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ERLAN DAUTBEKOVICH NURSULTANOV

(to the 60th birthday)

On May 25, 2017 was the 60th birthday of Yerlan Dautbekovich Nursultanov, Doctor of Physical and Mathematical Sciences (1999), Professor (2001), Head of the Department of Mathematics and Informatics of the Kazakhstan branch of the M.V. Lomonosov Moscow State University (since 2001), member of the Editorial Board of the Eurasian Mathematical Journal.

E.D. Nursultanov was born in the city of Karaganda. He graduated from the Karaganda State University (1979) and then completed his postgraduate studies at the M.V. Lomonosov Moscow State University.

Professor Nursultanov's scientific interests are related to various areas of the theory of functions and functional analysis.

He introduced the concept of multi-parameter Lorentz spaces, network spaces and anisotropic Lorentz spaces, for which appropriate interpolation methods were developed. On the basis of the apparatus introduced by him, the questions of reiteration in the off-diagonal case for the real Lyons-Petre interpolation method, the multiplier problem for trigonometric Fourier series, the lower and upper bounds complementary to the Hardy-Littlewood inequalities for various orthonormal systems were solved. The convergence of series and Fourier transforms were studied with sufficiently general monotonicity conditions. The lower bounds for the norm of the convolution operator are obtained, and its upper bounds are improved (a stronger result than the O'Neil inequality). An exact cubature formula with explicit nodes and weights for functions belonging to spaces with a dominated mixed derivative is constructed, and a number of other problems in this area are solved.

He has published more than 50 scientific papers in high rating international journals included in the lists of Thomson Reuters and Scopus. 2 doctor of sciences, 9 candidate of sciences and 4 PhD dissertations have been defended under his supervision.

His merits and achievements are marked with badges of the Ministry of Education and Science of the Republic of Kazakhstan "For Contribution to the Development of Science" (2007), "Honored Worker of Education" (2011), "Y. Altynsarin" (2017). He is a laureate of the award named after K. Satpaev in the field of natural sciences for 2005, the grant holder "The best teacher of the university" for 2006 and 2011, the grant holder of the state scientific scholarship for outstanding contribution to the development of science and technology of the Republic of Kazakhstan for years 2007-2008, 2008 -2009. In 2017 he got the Top Springer Author award, established by Springer Nature together with JSC "National Center for Scientific and Technical Information".

The Editorial Board of the Eurasian Mathematical Journal congratulates Erlan Dautbekovich Nursultanov on the occasion of his 60th birthday and wishes him good health and successful work in mathematics and mathematical education.

JAMALBEK TUSSUPOV

(to the 60th birthday)

On April 10, 2017 was the 60th birthday of Jamalbek Tussupov, Doctor of Physical and Mathematical Sciences, Professor, Head of the Information Systems Department of the L.N. Gumilyov Eurasian National University, member of the Kazakhstan and American Mathematical Societies, member of the Association of Symbolic Logic, member of the Editorial Board of the Eurasian Mathematical Journal.

J. Tussupov was born in Taraz (Jambyl region of the Kazakh SSR). He graduated from the Karaganda State University (Kazakhstan) in 1979 and later on completed his postgraduate studies at S.L. Sobolev Institute of Mathematics of the Academy of Sciences of Russia (Novosibirsk).

Professor Tussupov's research interests are in mathematical logic, computability, computable structures, abstract data types, ontology, formal semantics. He solved the following problems of computable structures:

- the problems of S.S. Goncharov and M.S. Manasse: the problem of characterizing relative categoricity in the hyperarithmetical hierarchy given levels of complexity of Scott families, and the problem on the relationship between categoricity and relative categoricity of computable structures in the arithmetical and hyperarithmetical hierarchies;
- the problem of Yu.L. Ershov: the problem of finite algorithmic dimension in the arithmetical and hyperarithmetical hierarchies;
- the problem of C.J. Ash and A. Nerode: the problem of the interplay of relations of bounded arithmetical and hyperarithmetical complexity in computable presentations and the definability of relations by formulas of given complexity;
- the problem of S. Lempp: the problem of structures having presentations in just the degrees of all sets X such that for algebraic classes as symmetric irreflexive graphs, nilpotent groups, rings, integral domains, commutative semigroups, lattices, structure with two equivalences, bipartite graphs.

Professor Tussupov has published about 100 scientific papers, five textbooks for students and one monograph. Three PhD dissertations have been defended under his supervision.

Professor Tussupov is a fellow of "Bolashak" Scholarship, 2011 (Notre Dame University, USA), "Erasmus+", 2016 (Poitiers University, France). He was awarded the title "The Best Professor of 2012" (Kazakhstan). In 2015 Jamalbek Tussupov was also awarded for the contribution to science in the Republic of Kazakhstan.

The Editorial Board of the Eurasian Mathematical Journal congratulates Dr. Professor Jamalbek Tussupov on the occasion of his 60th aniversary and wishes him strong health, new achievements in science, inspiration for new ideas and fruitfull results.

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INVESTIGATION OF MATHEMATICAL MODELS OF ONE-PHASE STEFAN PROBLEMS WITH UNKNOWN NONLINEAR COEFFICIENTS

N.L. Gol'dman

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Key words: inverse Stefan problems, parabolic equations, phase boundary, uniqueness theorems, duality principle.

AMS Mathematics Subject Classification: 35K59, 35R30, 35R35.

Abstract. One-phase models of inverse Stefan problems with unknown temperature-dependent convection coefficients are considered. The final observation is considered as an additional information on the solution of the direct Stefan problem. For such inverse problems we justify the corresponding mathematical statements allowing to determine coefficients multiplying the lowest order derivatives in quasilinear parabolic equations in a one-phase domain with an unknown moving boundary. On the basis of the duality principle conditions for the uniqueness of their smooth solution are obtained. The proposed approach allows one to clarity a relationship between the uniqueness property for coefficient inverse Stefan problems and the density property of solutions of the corresponding adjoint problems. It is shown that this density property follows, in turn, from the known inverse uniqueness for linear parabolic equations.

