computable structures, abstract data types, ontology, formal semantics. He solved the following problems of computable structures:

- the problems of S.S. Goncharov and M.S. Manasse: the problem of characterizing relative categoricity in the hyperarithmetical hierarchy given levels of complexity of Scott families, and the problem on the relationship between categoricity and relative categoricity of computable structures in the arithmetical and hyperarithmetical hierarchies;
- the problem of Yu.L. Ershov: the problem of finite algorithmic dimension in the arithmetical and hyperarithmetical hierarchies;
- the problem of C.J. Ash and A. Nerode: the problem of the interplay of relations of bounded arithmetical and hyperarithmetical complexity in computable presentations and the definability of relations by formulas of given complexity;
- the problem of S. Lempp: the problem of structures having presentations in just the degrees of all sets X such that for algebraic classes as symmetric irreflexive graphs, nilpotent groups, rings, integral domains, commutative semigroups, lattices, structure with two equivalences, bipartite graphs.

Professor Tussupov has published about 100 scientific papers, five textbooks for students and one monograph. Three PhD dissertations have been defended under his supervision.

Professor Tussupov is a fellow of "Bolashak" Scholarship, 2011 (Notre Dame University, USA), "Erasmus+", 2016 (Poitiers University, France). He was awarded the title "The Best Professor of 2012" (Kazakhstan). In 2015 Jamalbek Tussupov was also awarded for the contribution to science in the Republic of Kazakhstan.

The Editorial Board of the Eurasian Mathematical Journal congratulates Dr. Professor Jamalbek Tussupov on the occasion of his 60th aniversary and wishes him strong health, new achievements in science, inspiration for new ideas and fruitfull results.

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ON THE UNIFORM CONVERGENCE OF FOURIER SERIES ON A CLOSED DOMAIN

A.A. Rakhimov

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Key words: Riesz means, uniform convergence, eigenfunction expansions, spaces with mixed norm.

AMS Mathematics Subject Classification: 42C10.

Abstract. The uniform convergence on a closed domain is studied of eigenfunction expansions of continuous functions belonging to function spaces with mixed norm.

1 Introduction and preliminaries

Let Ω be a bounded domain in R^N with a smooth boundary $\partial\Omega$. Let $\hat{A}$ be the self-adjoint extension of a positive formally elliptic differential operator of order $2m$ with regular boundary conditions [1].

Denote by $\{u_n(x)\}$ a complete orthonormal in $L_2(\Omega)$ system of eigenfunctions of the operator $\hat{A}$ corresponding to the sequence of eigenvalues $0 < \lambda_1 < \lambda_2 < .... < \lambda_n \to \infty$. For any function $f \in L_2(\Omega)$ we introduce the Riesz means of order s of the partial sums of the Fourier series

$$E^s_\lambda f(x) = \sum_{\lambda_n<\lambda} \left(1 - \frac{\lambda_n}{\lambda}\right)^s f_n u_n(x). \tag{1.1}$$

Here $\lambda > 0$, $f_n = (f, u_n)$ are the Fourier coefficients of the function f with respect to the system $\{u_n(x)\}$.

Note that if $s = 0$, then (1.1) is just the partial sum of the Fourier series of the function f.

Precise conditions of uniform convergence on compact subsets of the domain Ω of Fourier series were established by V.A. Il'in (see [9]).

Theorem 1.1. *If*

$$\alpha \geq \frac{N-1}{2}, \quad \alpha p > N, \quad p \geq 1 \tag{1.2}$$

then the Fourier series via the eigenfunctions of the Laplace operator of any function with compact support belonging to the Sobolev space $W^\alpha_p(\Omega)$ converges uniformly on any compact subset of the domain Ω.

Convergence of the Riesz means (1.1) of smooth functions on compact subsets of the domain Ω requires modification of condition (1.2) in Theorem 1.1 as follows

$$\alpha + s \geq \frac{N-1}{2}, \quad \alpha p > N, \quad s \geq 0, \quad p \geq 1. \tag{1.3}$$

The sharpness of the first inequality in (1.3) for eigenfunction expansions associated with Laplace operator was proved by V.A. Il'in (see in [9]). The sharpness of the second inequality in (1.3) follows from the fact that the condition $\alpha p \le N$ implies the existence of an unbounded function with compact support belonging to the appropriate Sobolev space for which its Fourier series cannot converge uniformly.

