ISSN 2077–9879

Eurasian Mathematical Journal

2017, Volume 8, Number 3

Founded in 2010 by
the L.N. Gumilyov Eurasian National University

in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia
the University of Padua

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

EURASIAN MATHEMATICAL JOURNAL

Aims and Scope

The Eurasian Mathematical Journal (EMJ) publishes carefully selected original research papers in all areas of mathematics written by mathematicians, principally from Europe and Asia. However papers by mathematicians from other continents are also welcome.

From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

The contents of EMJ are indexed in Scopus, Web of Science (ESCI), Mathematical Reviews, MathSciNet, Zentralblatt Math (ZMATH), Referativnyi Zhurnal – Matematika, Math-Net.Ru.

The EMJ is included in the list of journals recommended by the Committee for Control of Education and Science (Ministry of Education and Science of the Republic of Kazakhstan) and in the list of journals recommended by the Higher Attestation Commission (Ministry of Education and Science of the Russian Federation).

Information for the Authors

Submission. Manuscripts should be written in LaTeX and should be submitted electronically in DVI, PostScript or PDF format to the EMJ Editorial Office via e-mail (eurasianmj@yandex.kz).

When the paper is accepted, the authors will be asked to send the tex-file of the paper to the Editorial Office.

The author who submitted an article for publication will be considered as a corresponding author. Authors may nominate a member of the Editorial Board whom they consider appropriate for the article. However, assignment to that particular editor is not guaranteed.

Copyright. When the paper is accepted, the copyright is automatically transferred to the EMJ. Manuscripts are accepted for review on the understanding that the same work has not been already published (except in the form of an abstract), that it is not under consideration for publication elsewhere, and that it has been approved by all authors.

Title page. The title page should start with the title of the paper and authors' names (no degrees). It should contain the Keywords (no more than 10), the Subject Classification (AMS Mathematics Subject Classification (2010) with primary (and secondary) subject classification codes), and the Abstract (no more than 150 words with minimal use of mathematical symbols).

Figures. Figures should be prepared in a digital form which is suitable for direct reproduction.

References. Bibliographical references should be listed alphabetically at the end of the article. The authors should consult the Mathematical Reviews for the standard abbreviations of journals' names.

Authors' data. The authors' affiliations, addresses and e-mail addresses should be placed after the References.

Proofs. The authors will receive proofs only once. The late return of proofs may result in the paper being published in a later issue.

Offprints. The authors will receive offprints in electronic form.

Publication Ethics and Publication Malpractice

For information on Ethics in publishing and Ethical guidelines for journal publication see http://www.elsevier.com/publishingethics and http://www.elsevier.com/journal-authors/ethics.

Submission of an article to the EMJ implies that the work described has not been published previously (except in the form of an abstract or as part of a published lecture or academic thesis or as an electronic preprint, see http://www.elsevier.com/postingpolicy), that it is not under consideration for publication elsewhere, that its publication is approved by all authors and tacitly or explicitly by the responsible authorities where the work was carried out, and that, if accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright-holder. In particular, translations into English of papers already published in another language are not accepted.

No other forms of scientific misconduct are allowed, such as plagiarism, falsification, fraudulent data, incorrect interpretation of other works, incorrect citations, etc. The EMJ follows the Code of Conduct of the Committee on Publication Ethics (COPE), and follows the COPE Flowcharts for Resolving Cases of Suspected Misconduct ($http : //publicationethics.org/files/u2/New_Code.pdf$). To verify originality, your article may be checked by the originality detection service CrossCheck http://www.elsevier.com/editors/plagdetect.

The authors are obliged to participate in peer review process and be ready to provide corrections, clarifications, retractions and apologies when needed. All authors of a paper should have significantly contributed to the research.

The reviewers should provide objective judgments and should point out relevant published works which are not yet cited. Reviewed articles should be treated confidentially. The reviewers will be chosen in such a way that there is no conflict of interests with respect to the research, the authors and/or the research funders.

The editors have complete responsibility and authority to reject or accept a paper, and they will only accept a paper when reasonably certain. They will preserve anonymity of reviewers and promote publication of corrections, clarifications, retractions and apologies when needed. The acceptance of a paper automatically implies the copyright transfer to the EMJ.

