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ERLAN DAUTBEKOVICH NURSULTANOV

(to the 60th birthday)

On May 25, 2017 was the 60th birthday of Yerlan Dautbekovich Nursultanov, Doctor of Physical and Mathematical Sciences (1999), Professor (2001), Head of the Department of Mathematics and Informatics of the Kazakhstan branch of the M.V. Lomonosov Moscow State University (since 2001), member of the Editorial Board of the Eurasian Mathematical Journal.

E.D. Nursultanov was born in the city of Karaganda. He graduated from the Karaganda State University (1979) and then completed his postgraduate studies at the M.V. Lomonosov Moscow State University.

Professor Nursultanov's scientific interests are related to various areas of the theory of functions and functional analysis.

He introduced the concept of multi-parameter Lorentz spaces, network spaces and anisotropic Lorentz spaces, for which appropriate interpolation methods were developed. On the basis of the apparatus introduced by him, the questions of reiteration in the off-diagonal case for the real Lyons-Petre interpolation method, the multiplier problem for trigonometric Fourier series, the lower and upper bounds complementary to the Hardy-Littlewood inequalities for various orthonormal systems were solved. The convergence of series and Fourier transforms were studied with sufficiently general monotonicity conditions. The lower bounds for the norm of the convolution operator are obtained, and its upper bounds are improved (a stronger result than the O'Neil inequality). An exact cubature formula with explicit nodes and weights for functions belonging to spaces with a dominated mixed derivative is constructed, and a number of other problems in this area are solved.

He has published more than 50 scientific papers in high rating international journals included in the lists of Thomson Reuters and Scopus. 2 doctor of sciences, 9 candidate of sciences and 4 PhD dissertations have been defended under his supervision.

His merits and achievements are marked with badges of the Ministry of Education and Science of the Republic of Kazakhstan "For Contribution to the Development of Science" (2007), "Honored Worker of Education" (2011), "Y. Altynsarin" (2017). He is a laureate of the award named after K. Satpaev in the field of natural sciences for 2005, the grant holder "The best teacher of the university" for 2006 and 2011, the grant holder of the state scientific scholarship for outstanding contribution to the development of science and technology of the Republic of Kazakhstan for years 2007-2008, 2008 -2009. In 2017 he got the Top Springer Author award, established by Springer Nature together with JSC "National Center for Scientific and Technical Information".

The Editorial Board of the Eurasian Mathematical Journal congratulates Erlan Dautbekovich Nursultanov on the occasion of his 60th birthday and wishes him good health and successful work in mathematics and mathematical education.

JAMALBEK TUSSUPOV

(to the 60th birthday)

On April 10, 2017 was the 60th birthday of Jamalbek Tussupov, Doctor of Physical and Mathematical Sciences, Professor, Head of the Information Systems Department of the L.N. Gumilyov Eurasian National University, member of the Kazakhstan and American Mathematical Societies, member of the Association of Symbolic Logic, member of the Editorial Board of the Eurasian Mathematical Journal.

J. Tussupov was born in Taraz (Jambyl region of the Kazakh SSR). He graduated from the Karaganda State University (Kazakhstan) in 1979 and later on completed his postgraduate studies at S.L. Sobolev Institute of Mathematics of the Academy of Sciences of Russia (Novosibirsk).

Professor Tussupov's research interests are in mathematical logic, computability, computable structures, abstract data types, ontology, formal semantics. He solved the following problems of computable structures:

- the problems of S.S. Goncharov and M.S. Manasse: the problem of characterizing relative categoricity in the hyperarithmetical hierarchy given levels of complexity of Scott families, and the problem on the relationship between categoricity and relative categoricity of computable structures in the arithmetical and hyperarithmetical hierarchies;
- the problem of Yu.L. Ershov: the problem of finite algorithmic dimension in the arithmetical and hyperarithmetical hierarchies;
- the problem of C.J. Ash and A. Nerode: the problem of the interplay of relations of bounded arithmetical and hyperarithmetical complexity in computable presentations and the definability of relations by formulas of given complexity;
- the problem of S. Lempp: the problem of structures having presentations in just the degrees of all sets X such that for algebraic classes as symmetric irreflexive graphs, nilpotent groups, rings, integral domains, commutative semigroups, lattices, structure with two equivalences, bipartite graphs.

Professor Tussupov has published about 100 scientific papers, five textbooks for students and one monograph. Three PhD dissertations have been defended under his supervision.

Professor Tussupov is a fellow of "Bolashak" Scholarship, 2011 (Notre Dame University, USA), "Erasmus+", 2016 (Poitiers University, France). He was awarded the title "The Best Professor of 2012" (Kazakhstan). In 2015 Jamalbek Tussupov was also awarded for the contribution to science in the Republic of Kazakhstan.

The Editorial Board of the Eurasian Mathematical Journal congratulates Dr. Professor Jamalbek Tussupov on the occasion of his 60th aniversary and wishes him strong health, new achievements in science, inspiration for new ideas and fruitfull results.