Moreover, conditions (1.3) are sufficient for functions in the Nikol'skii spaces $H_p^\alpha(\Omega)$. The last statement was proved in the case of expansions associated with the eigenfunctions of the Laplace operator by V.A. Il'in and Sh.A. Alimov, in the case of expansions associated with elliptic operators of second order with variable coefficients by V.A. Il'in and E.I. Moiseev. Finally, for general elliptic differential operators of order $2m$ Sh.A. Alimov has proved in [4] the following statement

Theorem 1.2. *If f belongs to the space $\overset{\circ}{H}{}_p^\alpha(\Omega)$ and has compact support in Ω, then under conditions (1.3) the Riesz means $E_\lambda^s f(x)$ converge as $\lambda \to +\infty$ to f uniformly on any compact $K \subset \Omega$.*

Here $\overset{\circ}{H}{}_p^\alpha(\Omega)$, $(\overset{\circ}{W}{}_p^\alpha(\Omega))$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm of the Nikol'skii (Sobolev) space $H_p^\alpha(\Omega)$ $(W_p^\alpha(\Omega))$.

In the case in which the second condition in (1.3) is replaced by $\alpha p = N$, it is necessary to assume that the function f is continuous (see [3]):

Theorem 1.3. *Let Ω_0 be an arbitrary open subset of Ω and let*

$$\alpha + s > \frac{N-1}{2}, \quad \alpha p = N, \quad s \ge 0, \quad p \ge 1. \tag{1.4}$$

Then for any function $f \in \overset{\circ}{W}{}_p^\alpha(\Omega)$ continuous on Ω_0

$$\lim_{\lambda\to\infty} E_\lambda^s f(x) = f(x). \tag{1.5}$$

uniformly on any compact set $K \subset \Omega_0$.

The first condition $\alpha + s > \frac{N-1}{2}$ in (1.4) is also precise [3].

Theorem 1.4. *Let x_0 be an arbitrary point of the domain Ω and let*

$$\alpha + s = \frac{N-1}{2}, \quad \alpha p = N, \quad s \ge 0, \quad p \ge 1. \tag{1.6}$$

Then there exists a function $f \in \overset{\circ}{W}{}_p^\alpha(\Omega)$, which is continuous in Ω, and such that

$$\overline{\lim_{\lambda\to\infty}} E_\lambda^s f(x_0) = +\infty. \tag{1.7}$$

These results were extended to the Nikol'skii spaces in [11].

2 Uniform convergence on closed domains

The uniform convergence of Fourier series on closed domains $\overline{\Omega}$ was studied by V.A. Il'in (see [10]). In [10] for the eigenfunction expansions associated with the first, second and third boundary conditions for the Laplace operator it was proved that if $f \in W_p^{\frac{N+2}{2}}, p > \frac{2N}{N-1}$ and the functions $f, \Delta f, ..., \Delta^\beta f$, up to a certain order β, satisfy the appropriate boundary conditions, then the Fourier series of f converges uniformly on closed domain $\overline{\Omega}$.

For the elliptic differential operator of order $2m$ with the regular boundary conditions G.I. Eskin (see [6]) proved that the eigenfunction expansion of a function in $\overset{\circ}{W}{}_p^{\frac{2N-1}{4}+\varepsilon}$ with any $\varepsilon > 0$ converges uniformly on closed domains.

E.I. Moiseev studied the problem for the elliptic operators of second order for the first boundary value problem. In [12] it is proved that if f is a function with compact support in the space $W_p^{\frac{N-1}{2}}, p > \frac{2N}{N-1}$, such that the series

$$\sum_{n=1}^{\infty} \lambda_n^{\frac{N-1}{2}} (\ln \lambda_n)^{2+\varepsilon} f_n^2$$

converges, then its expansion via eigenfunctions converges uniformly on the closed domain $\overline{\Omega}$.

Moreover, it was proved in [12] that the following estimate

$$\sum_{|\sqrt{\lambda_n}-\mu|\leq 1} u_n^2(x) = O(\mu^{N-1}\ln^2 \mu) \tag{2.1}$$

is valid uniformly on the closed domain $\overline{\Omega}$.

In [5] uniform convergence of expansions via eigenfunctions of the elliptic differential operator of order $2m$ with the Lopatinsky boundary condition was studied and the following result was proved.