1 Introduction

Inverse Stefan problems are inverse problems for parabolic equations in domains with free boundaries with material or energy balance conditions imposed on them. Such problems arise in the modeling and control of processes connected with heat and mass transfer. The goal is, by using some additional information, to determine the coefficients of the equation or the Stefan condition at the free boundary, the initial or boundary functions which must be given in the direct (classical) statement of the Stefan problem. Just like most of the inverse problems in mathematical physics, inverse Stefan problems are ill-posed. This is a result of the violation of the cause-effect relations in their statements. Unlike inverse problems for parabolic equations in domains with fixed boundaries, this class of ill-posed problems is unsufficiently studied, expecially for quasilinear equations and in the case when the time dependence of the moving boundary is unknown. The research on inverse Stefan problems is motivated by both theoretical interest in such formulations and by their numerous applications to thermophysics and mechanics of continuous media. The modern needs of technologies both in heat processes (e.g., metallurgy, astronautics, and power engineering) and in hydrology, exploitation of oil-gas fields, etc. lead to various formulations of inverse Stefan problems depending on the unknown model characteristic and the type of additional information, see, for example, [1–8].

This paper continues the investigation of quasilinear models of inverse Stefan problems begun in [9–11]. Such models arise, for example, in the modeling of the high temperature processes where it is necessary to take into account the dependence of thermophysical characteristics upon the temperature. In a thermophysical interpretation, the one-phase models of inverse

Stefan problems considered in this work consist of finding the temperature field, phase transition boundary (e.g. the melting front), and the temperature-dependent convection coefficient under the assumption that the temperature distribution and the phase boundary position are given at a final time. The corresponding mathematical formulation is to determine the unknown coefficient multiplying the lowest order derivative in a quasilinear parabolic equation in a one-phase domain whose external boundary is a phase front with an unknown time dependence. Additional information is given in the form of final overdetermination.

A characteristic feature of ill-posed inverse Stefan problems of this type is that they possibly do not have a solution or it is unstable with respect to errors in the input data [9]. However, if there exists a solution, it should be unique. In the present paper, we justify mathematical statements of the corresponding coefficient inverse Stefan problems and obtain sufficient conditions for the uniqueness of a solution in a class of smooth functions. To prove the uniqueness theorems we use the duality principle by analogy with [12], where it was applied to a parabolic equation with an unknown coefficient multiplying the lowest order derivative in a domain with fixed boundaries. For this end, the "straightening phase boundaries" substitution is carried out, which transforms the phase domain into a rectangular domain of fixed width. The proposed approach allows one to establish a relationship between the uniqueness property for inverse Stefan problems and the density property of solutions of the corresponding adjoint problems. These density properties follow, in turn, from the known inverse uniqueness for linear parabolic equations [13, 14].

The principles of constructing stable approximate solutions of ill-posed inverse Stefan problems are described in [9], and they are applicable to the coefficient inverse problems under study.

2 Justification of mathematical statements in Hölder classes

Suppose that the direct statement of a one-phase quasilinear Stefan problem consists of finding a function $u(x,t)$ in the domain $\overline{Q} = \{0 \le x \le \xi(t), 0 \le t \le T\}$ and a phase boundary $\xi(t)$ for $0 \le t \le T$ from the conditions

$$c(x,t,u)u_t - Lu = f(x,t), \quad (x,t) \in Q, \tag{2.1}$$

$$u|_{x=0} = v(t), \quad 0 < t \le T, \tag{2.2}$$

$$u|_{x=\xi(t)} = u^*(t), \quad 0 < t \le T, \tag{2.3}$$

$$u|_{t=0} = \varphi(x), \quad 0 \le x \le l_0, \tag{2.4}$$

$$a(x,t,u)u_x + \chi(x,t,u)|_{x=\xi(t)} = -\gamma(x,t,u)|_{x=\xi(t)}\xi_t(t), \quad 0 < t \le T, \tag{2.5}$$

$$\xi|_{t=0} = l_0, \quad l_0 > 0, \tag{2.6}$$

where Lu is a uniformly elliptic operator of the form

$$Lu \equiv (a(x,t,u)u_x)_x - b(x,t,u)u_x - d(x,t,u), \tag{2.7}$$

$a \ge a_{\min} > 0$, b, $c \ge c_{\min} > 0$, d, f, v, u^*, $\gamma \ge \gamma_{\min} > 0$, χ, and φ are known functions, $a_{\min}$, $c_{\min}$, $\gamma_{\min}$, and $l_0 = \text{const} > 0$.

If the function $b(x,t,u)$ in (2.7) is unknown but the additional information of the solution of the direct Stefan problem (2.1)–(2.6) is given at $t = T$

$$u|_{t=T} = g(x), \quad 0 \le x \le l, \quad \xi|_{t=T} = l, \quad l > 0, \tag{2.8}$$

then the following statement of a coefficient inverse problem with final overdetermination arises.

It is required to find a function $u(x,t)$ in the domain $\overline{Q}$, a phase boundary $\xi(t)$ for $0 \le t \le T$, and a coefficient $b(x,t,u)$ for $(x,t) \in \overline{Q}$ and $u \in [-M_0, M_0]$ (where $M_0 \ge \max_{(x,t)\in\overline{Q}} |u|$, M_0 is the constant from the maximum principle for the boundary value problem (2.1)–(2.4)) that satisfy conditions (2.1)–(2.7) and additional condition (2.8).

In what follows, we assume that $b(x,t,u)$ has one of the structures

$$\begin{aligned} b(x,t,u) &= p(u)b_0(x,t), \\ b(x,t,u) &= p(x,u)b_0(x,t) \end{aligned} \tag{2.9}$$

where $b_0(x,t)$ is a given function and p is an unknown coefficient.

By using the standard notation for function classes in [15], we formulate requirements on the input data, which imply the assumptions for the corresponding inverse Stefan problem.

(i) For $(x,t) \in \overline{Q}$, $|u| < \infty$, the functions a, a_x, a_u, b_0, c, d, and f are uniformly bounded, $a \ge a_{\min} > 0$, $c \ge c_{\min} > 0$.