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FRACTIONAL OSCILLATORY INTEGRAL OPERATORS AND THEIR COMMUTATORS ON GENERALIZED ORLICZ-MORREY SPACES OF THE THIRD KIND

A. Eroglu

Communicated by V.S. Guliyev

Key words: generalized Orlicz-Morrey space, oscillatory integral, commutator, BMO spaces.

AMS Mathematics Subject Classification: 42B20, 42B25, 42B35.

Abstract. We deal with the generalized Orlicz-Morrey space $M_{\Phi,\varphi}$ of the third kind and consider the boundedness of the oscillatory integral operators and fractional oscillatory integral operators on $M_{\Phi,\varphi}$. Some integral estimates for generalized Orlicz-Morrey spaces of the third kind are also obtained by using weighted Hardy operators. The corresponding commutators generated by BMO-functions are also considered.

1 Introduction and main results

In the last decade, there is an evident increase of investigations related to both the theory of the generalized Orlicz-Morrey spaces and the operator theory in these spaces. This is caused by keen interest in this topic not only in real analysis, but also in partial differential equations and in applied mathematics.

In this paper, we are focused on the boundedness of the oscillatory singular integrals with standard and variable Calderón-Zygmund kernels on generalized Orlicz-Morrey spaces of the third kind.

Recall that a function $\Phi : [0,+\infty) \to [0,\infty)$ is called a Young function if it is a convex increasing function satisfying $\Phi(0) = 0$, $\Phi(t) > 0$ for all $t \in (0,\infty)$ and $\Phi(t) \to \infty$ as $t \to \infty$.

For a Young function Φ, its inverse Φ^{-1} is defined by setting, for all $t \in (0,\infty)$

$$\Phi^{-1}(t) := \inf\{s \in (0,\infty) : \Phi(s) > t\}.$$

Denote by Δ_2 the set of all convex bijections $\Phi : [0,\infty) \to [0,\infty)$ such that the *doubling condition*:

$$\Phi(2t) \le C\Phi(t) \quad (t \ge 0) \tag{1.1}$$

holds for some constant $C \ge 2$, which is called the doubling constant, and by ∇_2 the set of all convex functions $\Phi : [0,\infty) \to [0,\infty]$ such that the ∇_2-*condition*:

$$C'\Phi(t) \le \Phi(2t) \quad (t \ge 0) \tag{1.2}$$

holds for some $C' > 2$. Note that C in (1.1) exceeds 2 if $\Phi \in \Delta_2 \cap \nabla_2$ due to (1.2). Recall also that the *conjugate function* Ψ of Φ is defined by:

$$\Psi(t) \equiv \sup\{st - \Phi(s) \,:\, s \ge 0\} \quad (t \ge 0).$$

Let Φ be a Young function. Recall that the *Orlicz norm* $\|f\|_{L_\Phi(E)}$ over a measurable set E in $\mathbb{R}^n$ is defined by:

$$\|f\|_{L_\Phi(E)} \equiv \inf\left\{\lambda > 0 : \int_E \Phi\Big(\frac{|f(x)|}{\lambda}\Big)dx \le 1\right\}.$$

Define $L_\Phi^{\rm loc}(\mathbb{R}^n)$ as the set of all measurable functions f for which $f \in L_\Phi(K)$ for all compact sets K in $\mathbb{R}^n$.

A natural step in the theory of functions spaces was to study Orlicz-Morrey spaces where the "Morrey-type measuring"of regularity of functions is realized with respect to the Orlicz norm over balls instead of the Lebesgue one. Such spaces were first introduced and studied by Nakai [23]. Then another kind of generalized Orlicz-Morrey spaces were introduced by Sawano *et al.* [28]. Our definition of generalized Orlicz-Morrey spaces introduced in [4] (see [12]) and used here is different from those of the papers [23] and [28].

We now define generalized Orlicz-Morrey spaces of the third kind. Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and Φ any Young function. The generalized Orlicz-Morrey space (of the third kind) $M_{\Phi,\varphi}(\mathbb{R}^n)$ is the space of functions $f \in L_\Phi^{\rm loc}(\mathbb{R}^n)$ with finite norm

$$\|f\|_{M_{\Phi,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\Phi^{-1}(|B(x,r)|^{-1})\|f\|_{L_\Phi(B(x,r))}.$$

Note that $M_{\Phi,\varphi}(\mathbb{R}^n)$ covers many classical function spaces.

Example 1. Let $1 \le q \le p < \infty$ and $\Phi \in \Delta_2 \cap \nabla_2$. From the following special cases, we see that our results will cover the Lebesgue space $L_p(\mathbb{R}^n)$, the *classical Morrey space* $M_q^p(\mathbb{R}^n)$, the generalized Morrey space $M_{p,\varphi}(\mathbb{R}^n)$ and the Orlicz space $L_\Phi(\mathbb{R}^n)$ with norm coincidence:

1. If $\Phi(r) = r^p$ and $\varphi(x,r) = |B(x,r)|^{-\frac{1}{p}}$, then $M_{\Phi,\varphi}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ with norm equivalence.
2. If $\Phi(r) = r^q$ and $\varphi(x,r) = |B(x,r)|^{-\frac{1}{p}}$, then $M_{\Phi,\varphi}(\mathbb{R}^n)$, which is denoted by $M_q^p(\mathbb{R}^n)$, is the classical Morrey space (see [21]).
3. If $\Phi(r) = r^p$, then $M_{\Phi,\varphi}(\mathbb{R}^n) = M_{p,\varphi}(\mathbb{R}^n)$ is the generalized Morrey space which were discussed in [6, 7, 20, 22, 27, 29].
4. If $\varphi(x,r) = \Phi^{-1}(|B(x,r)|^{-1})$, then $M_{\Phi,\varphi}(\mathbb{R}^n) = L_\Phi(\mathbb{R}^n)$.