Theorem 2.1. *Let f be an arbitrary continuous function with compact support in Ω. Then the Riesz means $E_\lambda^s f$ of order $s > \frac{N}{2}$ converge to f uniformly on the closed domain $\overline{\Omega}$.*

In [14] by using estimate (2.1) the condition $s > \frac{N}{2}$ in Theorem 2.1 was replaced by $s > \frac{N-1}{2}$
.

We mention also the following result (see [13]).

Theorem 2.2. *Let*

$$\alpha + s > \frac{N-1}{2}, \quad \alpha p \geq N, \quad s \geq 0, \quad p \geq 1. \tag{2.2}$$

Then for any continuous function $f \in \overset{\circ}{H}{}_p^\alpha(\Omega)$ with compact support in the domain Ω uniformly in $\overline{\Omega}$

$$\lim_{\lambda\to\infty} E_\lambda^s f(x) = f(x). \tag{2.3}$$

Note that it follows by Theorem 1.4 that in the case $\alpha p = N$ the condition $\alpha + s > (N-1)/2$ is precise. In the case $\alpha p > N$ this problem is still open.

3 Convergence of expansions via eigenfunctions in the spaces with mixed norm

The space of all measurable functions with finite norm

$$\|f\|_{L_{pq}(\mathcal{R}^N)} = \|\|f\|_{L_p(\mathcal{R}^k)}\|_{L_q(\mathcal{R}^{N-k})}$$

is called the space with mixed norm $L_{pq}(\mathcal{R}^N)$. If a function is defined in the domain Ω then the corresponding space can be defined by extending a function by zero outside of the domain Ω.

By H_{pq}^α we denote the Banach space of all measurable functions f with respect to the norm

$$\|f\|_{H_{pq}^\alpha(\Omega)} = \|f\|_{L_{pq}(\Omega)} + \sum_{|k|=\ell} \sup_z |z|^{-\kappa} \|\Delta_z^2 \partial^k f(y)\|_{L_{pq}(\Omega_{|z|})}$$

Here $\alpha = \ell + \kappa$, ℓ is a non negative integer, $0 < \kappa \le 1$, $p, q \ge 1$, $k = (k_1, k_2, ..., k_n)$ multi-index, $|k| = k_1 + k_2 + ... + k_n$, and $\partial^k f$ denotes the weak derivative

$$\partial^k f(y) = \frac{\partial^{|k|} f(y)}{\partial y_1^{k_1}, \partial y_2^{k_2}, ..., \partial y_n^{k_n}}.$$

The symbol $\Delta_z^2 \partial^k f(y)$ denotes the second difference of the function $\partial^k f(y)$:

$$\Delta_z^2 \partial^k f(y) = \partial^k f(y+z) - 2\partial^k f(y) + \partial^k f(y).$$

$\|f\|_{L_{pq}(\Omega)}$ denotes the norm in the space L_{pq} and, for $h > 0$, $\Omega_h = \{x \in \Omega : dist(x, \partial\Omega) > h\}$.

By $\overset{\circ}{H}{}_{pq}^{\alpha}(\Omega)$ denote the closure of $C_0^{\infty}(\Omega)$ with respect to the norm of the space $H_{pq}^{\alpha}(\Omega)$.

Using the methods of [3]-[4] for functions in the spaces with the mixed norm the appropriate theorems on convergence of the spectral expansions associated with the Laplace operator on compact subsets of the domain were obtained in [15].

The main result of the present paper is the following theorem on the uniform convergence on the closed domain $\overline{\Omega}$

Theorem 3.1. *Let f be a continuous function with compact support in the domain Ω belonging to the space $\overset{\circ}{H}{}_{pq}^{\alpha}(\Omega)$ and*

$$\alpha > \frac{N-1}{2} - s, \quad \alpha = \frac{N-k}{q} + \frac{k}{p}, \quad 2 \le p < q, \quad 0 < k < N. \tag{3.1}$$

Then uniformly on $\overline{\Omega}$

$$\lim_{\lambda \longrightarrow \infty} E_{\lambda}^{s} f(x) = f(x).$$

4 Preliminary statements

Let h be a small positive number so that Ω_h is a non-empty proper subset of Ω.