(ii) For $(x,t,u) \in \overline{D} = \overline{Q} \times [-M_0, M_0]$ the function a, its derivatives a_x and a_u, and the functions c, d, γ, and χ belong to $H^{1,\lambda/2,1}(\overline{D})$; the functions b_0 and f belong to $H^{1,\lambda/2}(\overline{Q})$, $0 < \lambda < 1$, $\gamma \ge \gamma_{\min} > 0$.

(iii) The functions v, u^*, and φ belong to $H^{1+\lambda/2}[0,T]$ and $H^{2+\lambda}[0,l_0]$, respectively, and satisfy the matching conditions

$$\begin{aligned} c(x,0,\varphi)v_t - L\varphi|_{x=0,t=0} &= f(x,0)|_{x=0}, \\ c(x,0,\varphi)u_t^* - L\varphi|_{x=l_0,t=0} &= f(x,0)|_{x=l_0}. \end{aligned} \tag{2.10}$$

(iv) The input data provide the nondegeneracy of the domain $\overline{Q}$; i.e., the phase boundary does not intersect the external boundary $x = 0$: $\beta_0 < \xi(t)$ for $0 \le t \le T$, where $\beta_0 = \text{const} > 0$ (for details, see [9]).

(v) The final function g belongs to $H^{2+\lambda}[0,l]$ and satisfies the matching conditions $g|_{x=0} = v|_{t=T}$, $g|_{x=l} = u^*|_{t=T}$.

According to [9], for any coefficient p (see (2.9)) that belongs to the corresponding class

$$p(u) \in C^1[-M_0, M_0], \quad p(x,u) \in C^{1,1}(\overline{\Omega}),$$

$$\overline{\Omega} = [0, \beta_1] \times [-M_0, M_0], \quad \beta_1 = \max_{0 \le t \le T} \xi(t),$$

and satisfies the matching conditions (2.10), conditions (i)–(iv) ensure the unique solvability of the direct quasilinear Stefan problem (2.1)–(2.6) in the Hölder spaces $u(x,t) \in H^{2+\lambda,1+\lambda/2}(\overline{Q})$, $\xi(t) \in H^{1+\lambda/2}[0,T]$ and the fulfilment of the uniform estimates

$$|u|_{\overline{Q}}^{2+\lambda,1+\lambda/2} \le M, \quad |\xi|_{[0,T]}^{1+\lambda/2} \le \mathcal{M}, \quad M, \mathcal{M} = \text{const } > 0. \tag{2.11}$$

For this reason, we define a solution of the corresponding coefficient inverse Stefan problem as a collection of functions $\{u(x,t), \xi(t), p(u)\}$ or $\{u(x,t), \xi(t), p(x,u)\}$ that belong to the above-mentioned classes and satisfy relations (2.1)–(2.8) in the usual sense. For this ill-posed problem we eximane the conditions under which its solution (if it exists) is uniquely determined.

3 Uniqueness of the solution of the inverse Stefan problem with the unknown coefficient $p(u)$

3.1. Preliminaries. The suggested approach to the proof of the corresponding uniqueness conditions for the solution $\{u(x,t), \xi(t), p(u)\}$ (provided that it exists) is as follows.

Let $\{u_1, \xi_1, p_1\}$ and $\{u_2, \xi_2, p_2\}$ be two solutions of the inverse problem in the classes $H^{2+\lambda,1+\lambda/2}(\overline{Q}) \times H^{1+\lambda/2}[0,T] \times C^1[-M_0, M_0]$. The functions $\{u_1, \xi_1\}$ and $\{u_2, \xi_2\}$ can be treated as the solutions of the direct Stefan problem (2.1)–(2.6) that correspond to the coefficients p_1 and p_2 in the operator Lu (see (2.7) and (2.9)). Therefore, they satisfy estimates (2.11) in the Hölder classes $H^{2+\lambda,1+\lambda/2}(\overline{Q}) \times H^{1+\lambda/2}[0,T]$.

Before proving that $u_1(x,t) \equiv u_2(x,t)$ in $\overline{Q}$, $\xi_1(t) \equiv \xi_2(t)$ for $0 \le t \le T$, and $p_1(u) \equiv p_2(u)$ for $u \in [-M_0, M_0]$, we make "straightening phase boundary" substitution $y = x\xi^{-1}(t)$. This substitution transforms the phase domain $\overline{Q}$ into a rectangular domain of fixed width $\overline{\Pi} = \{0 \le y \le 1, 0 \le t \le T\}$.

In variables (y,t) the inverse Stefan problem (2.1)–(2.8) becomes

$$cu_t - \xi^{-2}(t)(au_y)_y + \xi^{-1}(t)\{pb_0 + cy\xi_t(t)\}u_y + d = f, \quad (y,t) \in \Pi, \tag{3.1}$$

$$u|_{y=0} = v(t), \quad u|_{y=1} = u^*(t), \quad 0 < t \le T, \tag{3.2}$$

$$u|_{t=0} = \varphi(yl_0), \quad \xi|_{t=0} = l_0, \quad 0 \le y \le 1, \tag{3.3}$$

$$\xi^{-1}(t)au_y + \chi|_{y=1} = -\gamma|_{y=1}\xi_t(t), \quad 0 < t \le T, \tag{3.4}$$

$$u|_{t=T} = g(yl), \quad \xi|_{t=T} = l, \quad 0 \le y \le 1. \tag{3.5}$$

The coefficients in the equation (3.1) and in the Stefan condition (3.4) are the values of the corresponding functions at the point $(y\xi(t), t, u)$. In view of (3.1)–(3.5) the differences $\Delta u = u_2 - u_1$, $\Delta\xi = \xi_2 - \xi_1$, and $\Delta p = p_2 - p_1$ satisfy relations that can be represented in the form

$$c\Delta u_t - \xi_2^{-2}(t)(a\Delta u_y)_y + \mathcal{A}\Delta u_y + \mathcal{B}\Delta u =$$

$$\mathcal{C}\Delta\xi(t) + \mathcal{D}\Delta\xi_t(t) - \xi_2^{-1}(t)b_0 u_{2y}\Delta p(u_2), \ (y,t) \in \Pi, \tag{3.6}$$

$$\Delta u|_{y=0} = 0, \quad \Delta u|_{y=1} = 0, \quad 0 < t \le T, \tag{3.7}$$

$$\Delta u|_{t=0} = 0, \quad 0 \le y \le 1, \tag{3.8}$$

$$\xi_2^{-1}(t)a\Delta u_y|_{y=1} = -\gamma|_{y=1}\Delta\xi_t(t) + \mathcal{F}|_{y=1}\Delta\xi(t), \quad 0 < t \le T, \quad \Delta\xi|_{t=0} = 0, \tag{3.9}$$

with additional conditions at $t = T$

$$\Delta u|_{t=T} = 0, \quad 0 \le y \le 1, \quad \Delta\xi|_{t=T} = 0. \tag{3.10}$$