The theory of boundedness of classical operators of the real analysis, such as the maximal operator, fractional maximal operator, Riesz potential and the singular integral operators etc, from one generalized Orlicz-Morrey space to another one is well studied, see for example, [4, 11, 14, 15, 23, 24, 28].

Let $f \in L_1^{\rm loc}(\mathbb{R}^n)$. The Riesz potential I_α is defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)dy}{|x-y|^{n-\alpha}}, \qquad 0 < \alpha < n.$$

Here and subsequently, C will denote a positive constant which may vary from line to line but will remain independent of the relevant quantities.

The Calderón-Zygmund singular integral operator is defined by

$$\widetilde{T}f(x) = p.v. \int_{\mathbb{R}^n} K(x-y)f(y)dy, \tag{1.3}$$

where K is a Calderón-Zygmund kernel (CZK). We say a kernel $K \in C^1(\mathbb{R}^n \setminus \{0\})$ is a CZK if it satisfies $|K(x)| \le \frac{C}{|x|^n}$, $|\nabla K(x)| \le \frac{C}{|x|^{n+1}}$ and $\int_{a<|x|<b} K(x)dx = 0$, for all a, b with $0 < a < b$.

It is worth pointing out that the kernel in (1.3) is a convolution kernel. However, there were many kinds of operators with non-convolution kernels, such as Fourier transform and Radon transform [25] which both are versions of oscillatory integrals. The object we consider in this paper is a class of oscillatory integrals due to Ricci and Stein [26]

$$Tf(x) = p.v. \int_{\mathbb{R}^n} e^{iP(x,y)} K(x-y) f(y) dy, \tag{1.4}$$

where $P(x,y)$ is a real valued polynomial defined on $\mathbb{R}^n \times \mathbb{R}^n$, and K is a CZK.

It is well known that the oscillatory factor $e^{iP(x,y)}$ makes it impossible to establish the L_p norm inequalities of (1.4) by the method as in the case of Calderón-Zygmund operators or fractional integrals. In [2], S. Chanillo and M. Christ established the weak $(1,1)$ type estimate of T.

A distribution kernel K is called a standard Calderón-Zygmund kernel (SCZK) if it satisfies the following hypotheses

$$|K(x,y)| \le \frac{C}{|x-y|^n}, \quad x \ne y \tag{1.5}$$

and

$$|\nabla_x K(x,y)| + |\nabla_y K(x,y)| \le \frac{C}{|x-y|^{n+1}}, \quad x \ne y. \tag{1.6}$$

The corresponding Calderón-Zygmund integral operator $\widetilde{S}$ and oscillatory integral operator S are defined by

$$\widetilde{S} f(x) = p.v. \int_{\mathbb{R}^n} K(x,y) f(y) dy$$

and

$$Sf(x) = p.v. \int_{\mathbb{R}^n} e^{iP(x,y)} K(x,y) f(y) dy,$$

where $P(x,y)$ is a real valued polynomial defined on $\mathbb{R}^n \times \mathbb{R}^n$. In [19], Lu and Zhang proved that S is bounded on L_p with $1 < p < \infty$.

In [26], Ricci and Stein also introduced the standard fractional Calderón-Zygmund kernel (SFCZK) K_α with $0 < \alpha < n$, where conditions (1.5) and (1.6) were replaced by

$$|K_\alpha(x,y)| \le \frac{C}{|x-y|^{n-\alpha}}, \quad x \ne y$$

and

$$|\nabla_x K_\alpha(x,y)| + |\nabla_y K_\alpha(x,y)| \le \frac{C}{|x-y|^{n+1-\alpha}}, \quad x \ne y.$$

The corresponding fractional oscillatory integral operator is defined by

$$S_\alpha f(x) = \int_{\mathbb{R}^n} e^{iP(x,y)} K_\alpha(x,y) f(y) dy,$$

where $P(x,y)$ is also a real valued polynomial defined on $\mathbb{R}^n \times \mathbb{R}^n$. Obviously, when $\alpha = 0$, $S_0 = S$ and $K_0 = K$. Partly motivated by the idea from [4], [10] and the results of [11], we now give the results of this paper in the following.