Let $x \in \Omega_h$ and $y \in \overline{\Omega}$. Consider the following function of the distance $r = |x - y|$:

$$V(x, y, \lambda) = \begin{cases} \Gamma(s+1) z^s (2\pi)^{\frac{-N}{2}} \lambda^{\frac{N-2s}{4}} \frac{J_{\frac{N}{4}-s}(r\sqrt{\lambda}}{r^{\frac{N}{2}s}}, & r \le R \\ 0, & r > R \end{cases}, \tag{4.1}$$

where R is a positive constant less than $\frac{h}{4}$ and $J_{\nu}(t)$ the Bessel function of the first kind of order ν.

For the eigenfunctions $u_n(x)$ we use the following mean value formula in a sphere $\{r < R\}$ centred at $x \in \Omega_h$ [10]:

$$\int_{r<R} u_n(y) dy \;=\; (2\pi R)^{\frac{N}{2}} J_{\frac{N}{2}}(R\sqrt{\lambda_n}) \lambda_n^{\frac{-N}{4}} u_n(x) \tag{4.2}$$

By applying mean value formula (4.2) we get the following formula for the Fourier coefficients of the function $V(x, y, \lambda)$

$$v_n^{\lambda}(x) = 2^s \Gamma(s+1) \lambda_n^{\frac{2-N}{4}} \lambda^{\frac{N-2s}{4}} u_n(x) \int_0^R J_{\frac{N}{2}+s}(\sqrt{\lambda} r) J_{\frac{N}{2}-1}(\sqrt{\lambda_n} r) r^{-s} dr. \tag{4.3}$$

In order to evaluate the integral on the right-hand side of (4.3) we use following well-known formula for the Bessel functions [16]:

$$\int_0^{\infty} J_{a+s}(\sqrt{\lambda} r) J_{a-s}(\sqrt{\lambda_n} r) r^{-s} dr = \begin{cases} \frac{(1-\frac{\lambda_n}{\lambda})^s \lambda^s \lambda_n^{\frac{a-1}{2}}}{2^s \Gamma(s+1) \lambda^{\frac{a+s}{2}}}, & \lambda_n \le \lambda \\ 0, & \lambda_n > \lambda \end{cases}. \tag{4.4}$$

Then by splitting the integral in the right-hand side of (4.3) in two integrals

$$\int_0^R = \int_0^\infty - \int_R^\infty$$

we get

$$v_n^\lambda(x) = \delta_n^\lambda u_n(x)(1-\frac{\lambda_n}{\lambda})^s - 2^s\Gamma(s+1)\lambda_n^{\frac{1-N}{4}}\lambda^{\frac{N-1}{4}-\frac{s}{2}}u_n(x)I_0(\lambda_n,\lambda), \tag{4.5}$$

where $I_0(\lambda_n,\lambda)$ defined by

$$I_m(\lambda_n,\lambda) = (\lambda\lambda_n)^{\frac{1}{4}}\int_R^\infty J_{\frac{N}{2}+s-m}(\sqrt{\lambda}r)J_{\frac{N}{2}-1-m}(\sqrt{\lambda_n}r)r^{-s}dr, \quad m=\overline{0,[\frac{N-1}{2}]} \tag{4.6}$$

when $m=0$ and $\delta_n^\lambda = \begin{cases} 1, & \lambda_n < \lambda \\ 0, & \lambda_n \geq \lambda. \end{cases}$

Multiplying both sides of (4.5) by $u_n(x)$ and summing in n, we get the following equality in L_2 - sense with the respect to the variable y

$$V(x,y,\lambda) = \Theta^s(x,y,\lambda) - 2^s\Gamma(s+1)\lambda^{\frac{N-1}{4}-\frac{s}{2}}\sum_{n-1}^\infty \lambda_n^{\frac{1-N}{4}}I_0(\lambda,\lambda_n)u_n(x)u_n(y), \tag{4.7}$$

where $\Theta^s(x,y,\lambda) = \sum_{\lambda_n<\lambda}u_n(x)u_n(y)$ is known as the spectral function [1].

Lemma 4.1. *For any function $f\in L_2(\Omega)$ the integral*

$$\int_\Omega f(x)V(x,y,\lambda)dx$$

is continuous in y on the closed domain $\overline{\Omega}$.