The Editorial Board of the EMJ will monitor and safeguard publishing ethics.

The procedure of reviewing a manuscript, established by the Editorial Board of the Eurasian Mathematical Journal

1. **Reviewing procedure**

1.1. All research papers received by the Eurasian Mathematical Journal (EMJ) are subject to mandatory reviewing.

1.2. The Managing Editor of the journal determines whether a paper fits to the scope of the EMJ and satisfies the rules of writing papers for the EMJ, and directs it for a preliminary review to one of the Editors-in-chief who checks the scientific content of the manuscript and assigns a specialist for reviewing the manuscript.

1.3. Reviewers of manuscripts are selected from highly qualified scientists and specialists of the L.N. Gumilyov Eurasian National University (doctors of sciences, professors), other universities of the Republic of Kazakhstan and foreign countries. An author of a paper cannot be its reviewer.

1.4. Duration of reviewing in each case is determined by the Managing Editor aiming at creating conditions for the most rapid publication of the paper.

1.5. Reviewing is confidential. Information about a reviewer is anonymous to the authors and is available only for the Editorial Board and the Control Committee in the Field of Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan (CCFES). The author has the right to read the text of the review.

1.6. If required, the review is sent to the author by e-mail.

1.7. A positive review is not a sufficient basis for publication of the paper.

1.8. If a reviewer overall approves the paper, but has observations, the review is confidentially sent to the author. A revised version of the paper in which the comments of the reviewer are taken into account is sent to the same reviewer for additional reviewing.

1.9. In the case of a negative review the text of the review is confidentially sent to the author.

1.10. If the author sends a well reasoned response to the comments of the reviewer, the paper should be considered by a commission, consisting of three members of the Editorial Board.

1.11. The final decision on publication of the paper is made by the Editorial Board and is recorded in the minutes of the meeting of the Editorial Board.

1.12. After the paper is accepted for publication by the Editorial Board the Managing Editor informs the author about this and about the date of publication.

1.13. Originals reviews are stored in the Editorial Office for three years from the date of publication and are provided on request of the CCFES.

1.14. No fee for reviewing papers will be charged.

2. **Requirements for the content of a review**

2.1. In the title of a review there should be indicated the author(s) and the title of a paper.

2.2. A review should include a qualified analysis of the material of a paper, objective assessment and reasoned recommendations.

2.3. A review should cover the following topics:

- compliance of the paper with the scope of the EMJ;

- compliance of the title of the paper to its content;

- compliance of the paper to the rules of writing papers for the EMJ (abstract, key words and phrases, bibliography etc.);

- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);

- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

2.4. The final part of the review should contain an overall opinion of a reviewer on the paper and a clear recommendation on whether the paper can be published in the Eurasian Mathematical Journal, should be sent back to the author for revision or cannot be published.

Web-page

The web-page of EMJ is www.emj.enu.kz. One can enter the web-page by typing Eurasian Mathematical Journal in any search engine (Google, Yandex, etc.). The archive of the web-page contains all papers published in EMJ (free access).

Subscription

For Institutions

- US$ 200 (or equivalent) for one volume (4 issues)
- US$ 60 (or equivalent) for one issue

For Individuals

- US$ 160 (or equivalent) for one volume (4 issues)
- US$ 50 (or equivalent) for one issue.

The price includes handling and postage.

The Subscription Form for subscribers can be obtained by e-mail:

eurasianmj@yandex.kz

The Eurasian Mathematical Journal (EMJ)
The Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana
Kazakhstan

ERLAN DAUTBEKOVICH NURSULTANOV

(to the 60th birthday)

On May 25, 2017 was the 60th birthday of Yerlan Dautbekovich Nursultanov, Doctor of Physical and Mathematical Sciences (1999), Professor (2001), Head of the Department of Mathematics and Informatics of the Kazakhstan branch of the M.V. Lomonosov Moscow State University (since 2001), member of the Editorial Board of the Eurasian Mathematical Journal.

E.D. Nursultanov was born in the city of Karaganda. He graduated from the Karaganda State University (1979) and then completed his post-graduate studies at the M.V. Lomonosov Moscow State University.

Professor Nursultanov's scientific interests are related to various areas of the theory of functions and functional analysis.