Theorem 1.1. *Let Φ any Young function, φ_1, φ_2 and Φ satisfy the condition*

$$\int_r^\infty \Big(\operatorname*{ess\,sup}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\big(|B(x,s)|^{-1}\big)} \Big) \Phi^{-1}\big(|B(x,t)|^{-1}\big) \frac{dt}{t} \le C\, \varphi_2(x,r), \tag{1.7}$$

where C does not depend on x and r. If K is a SCZK and the operator $\widetilde{S}$ is of type $\big(L_2(\mathbb{R}^n), L_2(\mathbb{R}^n)\big)$, then for $\Phi \in \Delta_2 \cap \nabla_2$ and any polynomial $P(x,y)$ the operator S is bounded from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M_{\Phi,\varphi_2}(\mathbb{R}^n)$.

Theorem 1.2. *Let $0 < \alpha < n$ and the functions (φ_1, φ_2) and (Φ, Ψ) satisfy the condition*

$$\int_r^\infty \operatorname*{ess\,sup}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\big(|B(x,s)|^{-1}\big)} \Psi^{-1}\big(|B(x,t)|^{-1}\big)\frac{dt}{t} \le C\,\varphi_2(x,r), \tag{1.8}$$

where C does not depend on x and r. Then for the conditions (2.1) and (2.2), S_α is bounded from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M_{\Psi,\varphi_2}(\mathbb{R}^n)$.

For a locally integrable function b, the commutator operator formed by S (or S_α) and b are defined by

$$S_b f(x) = b(x)Sf(x) - S(bf)(x)$$

and

$$S_{\alpha,b} f(x) = b(x)S_\alpha f(x) - S_\alpha(bf)(x).$$

Theorem 1.3. *Let Φ any Young function, $b \in BMO(\mathbb{R}^n)$ and (φ_1, φ_2) satisfies the condition*

$$\int_r^\infty \Big(1+\ln\frac{t}{r}\Big)\Big(\operatorname*{ess\,sup}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\big(|B(x,s)|^{-1}\big)}\Big) \Phi^{-1}\big(|B(x,t)|^{-1}\big)\frac{dt}{t} \le C\,\varphi_2(x,r), \tag{1.9}$$

where C does not depend on x and r. If K is a SCZK and the operator $\widetilde{S}$ is of type $\big(L_2(\mathbb{R}^n), L_2(\mathbb{R}^n)\big)$, then for any polynomial $P(x,y)$ the operator S_b is bounded from M_{Φ,φ_1} to M_{Φ,φ_2}.

Theorem 1.4. *Let $0 < \alpha < n$ and $b \in BMO(\mathbb{R}^n)$. Let Φ be a Young function and Ψ defined, via its inverse, by setting, for all $t \in (0,\infty)$, $\Psi^{-1}(t) := \Phi^{-1}(t)t^{-\alpha/n}$ and $\Phi, \Psi \in \Delta_2 \cap \nabla_2$. Let also (φ_1, φ_2) and (Φ, Ψ) satisfy the condition*

$$\int_r^\infty \Big(1+\ln\frac{t}{r}\Big) \operatorname*{ess\,sup}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\big(|B(x,s)|^{-1}\big)} \Psi^{-1}\big(|B(x,t)|^{-1}\big)\frac{dt}{t} \le C\,\varphi_2(x,r),$$

where C does not depend on x and r. Then the operator $S_{b,\alpha}$ is bounded from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M_{\Psi,\varphi_2}(\mathbb{R}^n)$.

Remark 1. Note that, in the case $\Phi(t) = t^p$ the Theorems 1.1 – 1.4 were proved in [6].

2 Some known results in generalized Orlicz-Morrey spaces $M_{\Phi,\varphi}(\mathbb{R}^n)$

The following interpolation result is from [5].

Lemma 2.1. *Let Φ any Young function and T be a sublinear operator of weak type (p,p) for any $p \in (1,\infty)$. Then T is bounded on $L^\Phi(\mathbb{R}^n)$ for $\Phi \in \Delta_2 \cap \nabla_2$.*

As a consequence of Lemma 2.1 and the L_p boundedness of the operator S [19], we get the following result.

Corollary 2.1. *If K is a SCZK and the operator $\widetilde{S}$ is of type $\big(L_2(\mathbb{R}^n), L_2(\mathbb{R}^n)\big)$, then for $\Phi \in \Delta_2 \cap \nabla_2$ and any polynomial $P(x,y)$ the operator S is bounded on $L_\Phi(\mathbb{R}^n)$.*

In [4] there were obtained sufficient conditions on weights φ_1 and φ_2 for the boundedness of the singular operator T from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M_{\Phi,\varphi_2}(\mathbb{R}^n)$, see also [15].

Theorem 2.1. *Let Φ any Young function, φ_1, φ_2 and Φ satisfy the condition (1.7). Then the operator T is bounded from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M_{\Phi,\varphi_2}(\mathbb{R}^n)$ for $\Phi \in \Delta_2 \cap \nabla_2$.*

We recall that, for functions Φ and Ψ from $[0,\infty)$ into $[0,\infty]$, the function Ψ is said to dominate Φ globally if there exists a positive constant c such that $\Phi(s) \le \Psi(cs)$ for all $s \ge 0$.