Proof. From estimate (2.1) it follows that for any positive number ε uniformly with respect to $y\in\overline{\Omega}$

$$\sum_{\lambda_n>\lambda} u_n^2(y)\lambda_n^{-\varepsilon-\frac{N}{2}} = O(\lambda^{-\varepsilon}\ln^2\lambda). \tag{4.8}$$

This can be proved if the sum in the left-hand side of (4.8) is represented in the form of a series:

$$\sum_{\lambda_n>\lambda} u_n^2(y)\lambda_n^{-\varepsilon-\frac{N}{2}} \leq \sum_{k=0}^\infty \sum_{\sqrt{\lambda}+k<\sqrt{\lambda_n}<\sqrt{\lambda}+k+1} u_n^2(y)\lambda_n^{-\varepsilon-\frac{N}{2}} \leq$$

$$\leq \sum_{k=0}^\infty(\sqrt{\lambda}+k)^{-2\varepsilon-N}\sum_{\sqrt{\lambda}+k<\sqrt{\lambda_n}<\sqrt{\lambda}+k+1} u_n^2(y).$$

Similarly from the estimate of integral (4.6) (see [16])

$$|I_m(\lambda,\lambda_n)| \leq \frac{c}{1+|\sqrt{\lambda}-\sqrt{\lambda_n}|} \tag{4.9}$$

(where $c>0$ is independent of λ and n) and estimate (4.8) by application of the Cauchy-Schwarz inequality it follows that the series

$$\sum_{n=1}^\infty f_n u_n(y)\lambda_n^{\frac{1-N}{4}}I_m(\lambda,\lambda_n)$$

converges uniformly on the closed domain $\overline{\Omega}$. □

Let a function $f \in L_2(\Omega)$ has compact support in Ω and $supp f \subset \Omega_h$. Then for $y \in \overline{\Omega}$ Riesz means (1.1) of the partial sums of the Fourier series of the function f via the eigenfunctions $u_n(x)$ can be written as

$$E_\lambda^s f(y) = \int_{\Omega_h} f(x)V(x,y,\lambda)dx + +2^s\Gamma(s+1)\lambda^{\frac{N-1}{4}-\frac{s}{2}} \sum_{n=1}^{\infty} f_n u_n(y)\lambda_n^{\frac{1-N}{4}} I_0(\lambda,\lambda_n) \tag{4.10}$$

Denote by $B(R,y)$ the sphere of radius R centred at $y \in \overline{\Omega}$. Then, taking into consideration the continuity of the function

$$\int_{\Omega_h} f(x)v_n^\lambda(|x-y|)dx$$

with respect to y on the closed domain $\overline{\Omega}$ and the fact that it is equal to the first term in the right-hand side of (4.10) we obtain following representation for the Riesz means (1.1) in the closed domain $\overline{\Omega}$

$$E_\lambda^s f(y) = \int_{\Omega_h} f(x)v_n^\lambda(|x-y|)dxdx + 2^s\Gamma(s+1)\lambda^{\frac{N-1}{4}-\frac{s}{2}} \sum_{n=1}^{\infty} f_n u_n(y)\lambda_n^{\frac{1-N}{4}} I_0(\lambda,\lambda_n) \tag{4.11}$$

We will study (4.11) when $\lambda \to \infty$. Thus, further let $\lambda \geq e$.

Lemma 4.2. *Uniformly with respect to* $y \in \overline{\Omega}$

$$\sum_{n=1}^{\infty} u_n^2(y)\ \lambda_n^{\frac{1-N}{2}}\ [I_m(\lambda,\lambda_n)]^2\ \leq\ C\ \ln^2\lambda, \tag{4.12}$$

where $\lambda \geq e$ *and* $C > 0$ *is independent of* λ *and* y.

Proof. Similarly to (4.9) from (2.1) it follows

$$\sum_{\lambda_n<\lambda} u_n^2(y)\lambda_n^{\varepsilon-\frac{N}{2}} = O(\lambda^\varepsilon \ln^2 \lambda). \tag{4.13}$$

uniformly with respect to $y \in \overline{\Omega}$. Here ε is an arbitrary positive number.