Here a, b_0, c, γ, etc., are the values of these functions at the point $(y\xi_2(t), t, u_2)$. The coefficients $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{F}$ depend appropriately on u_2, its derivatives u_{2y}, u_{2yy}, and u_{2t}. Moreover, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{F}$ depend appropriately on the y- and u-derivatives of the coefficients in the equation (3.1) and the Stefan condition (3.4) at the intermediate point $(y\xi(t), t, u)$ with $\xi(t) = \sigma\xi_1(t) + (1-\sigma)\xi_2(t)$ and $u = \theta u_1 + (1-\theta)u_2$ for $0 < \sigma < 1$ and $0 < \theta < 1$. All these coefficients regarded as functions of (y,t) are in $H^{\lambda,\lambda/2}$ in the domain $\overline{\Pi} = \{0 \le y \le 1, 0 \le t \le T\}$ in view of smoothness conditions (i)–(iii) and estimates (2.11) in the Hölder classes. In particular, the coefficient $\mathcal{A}(y,t)$ has the form $\mathcal{A}(y,t) = \xi^{-1}(t)\{p_2 b_0 + cy\xi_t(t) - a_u u_y\}$ and is in $H^{\lambda,\lambda/2}(\overline{\Pi})$ in view of condition (ii) on b_0, c, and a_u, estimates (2.11) for u_1, u_2, and since $p_2 \in C^1[-M_0, M_0]$.

3.2. The duality principle and properties of adjoint problems. Let us proceed to the proof of the assertion that $\Delta u \equiv 0$ in $\overline{\Pi}$, $\Delta\xi \equiv 0$ for $0 \le t \le T$, and $\Delta p \equiv 0$ for $u \in [-M_0, M_0]$.

For this end, we use the duality principle by analogy with [12], where it was applied for the coefficient inverse problem in a domain with fixed boundary.

Namely, we remark that the relations (3.6)–(3.8) are linear with respect to Δu, $\Delta\xi$, and Δp. This allows one to start with the study of the corresponding boundary value problem for the equation

$$c\Delta u_t - \mathcal{L}\Delta u = -\xi_2^{-1}(t) b_0 u_{2y} \Delta p(u_2), \ (y,t) \in \Pi, \tag{3.11}$$
$$\mathcal{L}\Delta u \equiv \xi_2^{-2}(t)(a\Delta u_y)_y - \mathcal{A}\Delta u_y - \mathcal{B}\Delta u.$$

Consider the boundary value problem adjoint to (3.11), (3.7), (3.8),

$$(c\psi)_t + \mathcal{L}^*\psi = 0, \quad 0 < y < 1, \quad 0 \le t < T, \tag{3.12}$$

$$\psi|_{y=0} = 0, \quad \psi|_{y=1} = 0, \quad 0 \le t < T, \tag{3.13}$$

$$\psi|_{t=T} = \eta(y), \quad 0 \le y \le 1, \tag{3.14}$$

where $\eta(y)$ is an arbitrary function from $\overset{0}{C}[0,1]$ and

$$\mathcal{L}^*\psi \equiv \xi_2^{-2}(t)(a\psi_y)_y + (\mathcal{A}\psi)_y - \mathcal{B}\psi$$

is the operator adjoint to the operator $\mathcal{L}\Delta u$.

The solution of this linear boundary value problem is defined by $\psi(y,t;\eta)$. Next we investigate the properties of $\psi(y,t;\eta)$.

Lemma 3.1. *Let conditions* (i)–(v) *hold and, moreover, the derivative* c_t *be in* $H^{\lambda,\lambda/2,1}(\overline{D})$, *the derivative* b_{0x} *be in* $H^{\lambda,\lambda/2}(\overline{Q})$. *Then, for any function* $\eta(y) \in \overset{0}{C}[0,1]$, *the corresponding solution* $\psi(y,t;\eta)$ *of adjoint problem* (3.12)–(3.14) *belongs to* $C(\overline{\Pi}) \cap C^{2,1}(\Pi)$ *and satisfies the relation*

$$\int_0^T\int_0^1 \psi(y,t;\eta)h(y,t)\,dy\,dt = 0 \quad \forall \eta \in \overset{0}{C}[0,1], \qquad h(y,t) = -\xi_2^{-1}(t) b_0 u_{2y} \Delta p(u_2). \tag{3.15}$$

Proof. Unique solvability of problem (3.12)–(3.14) in $C(\overline{\Pi}) \cap C^{2,1}(\Pi)$ for any $\eta \in \overset{0}{C}[0,1]$ follows from [15] thanks to the corresponding smoothness of the coefficients in the equation (3.12); in particular, y-derivative of the coefficient $\mathcal{A}(y,t)$ belongs to $H^{\lambda,\lambda/2}(\overline{\Pi})$.