In the theorems below we also use the notation

$$\widetilde{\Psi_P}(s) = \int_0^s r^{P'-1}(\mathcal{B}_P^{-1}(r^{P'}))^{P'}\,dr,$$

where $1 < P \le \infty$ and $\widetilde{\Psi_P}(s)$ is the Young conjugate funtion to $\Psi_P(s)$, and

$$\Phi_P(s) = \int_0^s r^{P'-1}(\mathcal{A}_P^{-1}(r^{P'}))^{P'}\,dr,$$

where $\mathcal{B}_P^{-1}(s)$ and $\mathcal{A}_P^{-1}(s)$ are inverses to

$$\mathcal{B}_P(s) = \int_0^s \frac{\Psi(t)}{t^{1+P'}}dt \quad \text{and} \quad \mathcal{A}_P(s) = \int_0^s \frac{\widetilde{\Phi}(t)}{t^{1+P'}}dt,$$

respectively. These functions $\Psi_P(s)$ and $\Phi_P(s)$ are used below with $P = \frac{n}{\alpha}$.

The following statements were proved by Cianchi [1].

Theorem 2.2. *Let Φ and Ψ Young functions and $0 < \alpha < n$. Then*

The Riesz potential I_α is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$ if and only if

$$\int_0^1 \widetilde{\Phi}(t)/t^{1+n/(n-\alpha)}dt < \infty, \quad \int_0^1 \Psi(t)/t^{1+n/(n-\alpha)}dt < \infty, \tag{2.1}$$

and

$$\Phi \text{ dominates } \Psi_{n/\alpha} \text{ globally and } \Phi_{n/\alpha} \text{ dominates } \Psi \text{ globally.} \tag{2.2}$$

The following statements were proved by Guliyev and Deringoz [10].

Theorem 2.3. *Let $0 < \alpha < n$ and the functions (φ_1, φ_2) and (Φ, Ψ) satisfy the condition (1.8). Then for the conditions (2.1) and (2.2), I_α is bounded from $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.*

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

3 The fractional oscillatory integral operators in the spaces $M_{\Phi,\varphi}(\mathbb{R}^n)$

In this section we are going to use the following statement on the boundedness of the weighted Hardy operator

$$H_w^* g(r) := \int_r^\infty g(s)w(s)ds, \quad r \in (0,\infty),$$

where w is a weight.

The following theorem was proved in [9].

Theorem 3.1. *Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{r>0} v_2(r) H_w^* g(r) \le C \sup_{r>0} v_1(r) g(r) \tag{3.1}$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{r>0} v_2(r) \int_r^\infty \frac{w(t)dt}{\sup_{t<s<\infty} v_1(s)} < \infty. \tag{3.2}$$

Moreover, the value $C = B$ is the best constant for (3.1).

Remark 2. In (3.1) and (3.2) it is assumed that $\frac{1}{\infty}=0$ and $0\cdot\infty=0$.

Lemma 3.1. [4] *For a Young function Φ, the following inequality is valid*

$$\int_{B(x,r)}|f(y)|dy\leq 2|B(x,r)|\Phi^{-1}\left(|B(x,r)|^{-1}\right)\|f\|_{L^{\Phi}(B(x,r))}$$

and $\|\chi_B\|_{L_\Phi}=\frac{1}{\Phi^{-1}\left(|B|^{-1}\right)}$.

Lemma 3.2. *Let Φ any Young function, and K is a SCZK and the Calderón-Zygmund singular integral operator $\widetilde{S}$ is of type $\big(L_2(\mathbb{R}^n),L_2(\mathbb{R}^n)\big)$. Then for $\Phi\in\Delta_2\bigcap\nabla_2$ and any polynomial $P(x,y)$ the inequality*

$$\|Sf\|_{L_\Phi(B(x_0,r))}\lesssim\frac{1}{\Phi^{-1}\big(|B(x_0,r)|^{-1}\big)}\int_{2r}^{\infty}\|f\|_{L_\Phi(B(x_0,t))}\Phi^{-1}\big(|B(x_0,t)|^{-1}\big)\frac{dt}{t}$$

holds for any ball $B(x_0,r)$ and for all $f\in L_\Phi^{loc}(\mathbb{R}^n)$.

Proof. Let $\Phi\in\Delta_2\bigcap\nabla_2$. For arbitrary $x_0\in\mathbb{R}^n$, set $B=B(x_0,r)$ for the ball centered at x_0 and radius r, $2B=B(x_0,2r)$. We represent f as

$$f=f_1+f_2,\quad f_1(y)=f(y)\chi_{2B}(y),\quad f_2(y)=f(y)\chi_{(2B)^\complement}(y)$$

and have

$$\|Sf\|_{L_\Phi(B)}\leq\|Sf_1\|_{L_\Phi(B)}+\|Sf_2\|_{L_\Phi(B)}.$$

If K is a SCZK and the operator $\widetilde{S}$ is of type $\big(L_2(\mathbb{R}^n),L_2(\mathbb{R}^n)\big)$, then from Corollary 2.1 for $\Phi\in\Delta_2\bigcap\nabla_2$ and any polynomial $P(x,y)$ the operator S is bounded on $L_\Phi(\mathbb{R}^n)$.