He introduced the concept of multi-parameter Lorentz spaces, network spaces and anisotropic Lorentz spaces, for which appropriate interpolation methods were developed. On the basis of the apparatus introduced by him, the questions of reiteration in the off-diagonal case for the real Lyons-Petre interpolation method, the multiplier problem for trigonometric Fourier series, the lower and upper bounds complementary to the Hardy-Littlewood inequalities for various orthonormal systems were solved. The convergence of series and Fourier transforms were studied with sufficiently general monotonicity conditions. The lower bounds for the norm of the convolution operator are obtained, and its upper bounds are improved (a stronger result than the O'Neil inequality). An exact cubature formula with explicit nodes and weights for functions belonging to spaces with a dominated mixed derivative is constructed, and a number of other problems in this area are solved.

He has published more than 50 scientific papers in high rating international journals included in the lists of Thomson Reuters and Scopus. 2 doctor of sciences, 9 candidate of sciences and 4 PhD dissertations have been defended under his supervision.

His merits and achievements are marked with badges of the Ministry of Education and Science of the Republic of Kazakhstan "For Contribution to the Development of Science" (2007), "Honored Worker of Education" (2011), "Y. Altynsarin" (2017). He is a laureate of the award named after K. Satpaev in the field of natural sciences for 2005, the grant holder "The best teacher of the university" for 2006 and 2011, the grant holder of the state scientific scholarship for outstanding contribution to the development of science and technology of the Republic of Kazakhstan for years 2007-2008, 2008 -2009. In 2017 he got the Top Springer Author award, established by Springer Nature together with JSC "National Center for Scientific and Technical Information".

The Editorial Board of the Eurasian Mathematical Journal congratulates Erlan Dautbekovich Nursultanov on the occasion of his 60th birthday and wishes him good health and successful work in mathematics and mathematical education.

JAMALBEK TUSSUPOV

(to the 60th birthday)

On April 10, 2017 was the 60th birthday of Jamalbek Tussupov, Doctor of Physical and Mathematical Sciences, Professor, Head of the Information Systems Department of the L.N. Gumilyov Eurasian National University, member of the Kazakhstan and American Mathematical Societies, member of the Association of Symbolic Logic, member of the Editorial Board of the Eurasian Mathematical Journal.

J. Tussupov was born in Taraz (Jambyl region of the Kazakh SSR). He graduated from the Karaganda State University (Kazakhstan) in 1979 and later on completed his postgraduate studies at S.L. Sobolev Institute of Mathematics of the Academy of Sciences of Russia (Novosibirsk).

Professor Tussupov's research interests are in mathematical logic, computability, computable structures, abstract data types, ontology, formal semantics. He solved the following problems of computable structures:

- the problems of S.S. Goncharov and M.S. Manasse: the problem of characterizing relative categoricity in the hyperarithmetical hierarchy given levels of complexity of Scott families, and the problem on the relationship between categoricity and relative categoricity of computable structures in the arithmetical and hyperarithmetical hierarchies;
- the problem of Yu.L. Ershov: the problem of finite algorithmic dimension in the arithmetical and hyperarithmetical hierarchies;
- the problem of C.J. Ash and A. Nerode: the problem of the interplay of relations of bounded arithmetical and hyperarithmetical complexity in computable presentations and the definability of relations by formulas of given complexity;
- the problem of S. Lempp: the problem of structures having presentations in just the degrees of all sets X such that for algebraic classes as symmetric irreflexive graphs, nilpotent groups, rings, integral domains, commutative semigroups, lattices, structure with two equivalences, bipartite graphs.

Professor Tussupov has published about 100 scientific papers, five textbooks for students and one monograph. Three PhD dissertations have been defended under his supervision.

Professor Tussupov is a fellow of "Bolashak" Scholarship, 2011 (Notre Dame University, USA), "Erasmus+", 2016 (Poitiers University, France). He was awarded the title "The Best Professor of 2012" (Kazakhstan). In 2015 Jamalbek Tussupov was also awarded for the contribution to science in the Republic of Kazakhstan.

The Editorial Board of the Eurasian Mathematical Journal congratulates Dr. Professor Jamalbek Tussupov on the occasion of his 60th aniversary and wishes him strong health, new achievements in science, inspiration for new ideas and fruitfull results.