To prove (3.15) we consider the expression

$$I = \int_0^T\int_0^1 \psi\{c\Delta u_t - \mathcal{L}\Delta u\}\,dy\,dt + \int_0^T\int_0^1 \Delta u\{(c\psi)_t + \mathcal{L}^*\psi\}\,dy\,dt.$$

On the one hand, from (3.11) and (3.12) it follows that

$$I = \int_0^T\int_0^1 \psi(y,t;\eta)h(y,t)\,dy\,dt.$$

On the other hand, integrating by parts and taking into account (3.7), (3.8) and (3.13), (3.14), and final condition (3.10) for $\Delta u|_{t=T}$, we obtain

$$I = \int_0^1 \{c\psi\Delta u\}\Big|_{t=0}^{t=T} dy = 0,$$

which yields relation (3.15). □

It should be noted that the condition $\Delta u|_{t=T} = 0$ is just what $\eta(y)$ in (3.14) can be an arbitrary function from $\overset{0}{C}[0,1]$. As a result, adjoint problem (3.12)–(3.14) have the same properties as a control problem with a control function in the initial condition. The role of this function is played by $\eta(y)$. The change of variable $t' = T - t$ in (3.12)–(3.14) gives a usual control problem for a linear parabolic equation.

The following lemmas show that the function $\psi(y,t;\eta)$ possesses density properties (by analogy with a solution of the control problem).

Lemma 3.2. *Let the conditions of Lemma* 3.1 *be satisfied; in addition, let the derivative a_t be continuous in the domain $\overline{D}$. Then, as the function $\eta(y)$ ranges over the space $\overset{0}{C}[0,1]$, the corresponding set of values $\{\psi(y,t;\eta)|_{t=\tau}\}$ is everywhere dense in $L_2[0,1]$ at any time $t=\tau$; i.e., the relation*

$$\int_0^1 \psi(y,t;\eta)|_{t=\tau}\, w(y)\, dy = 0, \quad 0 < \tau \le T,$$

for some function $w(y) \in \overset{0}{C}[0,1]$ implies that $w(y) = 0$ for $0 \le y \le 1$.

Proof. To prove Lemma 3.2 we again use the duality principle but now for problem (3.12)–(3.14). Namely, we consider the linear boundary value problem adjoint to (3.12)–(3.14) in the domain $\overline{\Pi}_\tau = \{0 \le y \le 1, \tau \le t \le T\}$

$$cz_t - \mathcal{L}z = 0, \quad 0 < y < 1, \quad \tau < t \le T, \tag{3.16}$$

$$z|_{y=0} = 0, \quad z|_{y=1} = 0, \quad \tau < t \le T, \tag{3.17}$$

$$z|_{t=\tau} = \theta(y;\tau), \quad 0 \le y \le 1, \tag{3.18}$$

where the operator $\mathcal{L}z$ has the same form as $\mathcal{L}\Delta u$ and

$$\theta(y;\tau) = \{c(y\xi_2(t),t,u_2)|_{t=\tau}\}^{-1} w(y).$$

Its solution $z(y,t;\tau)$ belongs to $\overset{0}{C}(\overline{\Pi}_\tau) \cap C^{2,1}(\Pi_\tau)$ and is a continuous function of the parameter τ in view of its stability with respect to the input data [15]. For it we obtain the additional final condition $z(y,t;\tau)|_{t=T} = 0$ with the use of the continuous function

$$F(\tau) = \int_\tau^T \int_0^1 z\{(c\psi)_t + \mathcal{L}^*\psi\}\, dy\, dt + \int_\tau^T \int_0^1 \psi\{cz_t - \mathcal{L}z\}\, dy\, dt.$$

In fact, by virtue of (3.12)–(3.14) and (3.16)–(3.18), $F(\tau)$ can be reduced to the form

$$F(\tau) = \int_0^1 c|_{t=T} z(y,T;\tau)\eta(y)\, dy - \int_0^1 c|_{t=\tau}\theta(y;\tau)\psi(y,\tau;\eta)\, dy = 0 \tag{3.19}$$

for any $\eta \in \overset{0}{C}[0,1]$. From here, taking into account the form of $\theta(y;\tau)$ and the assertion about $w(y)$, we conclude that $z(y,t;\tau)|_{t=T} = 0$ (thanks to the assumption $c \ge c_{\min} > 0$ and density of the space $\overset{0}{C}[0,1]$ in $L_2[0,1]$).

This final condition permits one to treat equation (3.16) with conditions (3.17) as a homogeneous boundary value problem for a linear parabolic equation in inverse time. By smoothness and uniform boundedness in $\overline{\Pi}_\tau$, the coefficients of equation (3.16) considered as functions of (y,t) satisfy the requirements [13, 14] that provide the so-called inverse uniqueness property for such a problem. Hence $z(y,t;\tau) \equiv 0$ in $\overline{\Pi}_\tau$ including $t=\tau$; i.e., $\theta(y;\tau)=0$ and $w(y)=0$ for $0 \le y \le 1$. Thus, the fact that the set $\{\psi(y,t;\eta)|_{t=\tau}\}$ is dense follows from the inverse uniqueness property. □

The following result is a generalization of Lemma 3.2 for an arbitrary time interval $[0,T_0]$, $0 < T_0 \le T$.

Lemma 3.3. *Let the conditions of Lemma 3.2 for the input data hold. Assume that for any function* $\eta \in \overset{0}{C}[0,1]$, *the corresponding solution* $\psi(y,t;\eta)$ *of the adjoint problem satisfies the relation on some interval* $[0,T_0]$, $0<T_0\le T$,

$$\int_0^{T_0}\int_0^1 \psi(y,t;\eta)\,\alpha(y,t)\,dy\,dt = 0 \quad \forall \eta \in \overset{0}{C}[0,1], \tag{3.20}$$

where $\alpha(y,t)$ *is a function of constant signs with respect to* $t\in[0,T]$ *and, moreover,* $\alpha(y,t)$ *is in* $H^{\lambda,\lambda/2}(\overline{\Pi})$. *Then* $\alpha(y,T_0)=0$ *for* $0\le y\le 1$.

Proof. Just as in the proof of Lemma 3.2, consider problem (3.16)–(3.18) in the domain $\overline{\Pi}_\tau$ but for $\theta(y;\tau)$ of the form

$$\theta(y;\tau) = \left\{c(y\xi_2(t),t,u_2)|_{t=\tau}\right\}^{-1}\alpha(y,\tau)$$

and for all τ such that $0\le\tau\le T_0$.