Since $f_1\in L_\Phi(\mathbb{R}^n)$, $Sf_1\in L_\Phi(\mathbb{R}^n)$ and boundedness of S in $L_\Phi(\mathbb{R}^n)$ (see [26]) it follows that

$$\|Sf_1\|_{L_\Phi(B)}\leq\|Sf_1\|_{L_\Phi(\mathbb{R}^n)}\leq C\|f_1\|_{L_\Phi(\mathbb{R}^n)}=C\|f_1\|_{L_\Phi(2B)},$$

where constant $C>0$ is independent of f.

It's clear that $x\in B,y\in(2B)^\complement$ implies $\frac{1}{2}|x_0-y|\leq|x-y|\leq\frac{3}{2}|x_0-y|$. We get

$$|Sf_2(x)|\leq c_0\int_{(2B)^\complement}\frac{|f(y)|}{|x_0-y|^n}dy.$$

By Fubini's theorem we have

$$\int_{(2B)^\complement}\frac{|f(y)|}{|x_0-y|^n}dy\approx\int_{(2B)^\complement}|f(y)|\int_{|x_0-y|}^{\infty}t^{-1-n}dtdy\lesssim\int_{2r}^{\infty}\|f\|_{L_1(B(x_0,t))}\frac{dt}{t^{n+1}}.\tag{3.3}$$

Applying the Hölder's inequality (see, Lemma 3.1), we get

$$\begin{aligned}\int_{(2B)^\complement}\frac{|f(y)|}{|x_0-y|^n}dy&\lesssim\int_{2r}^{\infty}\|f\|_{L_\Phi(B(x_0,t))}\|1\|_{L_{\widetilde{\Phi}}(B(x_0,t))}\frac{dt}{t^{n+1}}\\&=\int_{2r}^{\infty}\|f\|_{L_\Phi(B(x_0,t))}\frac{1}{\widetilde{\Phi}^{-1}(|B(x_0,t)|^{-1})}\frac{dt}{t^{n+1}}\\&\approx\int_{2r}^{\infty}\|f\|_{L_\Phi(B(x_0,t))}\Phi^{-1}\big(|B(x_0,t)|^{-1}\big)\frac{dt}{t}.\end{aligned}$$

Moreover, for all $\Phi \in \Delta_2$ the inequality

$$\|Sf_2\|_{L_\Phi(B)} \lesssim \frac{1}{\Phi^{-1}\big(|B(x_0,r)|^{-1}\big)} \int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))} \Phi^{-1}\big(|B(x_0,t)|^{-1}\big) \frac{dt}{t}$$

is valid. Thus

$$\|Sf\|_{L_\Phi(B)} \lesssim \|f\|_{L_\Phi(2B)} + \frac{1}{\Phi^{-1}\big(|B(x_0,r)|^{-1}\big)} \int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))} \Phi^{-1}\big(|B(x_0,t)|^{-1}\big) \frac{dt}{t}.$$

On the other hand, by

$$r \le \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \le 2r \quad \text{for} \quad r \ge 0 \tag{3.4}$$

we get

$$\Phi^{-1}\big(|B|^{-1}\big) \approx \Phi^{-1}\big(|B|^{-1}\big) r^n \int_{2r}^{\infty} \frac{dt}{t^{n+1}} \lesssim \int_{2r}^{\infty} \Phi^{-1}\big(|B(x_0,t)|^{-1}\big) \frac{dt}{t}$$

and then

$$\|f\|_{L_p(2B)} \lesssim \frac{1}{\Phi^{-1}\big(|B(x_0,r)|^{-1}\big)} \int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))} \Phi^{-1}\big(|B(x_0,t)|^{-1}\big) \frac{dt}{t}. \tag{3.5}$$

Hence

$$\|Sf\|_{L_p(B)} \lesssim \frac{1}{\Phi^{-1}\big(|B(x_0,r)|^{-1}\big)} \int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))} \Phi^{-1}\big(|B(x_0,t)|^{-1}\big) \frac{dt}{t}.$$

□

Proof of Theorem 1.1.

By Lemma 3.2 and Theorem 3.1 we have

$$\begin{aligned} \|Sf\|_{M_{\Phi,\varphi_2}(\mathbb{R}^n)} &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \int_r^{\infty} \Phi^{-1}\big(|B(x,t)|^{-1}\big) \|f\|_{L_\Phi(B(x,t))} \frac{dt}{t} \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1} \Phi^{-1}\big(|B(x,r)|^{-1}\big) \|f\|_{L_\Phi(B(x,r))} = \|f\|_{M_{\Phi,\varphi_1}}. \end{aligned}$$

Proof of Theorem 1.2. The proof of Theorem 1.2 follows from the Theorem 2.3 and the following observation

$$|S_\alpha f(x)| \le I_\alpha(|f|)(x).$$

4 Commutators of fractional oscillatory integral operators in the spaces $M_{p,\varphi}(\mathbb{R}^n)$

Let T be a Calderón-Zygmund singular integral operator and $b \in BMO(\mathbb{R}^n)$. A well known result of Coifman, Rochberg and Weiss [3] states that the commutator operator $[b,T]f = T(bf) - bTf$ is bounded on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$.