When $\sqrt{\lambda_n} \leq \frac{\sqrt{\lambda}}{2}$ from (4.9) we have $I_m(\lambda,\lambda_n) = O\big(\frac{1}{\sqrt{\lambda}}\big)$. Then from (4.12) with $\varepsilon = 1/2$ obtain the following estimate

$$\sum_{1\leq\ \sqrt{\lambda_n}\leq\frac{\sqrt{\lambda}}{2}} u_n^2(y)\ \lambda_n^{\frac{1-N}{2}}\ [I_m(\lambda,\lambda_n)]^2\ = O\big(\frac{\ln^2\lambda}{\sqrt{\lambda}}\big) \tag{4.14}$$

Similarly if $3\sqrt{\lambda} \leq 2\sqrt{\lambda_n}$ from (4.9) we have $I_m(\lambda,\lambda_n) = O\big(\frac{1}{\sqrt{\lambda_n}}\big)$. Then from (4.8) with $\varepsilon = 1/2$ obtain

$$\sum_{\sqrt{3\lambda}\leq 2\sqrt{\lambda_n}} u_n^2(y)\ \lambda_n^{\frac{1-N}{2}}\ [I_m(\lambda,\lambda_n)]^2\ = O\big(\frac{\ln^2\lambda}{\sqrt{\lambda}}\big) \tag{4.15}$$

The last interval for the eigenvalues $|\ \sqrt{\lambda_n}-\sqrt{\lambda}\ |\ \leq\ \frac{\sqrt{\lambda}}{2}$ we represent (see below) as union of the intervals $2^{m-1} \leq |\ \sqrt{\lambda_n}-\sqrt{\lambda}\ |\ \leq\ 2^m$. In the interval $2^{m-1} \leq |\ \sqrt{\lambda_n}-\sqrt{\lambda}\ |\ \leq\ 2^m$ from (4.9) we get $I_m(\lambda,\lambda_n) = O\big(\frac{1}{2^{m-1}}\big)$. Then from (2.1) obtain

$$\sum_{|\ \sqrt{\lambda_n}-\sqrt{\lambda}\ |\ \leq\ \frac{\sqrt{\lambda}}{2}} u_n^2(y)\ \lambda_n^{\frac{1-N}{2}}\ [I_m(\lambda,\lambda_n)]^2\ \leq$$

$$\leq c \sum_{m=1}^{k} \sum_{2^{m-1} \leq | \sqrt{\lambda_n} - \sqrt{\lambda} \ | \ \leq \ 2^m} u_n^2(y) \ \lambda_n^{\frac{1-N}{2}} \ 4^{1-m} \ \leq \ C \ ln^2 \lambda,$$

where k is the smallest number satisfying $2^{k+1} \geq \lambda$ and $C > 0$ is independent of λ. □

Lemma 4.3. *Let $f \in \overset{\circ}{H}{}_2^\alpha(\Omega)$, $\alpha = \ell + \kappa$, where ℓ is a non negative integer, $\kappa \in (0, 1]$.*
Then uniformly with respect to $y \in \overline{\Omega}$

$$\sum_{n=1}^{\infty} f_n u_n(y) \ \lambda_n^{\frac{\ell}{2} - \frac{N-1}{4}} \ I_\ell(\lambda, \lambda_n) \ = \ O\Big(\frac{\ln\lambda}{\lambda^{\frac{\kappa}{2}}}\Big) \|f\|_{H_2^\alpha}. \tag{4.16}$$

Proof. Note, that for any $\varepsilon > 0$ [3]

$$\sum_{\lambda < \lambda_n < 4\lambda} f_n^2 \ \lambda_n^{\alpha - \varepsilon} \ \leq \ c_\varepsilon \ \|f\|_{H_2^\alpha}^2, \tag{4.17}$$

where $c_\varepsilon > 0$ is independent of f and λ. Then from (4.9) and (4.13) it follows that

$$\sum_{1 < \lambda_n < \frac{\lambda}{4}} f_n u_n(y) \ \lambda_n^{\frac{\ell}{2} - \frac{N-1}{4}} \ I_\ell(\lambda, \lambda_n) \ \leq \ c \frac{\ln\lambda}{\lambda^{\frac{\kappa}{2}}} \|f\|_{H_2^\alpha}, \tag{4.18}$$

where $c > 0$ is independent of f and λ. If $\lambda_n > \frac{9\lambda}{4}$ instead of (4.13) we use (4.8) and obtain the estimate for such n.