The function $F(\tau)$ (see (3.19)) satisfies the relation

$$\int_0^{T_0} F(\tau)\,d\tau = \int_0^1\int_0^{T_0} z(y,T;\tau)\,d\tau\, c|_{t=T}\eta(y)\,dy - \int_0^{T_0}\int_0^1 \psi(y,\tau;\eta)c|_{t=\tau}\theta(y;\tau)\,dy\,d\tau = 0.$$

In view of the form of $\theta(y;\tau)$ this means (together with (3.20), the arbitrary choice of the function $\eta(y)$, and positiveness of the coefficient c) that

$$\int_0^{T_0} z(y,T;\tau)\,d\tau = 0, \quad 0\le y\le 1,$$

where the integrand $z(y,T;\tau)$ is the solution of problem (3.16)–(3.18) at the final time $t=T$. By using Green's function $G(y,x,t,\tau)$ [15] for representation of the solution $z(y,t;\tau)$ of this problem, we obtain

$$\int_0^{T_0} z(y,T;\tau)\,d\tau = \int_0^{T_0}\int_0^1 G(y,x,T,\tau)\theta(x;\tau)\,dx\,d\tau = 0, \quad 0\le y\le 1.$$

We can write this equality in the form

$$\int_0^{T}\int_0^1 G(y,x,T,\tau)\Theta(x;\tau)\,dx\,d\tau = 0, \quad 0\le y\le 1, \tag{3.21}$$

where $\Theta(x;\tau) = \begin{cases} \theta(x;\tau) & \text{for } 0<\tau\le T_0, \\ 0 & \text{for } T_0<\tau\le T. \end{cases}$

Now we consider the boundary value problem in the domain $\overline{\Pi} = \{0 \le y \le 1, 0 \le t \le T\}$ for the nonhomogeneous equation

$$cZ_t - \mathcal{L}Z = \Theta(y,\tau), \quad 0 < y < 1, \quad 0 < t \le T, \tag{3.22}$$

$$Z|_{y=0} = 0, \quad Z|_{y=1} = 0, \quad 0 < t \le T, \tag{3.23}$$

$$Z|_{t=0} = 0, \quad 0 \le y \le 1, \tag{3.24}$$

and show that its solution $Z(y,t)$ is a smooth function in $\overline{\Pi}$.

In fact, for $0 < y < 1$, $0 < t \le T_0$ we have $\Theta(y,t) = \theta(y,t)$ and $\theta(y,t) \in H^{\lambda,\lambda/2}$, hence $Z(y,t)$ belongs to $C^{2,1}$ for such values of y and t [15]. On the other hand, for $T_0 < t \le T$ the function $\Theta(y,t) = 0$. This means that for $T_0 < t \le T$ $Z(y,t)$ can be represented as a solution $z(y,t;T_0)$ of the boundary value problem in the domain $\overline{\Pi}_{T_0} = \{0 \le y \le 1, T_0 \le t \le T\}$ for the homogeneous equation

$$cz_t - \mathcal{L}z = 0, \quad 0 < y < 1, \quad T_0 < t \le T,$$

with the homogeneous boundary conditions at $y = 0$, $y = 1$, and with the initial condition

$$z|_{t=T_0} = Z(y,T_0), \quad 0 \le y \le 1,$$

where $Z(y,T_0)$ is a solution of problem (3.22)–(3.24) obtained at $t = T_0$. Since $Z(y,T_0) \in \overset{0}{C} [0,1] \cap C^2(0,1)$ then $z(y,t;T_0)$ belongs to $\overset{0}{C} (\overline{\Pi}_{T_0}) \cap C^{2,1}(\Pi_{T_0})$ [15]. This allows one to conclude that $Z(y,t)$ also belongs to $\overset{0}{C} (\overline{\Pi}_{T_0}) \cap C^{2,1}(\Pi_{T_0})$ as $Z(y,t)$ coincides with $z(y,t;T_0)$ in this domain. Thus, the solution $Z(y,t)$ of problem (3.22)–(3.24) is continuous everywhere in the domain $\overline{\Pi} = \{0 \le y \le 1, 0 \le t \le T\}$, and $Z(y,t)$ belongs to $C^{2,1}$ in the above-mentioned subdomains of this domain.

Since equality (3.21) is a representation of this solution at the final time $t = T$ [15], then from (3.21) it follows that $Z(y,T) = 0$ for $0 \le y \le 1$. But $Z(y,T) = z(y,T;T_0)$, hence $z(y,T;T_0)$ is also equal to 0 for $0 \le y \le 1$. Thus, in the domain $\overline{\Pi}_{T_0}$ the solution of the homogeneous equation with the homogeneous boundary conditions satisfies the final condition $z(y,t;T_0)|_{t=T} = 0$ for $0 \le y \le 1$. Just as in the proof of Lemma 3.2 we can use results of [13, 14] on the inverse uniqueness property; i.e., $z(y,t;T_0) \equiv 0$ in $\overline{\Pi}_{T_0}$. Then it follows from the initial condition $z|_{t=T_0} = Z(y,T_0)$ that $Z(y,T_0) = 0$ for $0 \le y \le 1$. But $Z(y,T_0)$ satisfies nonhomogeneous equation (3.22) with the right hand side $\Theta(y,t) = \theta(y,t)$ for $t = T_0$. Hence, $\theta(y,T_0) = 0$ for $0 \le y \le 1$. This means (see the form of the function $\theta(y,t)$) that $\alpha(y,T_0) = 0$ for $0 \le y \le 1$. □

3.3. Conditions of unique identification of $p(u)$**.** The density properties for adjoint problem (3.12)–(3.14) established with the help of the duality principle permit one to investigate the uniqueness of a solution of inverse Stefan problem (2.1)–(2.8) with an unknown coefficient $p(u)$.