First we recall the definition of the space $BMO(\mathbb{R}^n)$.

Definition 1. Suppose that $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, let

$$\|f\|_* = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}| dy,$$

where $f_{B(x,r)} = |B(x,r)|^{-1} \int_{B(x,r)} f(y) dy$. Define

$$BMO(\mathbb{R}^n) = \{f \in L^1_{\text{loc}}(\mathbb{R}^n) \ : \ \|f\|_* < \infty\}.$$

Before proving our theorems, we need the following lemmas.

Lemma 4.1. [13] *Let $b \in BMO(\mathbb{R}^n)$. Then there is a constant $C > 0$ such that*

$$\left|b_{B(x,r)} - b_{B(x,t)}\right| \le C\|b\|_* \ln \frac{t}{r} \quad \textit{for} \quad 0 < 2r < t, \tag{4.1}$$

where C is independent of b, x, r and t.

Lemma 4.2. [10, 16] *Let $f \in BMO(\mathbb{R}^n)$ and Φ be a Young function with $\Phi \in \Delta_2$, then*

$$\|f\|_* \approx \sup_{x\in\mathbb{R}^n, r>0} \Phi^{-1}\big(|B(x,r)|^{-1}\big) \left\|f(\cdot) - f_{B(x,r)}\right\|_{L^\Phi(B(x,r))}$$

We will use the following statement on the boundedness of the weighted Hardy operator

$$\mathcal{H}_w^* g(t) := \int_t^\infty \Big(1 + \ln \frac{s}{t}\Big) g(s) w(s) ds, \quad 0 < t < \infty,$$

where w is a weight.

The following lemma was proved in [8].

Lemma 4.3. *Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\operatorname*{ess\,sup}_{t>0} v_2(t) \mathcal{H}_w^* g(t) \le C \operatorname*{ess\,sup}_{t>0} v_1(t) g(t) \tag{4.2}$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \operatorname*{ess\,sup}_{t>0} v_2(t) \int_t^\infty \Big(1 + \ln \frac{s}{t}\Big) \frac{w(s) ds}{\operatorname*{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty. \tag{4.3}$$

Moreover, the value $C = B$ is the best constant for (4.2).

Remark 3. In (4.2) and (4.3) it is assumed that $\frac{1}{\infty} = 0$ and $0 \cdot \infty = 0$.

Lemma 4.4. *Let Let Φ be a Young function, $b \in BMO(\mathbb{R}^n)$, K is a SCZK and the Calderón-Zygmund singular integral operator $\widetilde{S}$ is of type $\big(L_2(\mathbb{R}^n), L_2(\mathbb{R}^n)\big)$. Then for $\Phi \in \Delta_2 \bigcap \nabla_2$ and any polynomial $P(x,y)$ the inequality*

$$\|S_b f\|_{L_\Phi(B(x_0,r))} \lesssim \frac{\|b\|_*}{\Phi^{-1}\big(|B(x_0,r)|^{-1}\big)} \int_{2r}^\infty \Big(1 + \ln \frac{t}{r}\Big) \|f\|_{L_\Phi(B(x_0,t))} \Phi^{-1}\big(|B(x_0,t)|^{-1}\big) \frac{dt}{t}$$

holds for any ball $B(x_0,r)$ and for all $f \in L_\Phi^{\rm loc}(\mathbb{R}^n)$.

Proof. Let $\Phi \in \Delta_2 \bigcap \nabla_2$. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0,r)$ for the ball centered at x_0 and radius r, $2B = B(x_0,2r)$. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2B}(y), \quad f_2(y) = f(y)\chi_{(2B)^{\complement}}(y)$$

and have

$$\|S_b f\|_{L_\Phi(B)} \le \|S_b f_1\|_{L_\Phi(B)} + \|S_b f_2\|_{L_\Phi(B)}.$$

It is known that (see [26], see also [17, 18, 19]), if K is a SCZK and the operator $\widetilde{S}$ is of type $\big(L_2(\mathbb{R}^n), L_2(\mathbb{R}^n)\big)$, then for $1 < p < \infty$ and any polynomial $P(x,y)$ the commutator operator S_b is bounded on $L_p(\mathbb{R}^n)$. Hence, by Lemma 2.1 for $\Phi \in \Delta_2 \bigcap \nabla_2$ and any polynomial $P(x,y)$

the commutator operator S_b is bounded on $L_\Phi(\mathbb{R}^n)$. Since $f_1 \in L_\Phi(\mathbb{R}^n)$, $Sf_1 \in L_\Phi(\mathbb{R}^n)$ and boundedness of S_b in $L_\Phi(\mathbb{R}^n)$ it follows that

$$\|S_b f_1\|_{L_\Phi(B)} \le \|S_b f_1\|_{L_\Phi(\mathbb{R}^n)} \le C\|b\|_* \, \|f_1\|_{L_\Phi(\mathbb{R}^n)} = C\|b\|_* \, \|f_1\|_{L_\Phi(2B)},$$

where constant $C > 0$ is independent of f.