Proof for the numbers n for which $\frac{\lambda}{4} < \lambda_n < \frac{9\lambda}{4}$ is as follows. Let k be the least natural number satisfying $2^{k+1} \geq \sqrt{\lambda}$. Then using (2.1), (4.9) and (4.17) we obtain

$$\Big| \sum_{\frac{\lambda}{4} < \lambda_n < \frac{9\lambda}{4}} f_n u_n(y) \ \lambda_n^{\frac{\ell}{2} - \frac{N-1}{4}} \ I_\ell(\lambda, \lambda_n) \ \Big| \leq$$

$$\leq \ \Big(\sum_{m=1}^{k} \sum_{2^{m-1} \leq |\sqrt{\lambda_n} - \sqrt{\lambda}| \leq 2^m} u_n^2(y) \ \lambda_n^{\frac{1-N}{2} - \kappa} \ [I_\ell(\lambda, \lambda_n)]^2\Big)^{\frac{1}{2}} \Big(\sum_{\frac{\lambda}{4} < \lambda_n < \frac{9\lambda}{4}} f_n^2 \ \lambda_n^{\alpha}\Big)^{\frac{1}{2}} \ \leq$$

$$\leq \ c \ \lambda^{\frac{\kappa}{2}} \ \ln\lambda \ \|f\|_{H_2^\alpha}.$$

where $c > 0$ is independent from f and λ. □

Let a function f belong to the class $C_0^\infty(\Omega)$ and let the support of this function be contained in Ω_h.

Denote by D the differential operator defined as

$$D\psi(r) = \frac{d}{dr}\Big[\frac{1}{r}\psi(r)\Big], \quad D^k \psi = D^{k-1}[D\psi]$$

Let $y \in \overline{\Omega}$ and $r \in (0, R)$, where $R < \frac{h}{4}$. Let the function $F(r)$ be defined as

$$F(r) = \frac{r^{N-1}}{\omega_N} \int_\theta f(y + r\theta) d\theta,$$

where ω_N is surface area of the unite sphere in R^N and the integral is taken over the sphere of radius r with the center at $x \in \Omega_r = \{y \in \Omega : \mathrm{dist}(y, \partial\Omega)\} > r$.

Denote $F_m(r) = \frac{1}{r}\, D^{m-1}F(r)$. Then (see [3])

$$F_m(r) = \frac{(2\pi)^{\frac{N}{2}}}{\omega_N}\, r^{\frac{N}{2}-m} \sum_{n=1}^{\infty} f_n\, u_n(y)\, \frac{J_{\frac{N}{2}-m}(r\sqrt{\lambda_n})}{\lambda_n^{\frac{N}{4}-\frac{m}{2}}}. \tag{4.19}$$

From the recurrent relations for the Bessel functions $J_\mu(t)$ we obtain

$$I_0(\lambda,\lambda_) \;=\; \Big(\frac{\lambda_n}{\lambda}\Big)^{\frac{\ell}{2}}\, I_\ell(\lambda,\lambda_n) \;+\; \Big(\frac{\lambda}{\lambda_n}\Big)^{\frac{1}{4}} \sum_{m=1}^{\ell} \Big(\frac{\lambda_n}{\lambda}\Big)^{m} J_{\frac{N}{2}+s-m}(R\sqrt{\lambda})\, J_{\frac{N}{2}-m}(R\sqrt{\lambda_n})\, R^{-s}. \tag{4.20}$$

In (4.11) we obtained a representation of Rieszz means (1.1) in the closed domain $\overline{\Omega}$ for any function $f\in L_2$. Transforming formula (4.11) by integrating by parts in the integral in the right-hand side and replacing $I_0(\lambda,\lambda_n)$ with the left-hand side of (4.20) we get

$$E_\lambda^s f(y) = 2^s\Gamma(s+1)\frac{\omega_N}{(2\pi)^{\frac{N}{2}}}\,\lambda^{\frac{N}{4}-\frac{s+\ell}{2}} \int_0^R \frac{J_{\frac{N}{2}+s-\ell}(r\sqrt{\lambda})}{r^{\frac{N}{2}+s-\ell}}\, D^\ell F(r)dr+$$

$$+\,2^s\Gamma(s+1)\,\lambda^{\frac{N-1}{4}-\frac{s+\ell}{2}} \sum_{n=1}^{\infty} f_n\, u_n(y)\lambda_n^{\frac{\ell}{2}-\frac{N-1}{4}}\, I_\ell(\lambda,\lambda_n)\,+$$

$$+2^s\Gamma(s+1)\frac{\omega_N}{(2\pi)^{\frac{N}{2}}}\,(R\sqrt{\lambda})^{-s}\sum_{m=1}^{\ell}\Big(\frac{\sqrt{\lambda}}{R}\Big)^{\frac{N}{2}-m}\Big[F_m(R)-\Big[\frac{1}{r}D^{m-1}F(r)\Big]\Big|_0^R\Big]$$

Note that the Fourier series the function $f\in C_0^\infty(\Omega)$ via the eigenfunctions of a converges uniformly and absolutely in the closed domain $\overline{\Omega}$. Thus the function $F_m(r)$ is continuous with respect to the variables y and r. Moreover, $F_m(r)$ tends to zero as $r\to 0$.