Theorem 3.1. *Let the following conditions be satisfied.*

1. *Assumptions* (i)–(v) *hold for the input data; in addition, the coefficient* b_0 *is positive for* $(x,t) \in \overline{Q}$*, the derivatives* a_t*,* c_t*, and* b_{0x} *belong to* $C(\overline{D})$*,* $H^{\lambda,\lambda/2,1}(\overline{D})$*, and* $H^{\lambda,\lambda/2}(\overline{Q})$*, respectively; the derivative of the final function* $g(x)$ *is a sign-definite function:* $|g_x(x)| > 0$ *for* $0 \le x \le l$*.*

2. *There exists a solution* $\{u(x,t), \xi(t), p(u)\}$ *of the considered inverse Stefan problem possessing the properties*

$$u(x,t) \in H^{2+\lambda,1+\lambda/2}(\overline{Q}), \quad p(u) \in C^1[-M_0, M_0], \quad 0 < \lambda < 1,$$

$$u(x,t)_x \text{ is a function of constant signs with respect to } t \in [0,T],$$
$$\xi(t) \in H^{1+\lambda/2}[0,T], \quad 0 < \beta_0 < \xi(t) \le \beta_1 \text{ for } 0 \le t \le T,$$

and satisfying relations (2.1)–(2.7)*, the final observation* (2.8)*, and matching conditions* (2.10)*.*

Then this solution is unique in the mentioned classes of smooth functions under one of the following conditions

(j) *$p(u)$ is defined for $u \in [-M_0, g_{\min})$ and $u \in (g_{\max}, M_0]$, where $g_{\min} = \min_{0\le x\le l} g(x)$ and $g_{\max} = \max_{0\le x\le l} g(x)$,*

(jj) *$p(u)$ is an analytic function for $u \in (-M_0, M_0)$.*

Proof. To prove this theorem, first we consider equation (3.11) with conditions (3.7), (3.8) and corresponding adjoint boundary value problem (3.12)–(3.14). The assumptions on the input data allow one to apply Lemma 3.3 to integral relation (3.15) of Lemma 3.1 with $\alpha(y,t) = h(y,t)$, where we propose that Δp is a function of constant signs with respect to $u \in [M_0, M_0]$ (see the form of the function $h(y,t)$). Hence, we conclude that

$$\{\xi_2^{-1}(t) b_0 u_{2y} \Delta p(u_2)\}\Big|_{t=T} = 0, \quad 0 \le y \le 1.$$

Since $\xi(t)|_{t=T} = l$ then taking into account this fact and the inequalities $b_0(yl,T) > 0$ and $|g_y(yl)| > 0$ for $0 \le y \le 1$, we obtain $\Delta p(g(yl)) = 0$ for $0 \le y \le 1$. Since the function $g(x)$ is continuous for $0 \le x \le l$, we have $\Delta p(g) = 0$ for $g \in [g_{\min}, g_{\max}]$. Under either of assumptions (j) and (jj), this means that $\Delta p(u) = 0$ for $u \in [-M_0, M_0]$. Then equation (3.11) together with conditions (3.7), (3.8) implies $\Delta u(x,t) \equiv 0$ in $\overline{Q}$ (in variables (x,t)) [15].

Now we return to equation (3.6) and consider its other linear part, namely

$$c\Delta u_t - \xi_2^{-2}(t)(a\Delta u_y)_y + \mathcal{A}\Delta u_y + \mathcal{B}\Delta u = \mathcal{C}\Delta\xi(t) + \mathcal{D}\Delta\xi_t(t), \quad (y,t) \in \Pi. \tag{3.25}$$

But from equation (3.25) and relations (3.7)–(3.9) it follows that $\Delta u(x,t) \equiv 0$ in $\overline{Q}$ (in variables (x,t)), $\Delta\xi(t) \equiv 0$ for $0 \le t \le T$ since the direct quasilinear Stefan problem (2.1)–(2.7) with the coefficient $b = p(u)b_0(x,t)$ has a unique solution (see [9]).

Thus, results obtained for equations (3.11) and (3.25) with the corresponding boundary and initial conditions allow one to complete the proof of Theorem 3.1. □

4 Uniqueness of the solution of the inverse Stefan problem with the unknown coefficient $p(x,u)$

Conditions for the uniqueness of $\{u(x,t), \xi(t), p(x,u)\}$ are established by the following theorem.

Theorem 4.1. *Let Assumption* 1 *of Theorem* 3.1 *hold. In addition, suppose that there exists a solution $\{u(x,t), \xi(t), p(x,u)\}$ satisfying relations* (2.1)–(2.7)*, final observation* (2.8)*, and matching conditions* (2.10) *and having the properties*

$$u(x,t) \in H^{2+\lambda,1+\lambda/2}(\overline{Q}), \quad \xi(t) \in H^{1+\lambda/2}[0,T], \quad p(x,u) \in C^{1+\lambda,1}(\overline{\Omega}), \quad 0 < \lambda < 1,$$

$$u(x,t)_x \text{ is a function of constant signs with respect to } t \in [0,T],$$
$$0 < \beta_0 < \xi(t) \le l_0 = \beta_1 \text{ for } 0 \le t \le T, \quad \overline{\Omega} = [0,\beta_1] \times [-M_0, M_0].$$

Then this solution is unique in the mentioned classes of smooth functions under one of the following conditions (jjj) *$p(x,u)$ is defined in $\overline{\Omega}$ outside the domain $\{(x,u) : 0 \le x \le l, g_{\min} \le u \le g_{\max}\}$, where $g_{\min} = \min_{0\le x\le l} g(x)$ and $g_{\max} = \max_{0\le x\le l} g(x)$,*

(jv) *$p(x,u)$ is an analytic function in the domain $\overline{\Omega}$.*

Proof. The proof of these claims is similar to that of Theorem 3.1. In particular, an analogue of equation (3.6) is given by the equation

$$c\Delta u_t - \xi_2^{-2}(t)(a\Delta u_y)_y + \mathcal{A}\Delta u_y + \mathcal{B}\Delta u =$$

$$\mathcal{C}\Delta\xi(t) + \mathcal{D}\Delta\xi_t(t) - \xi_2^{-1}(t)b_0 u_{2y}\Delta p(y\xi_2(t), u_2),\ \ (y,t) \in \Pi. \tag{4.1}$$