For $x \in B$ we have

$$|S_b f_2(x)| \lesssim \int_{\mathbb{R}^n} \frac{|b(y) - b(x)|}{|x-y|^n} |f_2(y)| dy \approx \int_{\complement_{(2B)}} \frac{|b(y) - b(x)|}{|x_0 - y|^n} |f(y)| dy.$$

Then

$$\begin{aligned} \|S_b f_2\|_{L_\Phi(B)} &\lesssim \Big\| \int_{\complement_{(2B)}} \frac{|b(y)-b(x)|}{|x_0-y|^n} |f(y)| dy \Big\|_{L_\Phi(B)} \\ &\lesssim \Big\| \int_{\complement_{(2B)}} \frac{|b(y)-b_B|}{|x_0-y|^n} |f(y)| dy \Big\|_{L_\Phi(B)} + \Big\| \int_{\complement_{(2B)}} \frac{|b(x)-b_B|}{|x_0-y|^n} |f(y)| dy \Big\|_{L_\Phi(B)} = I_1 + I_2. \end{aligned}$$

Let us estimate I_1. By Lemma 3.1 we have

$$\begin{aligned} I_1 &\approx \frac{1}{\Phi^{-1}\big(|B|^{-1}\big)} \int_{\complement_{(2B)}} \frac{|b(y)-b_B|}{|x_0-y|^n} |f(y)| dy \\ &\approx \frac{1}{\Phi^{-1}\big(|B|^{-1}\big)} \int_{\complement_{(2B)}} |b(y)-b_B||f(y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1}} dy \\ &\lesssim \frac{1}{\Phi^{-1}\big(|B|^{-1}\big)} \int_{2r}^{\infty} \int_{B(x_0,t)} |b(y)-b_B||f(y)| dy \frac{dt}{t^{n+1}}. \end{aligned}$$

Applying Hölder's inequality, by (3.4), (4.1) and Lemmas 3.1 and 4.2 we get

$$\begin{aligned} I_1 &\lesssim \frac{1}{\Phi^{-1}\big(|B|^{-1}\big)} \int_{2r}^{\infty} \big\|b(\cdot) - b_{B(x_0,t)}\big\|_{L^{\tilde{\Phi}}(B(x_0,t))} \|f\|_{L^\Phi(B(x_0,t))} \frac{dt}{t^{n+1}} \\ &+ \frac{1}{\Phi^{-1}\big(|B|^{-1}\big)} \int_{2r}^{\infty} |b_{B(x_0,r)} - b_{B(x_0,t)}| \|f\|_{L^\Phi(B(x_0,t))} \Phi^{-1}\big(|B(x_0,t)|^{-1}\big) \frac{dt}{t} \\ &\lesssim \frac{\|b\|_*}{\Phi^{-1}\big(|B|^{-1}\big)} \int_{2r}^{\infty} \Big(1 + \ln\frac{t}{r}\Big) \|f\|_{L^\Phi(B(x_0,t))} \Phi^{-1}\big(|B(x_0,t)|^{-1}\big) \frac{dt}{t}. \end{aligned}$$

In order to estimate I_2 note that

$$I_2 \approx \|b(\cdot) - b_B\|_{L^\Phi(B)} \int_{\complement_{(2B)}} \frac{|f(y)|}{|x_0-y|^n} dy.$$

By Lemma 4.2, we get

$$I_2 \lesssim \frac{\|b\|_*}{\Phi^{-1}\big(|B|^{-1}\big)} \int_{\complement_{(2B)}} \frac{|f(y)|}{|x_0-y|^n} dy. \tag{4.4}$$

Thus, by (3.3)

$$I_2 \lesssim \frac{\|b\|_*}{\Phi^{-1}\big(|B(x_0,r)|^{-1}\big)} \int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))} \Phi^{-1}\big(|B(x_0,t)|^{-1}\big) \frac{dt}{t}.$$

Summing up I_1 and I_2 we get

$$\|S_b f_2\|_{L_p(B)} \lesssim \frac{\|b\|_*}{\Phi^{-1}\big(|B(x_0,r)|^{-1}\big)} \int_{2r}^{\infty} \Big(1 + \ln\frac{t}{r}\Big) \|f\|_{L_\Phi(B(x_0,t))} \Phi^{-1}\big(|B(x_0,t)|^{-1}\big) \frac{dt}{t}. \tag{4.5}$$

Finally,

$$\|S_b f\|_{L_p(B)} \lesssim \|b\|_* \, \|f\|_{L_p(2B)} + \frac{\|b\|_*}{\Phi^{-1}\big(|B(x_0,r)|^{-1}\big)} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \, \|f\|_{L_\Phi(B(x_0,t))} \Phi^{-1}\big(|B(x_0,t)|^{-1}\big) \frac{dt}{t},$$

and statement of Lemma 4.4 follows by (3.5). □

Proof of Theorem 1.3. The statement of Theorem 1.3 follows by Lemma 4.4 and Theorem 3.1 in the same manner as in the proof of Lemma 4.3.

Proof of Theorem 1.4. The proof of Theorem 1.4 follows from the Theorem 34 in [10] and the following observation

$$|S_{\alpha,b} f(x)| \le I_{\alpha,b}(|f|)(x).$$

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