Then taking into account the last note, we obtain the following representation of the Riesz means

$$E_\lambda^s f(y) = 2^s\Gamma(s+1)\frac{\omega_N}{(2\pi)^{\frac{N}{2}}}\,\lambda^{\frac{N}{4}-\frac{s+\ell}{2}} \int_0^R \frac{J_{\frac{N}{2}+s-\ell}(r\sqrt{\lambda})}{r^{\frac{N}{2}+s-\ell}}\, D^\ell F(r)dr+$$

$$+\,2^s\Gamma(s+1)\,\lambda^{\frac{N-1}{4}-\frac{s+\ell}{2}} \sum_{n=1}^{\infty} f_n\, u_n(y)\lambda_n^{\frac{\ell}{2}-\frac{N-1}{4}}\, I_\ell(\lambda,\lambda_n). \tag{4.21}$$

5 Proof of Theorem 3.1

Proof. Let f be a continuous function with compact support in the domain Ω belonging to the space $\overset{\circ}{H}{}^\alpha_{pq}(\Omega)$ and (3.1) holds. Due to the density of $C_0^\infty(\Omega)$ in the space $\overset{\circ}{H}{}^\alpha_{pq}(\Omega)$, the statement of Theorem 3.1 follows from the inequality

$$|E_\lambda^s f(y)| \;\le\; c\big(\|f\|_{H^\alpha_{p,q}} \;+\; \|f\|_{C(\overline{\Omega})}\big). \tag{5.1}$$

It suffices to consider case $\alpha=\frac{N-1}{2}-s+\varepsilon$, where ε is a small positive number (smaller than κ).

If $2\le p<q$ from the embedding $H^\alpha_{p,q}\to H^\alpha_2$ and Lemma 4.3 we obtain the following estimate for the second term in the left-hand side of (4.21)

$$2^s\Gamma(s+1)\,\lambda^{\frac{N-1}{4}-\frac{s+\ell}{2}} \sum_{n=1}^{\infty} f_n\, u_n(y)\lambda_n^{\frac{\ell}{2}-\frac{N-1}{4}}\, I_\ell(\lambda,\lambda_n) \;=\; O\Big(\frac{\ln\lambda}{\lambda^{\frac{\kappa-\varepsilon}{2}}}\Big)\|f\|_{H^\alpha_{p,q}}. \tag{5.2}$$

Thus, for the Riesz means we have

$$E_\lambda^s f(y) = c_s \lambda^{\frac{2\kappa+1}{4}} \int_0^R \frac{J_\nu(r\sqrt{\lambda})}{r^\nu} D^\ell F(r)dr + O\Big(\frac{\ln\lambda}{\lambda^{\frac{\kappa-\varepsilon}{2}}}\Big)\|f\|_{H_{p,q}^\alpha}, \tag{5.3}$$

where $c_s = 2^s\Gamma(s+1)\frac{\omega_N}{(2\pi)^{\frac{N}{2}}}$ and $\nu = \frac{N}{2} + s - \ell$.

Note, as is shown in [3], there are constants $A_{m,\ell}$ such that

$$D^\ell F(r) = r^{N-1-2\ell} \sum_{m=0}^{\ell} A_{m,\ell} r^m \psi^{(m)}(r),$$

where $\psi(r) = \frac{1}{\omega_N}\int_\theta f(y + r\theta)d\theta$.

Then inequality (5.1) follows from (5.3) and the following estimate (see [15])

$$\int_0^R r^{m+\kappa-\frac{1}{2}} J_\nu(r\sqrt{\lambda})\psi^{(m)}dr \leq C\left(\|f\|_{H_{p,q}^\alpha} + \|f\|_{L_\infty}\right)\lambda^{\frac{-\kappa}{2}-\frac{1}{4}},$$

where $C > 0$ is independent of f and λ.

□

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