Hence, the corresponding form of equation (3.11) becomes

$$c\Delta u_t - \mathcal{L}\Delta u = -\xi_2^{-1}(t)b_0 u_{2y}\Delta p(y\xi_2(t), u_2),\ \ (y,t) \in \Pi. \tag{4.2}$$

Next, taking into account the assumptions of Theorem 4.1 on the input data and the solution of this inverse problem, we can apply Lemma 3.3 with $\alpha(y,t) = h(y,t)$ to the integral relation (3.15) of Lemma 3.1. Now the function $h(y,t)$ has the form

$$h(y,t) = -\xi_2^{-1}(t)b_0 u_{2y}\Delta p(y\xi_2(t), u_2),$$

where $y\xi_2(t) \in [0, l_0]$ for any $t \in [0,T]$ and Δp is a function of constant signs with respect to $u \in [M_0, M_0]$. This leads to

$$\{\xi_2^{-1}(t)b_0 u_{2y}\Delta p(y\xi_2(t), u_2)\}|_{t=T} = 0, \quad 0 \le y \le 1.$$

From here it follows that $\Delta p(yl, g(yl)) = 0$ for $0 \le y \le 1$ since $\xi(T) = l$, $b_0(yl,T) > 0$, and $|g_y(yl)| > 0$ for $0 \le y \le 1$. This, together with the continuity of the final function $g(x)$, implies that $\Delta p(x,g) \equiv 0$ for $0 \le x \le l$, $g \in [g_{\min}, g_{\max}]$. Hence, any of assumptions (iii) and (iv) allows one to conclude that $\Delta p(x,u) \equiv 0$ in the entire domain $\overline{\Omega}$. But this means that the equation (4.2) with conditions (3.7), (3.8) have a unique solution $\Delta u(x,t) \equiv 0$ in $\overline{Q}$ (in variables (x,t)) [15].

Investigation of the other linear part of the equation (4.1) completely repeats the corresponding claims for equation (3.25) and implies identities $\Delta u(x,t) \equiv 0$ in $\overline{Q}$ (in variables (x,t)), $\Delta\xi(t) \equiv 0$ for $0 \le t \le T$ since direct quasilinear Stefan problem (2.1)–(2.7) with the coefficient $b = p(x,u)b_0(x,t)$ has a unique solution (see [9]). □

Remark. The function spaces chosen for the input data and the solution $\{u, \xi, p\}$ of the considered inverse Stefan problems are natural in the sense that they are associated with the exact differential dependences in Hölder classes for the corresponding direct statement of one-phase quasilinear Stefan problem (2.1)–(2.7) [9]. However, if the set of admissible solutions is expanded by assuming that the desired coefficient p in (2.9) also depends on the variable t, the uniqueness property may be lost. This is illustrated by the following example.

Example. Two function sets

$$\begin{cases} u_1(x,t) &= x(2-t)(x+t^2), \\ \xi_1(t) &= 2-t^2, \\ p_1(x,t,u) &= \frac{u+2(t-2)+x(4t-3t^2-x)}{(t-2)(2x+t^2)}, \end{cases}$$

$$\begin{cases} u_2(x,t) &= x(2-t^2)(x+t), \\ \xi_2(t) &= 2-t, \\ p_2(x,t,u) &= \frac{u+2(t^2-2)+x(2-3t^2-2xt)}{(t^2-2)(2x+t)}, \end{cases}$$

are solutions of the following coefficient inverse Stefan problem in the domain $\overline{Q} = \{0 \le x \le \xi(t), 0 \le t \le 1\}$:

$$u_t - u_{xx} + p(x,t,u)u_x + u = 0, \quad (x,t) \in Q,$$

$$u|_{x=0} = 0, \quad u|_{x=\xi(t)} = 2(2-t^2)(2-t), \quad 0 < t \le 1,$$
$$u|_{t=0} = 2x^2, \quad 0 \le x \le 2, \quad \xi|_{t=0} = 2,$$
$$u_x + \chi(x,t)|_{x=\xi(t)} = \xi_t(t), \quad 0 < t \le 1,$$
$$u|_{t=1} = x(x+1), \quad 0 \le x \le 1, \quad \xi|_{t=1} = 1,$$

where the function $\chi(x,t)|_{x=\xi(t)}$ has the form

$$\chi(x,t)|_{x=\xi(t)} = (2t-1)\frac{\xi(t)-(2-t)}{t(t-1)} - 1 - (\xi(t)+2)(4-t^2-t-\xi(t)).$$

Therefore, the function sets

$$\{u(x,t), \xi(t), p(u)\} \in H^{2+\lambda,1+\lambda/2}(\overline{Q}) \times H^{1+\lambda/2}[0,T] \times C^1[-M_0, M_0],$$
$$\{u(x,t), \xi(t), p(x,u)\} \in H^{2+\lambda,1+\lambda/2}(\overline{Q}) \times H^{1+\lambda/2}[0,T] \times C^{1+\lambda,1}(\overline{\Omega})$$

form natural sets of admissible solutions in the corresponding statements of coefficient inverse Stefan problems.

5 Conclusions

The statements of one-phase inverse Stefan problems on the identification of nonlinear coefficients are investigated under the assumption that additional information is given in the form of final overdetermination. The following results of this analysis can be formulated.

1. The choice of function spaces for the input data and the solution of such inverse problems relies on unique solvability of the corresponding direct Stefan problems in Hölder classes.

2. For these statements the conditions ensuring the uniqueness of a solution (if it exists in the chosen spaces) are obtained. To this end on the basis of the duality principle, a relationship is proved between the uniqueness property for the considered inverse problems and the density properties for the corresponding adjoint problems. It is shown that such density properties follow, in turn, from the so-called inverse uniqueness for linear parabolic equations.

3. The sets of admissible solutions preserving the uniqueness property are indicated. It is shown that this property may be lost if the desired nonlinear coefficient also depends on the variable t.

4. Investigation of the uniqueness property for coefficient inverse Stefan problems is important not only for theory but also for mathematical modeling and numerical solving of complicated nonstationary processes.

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