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 8, Number 3 (2017), 28 – 35

EXISTENCE OF THE N-TH ROOT IN FINITE-DIMENSIONAL POWER-ASSOCIATIVE ALGEBRAS OVER REALS

A.A. Arutyunov, S.E. Zhukovskiy

Communicated by V.I. Burenkov

Keywords: real algebra, power-associative algebra, Cayley-Dickson construction.

AMS Mathematics Subject Classification (2010): 17A05, 13J30.

Abstract. The paper is devoted to the solvability of equations in finite-dimensional power-associative algebras over $\mathbb{R}$. Necessary and sufficient conditions for the existence of the n-th root in a power-associative $\mathbb{R}$-algebra are obtained. Sufficient solvability conditions for a specific class of polynomial equations in a power-associative $\mathbb{R}$-algebra are derived.

1 Introduction

The paper is devoted to the solvability of equations in finite-dimensional power-associative algebras over $\mathbb{R}$. In order to describe the considered problem in detail, recall some definitions.

An *algebra over* $\mathbb{R}$ (or simply $\mathbb{R}$-*algebra*) is a vector space V over $\mathbb{R}$ equipped with a *"product mapping"* (or *multiplication*) $V \times V \to V$, $(a, b) \mapsto ab$ such that

$$(\alpha a + \beta b)c = \alpha(ac) + \beta(bc), \quad a(\alpha b + \beta c) = \alpha(ab) + \beta(ac) \quad \forall\, a, b, c \in V, \quad \forall\, \alpha, \beta \in \mathbb{R}.$$

An $\mathbb{R}$-algebra is called *finite-dimensional* if the linear space V is finite-dimensional. Below we consider finite-dimensional $\mathbb{R}$-algebras only and denote them by $(\mathbb{R}^d, \cdot)$, $d \in \mathbb{N}$.

Given an algebra $(\mathbb{R}^d, \cdot)$, denote

$$a^1 := a, \quad a^{n+1} := a \cdot a^n \quad \forall\, a \in \mathbb{R}^d, \quad \forall\, n \in \mathbb{N}.$$

If $a^{n+m} = a^n a^m$ for each $a \in \mathbb{R}^d$, for every $n, m \in \mathbb{N}$, then the algebra $(\mathbb{R}^d, \cdot)$ is called *power-associative*. In this paper, we consider the equation

$$x^n = y \tag{1.1}$$

with the unknown x in a power-associative algebra $(\mathbb{R}^d, \cdot)$ and find necessary and sufficient conditions for this equation to have a solution $x \in \mathbb{R}^d$ for every right-hand side $y \in \mathbb{R}^d$, for every $n \in \mathbb{N}$. This problem can be considered as the problem of the existence of the n-th root in a power-associative $\mathbb{R}$-algebra. Moreover, we consider polynomial equations in power-associative $\mathbb{R}$-algebras and derive sufficient solvability conditions for such equations.

2 Main results

Let us start with solvability conditions for equation (1.1).

Theorem 2.1. *Let an $\mathbb{R}$-algebra $(\mathbb{R}^d,\cdot)$ be power-associative. If for any $y\in\mathbb{R}^d$ there exists a solution $x\in\mathbb{R}^d$ to the equation*

$$x^2 = y \tag{2.1}$$

then equation (1.1) has a solution $x\in\mathbb{R}^d$ for every right-hand side $y\in\mathbb{R}^d$, for every $n\in\mathbb{N}$.

Remark 1. Note that in this theorem we do not assume that the considered algebra is unitary. Recall that an $\mathbb{R}$-algebra $(\mathbb{R}^d,\cdot)$ is called *unitary,* if it has an *identity element*, i.e. an element $e\in\mathbb{R}^d$ such that

$$ex = xe = x \quad \forall\, x\in\mathbb{R}^d.$$

Proof of Theorem 2.1. **I.** Let us prove that *if $a^j\neq 0$, $j=\overline{1,m}$, and $a^{m+1}=0$ for some $a\in\mathbb{R}^d$, $m\geq 1$, then vectors a^j, $j=\overline{1,m}$ are linearly independent.* Assume that $\sum\limits_{j=1}^{m}\lambda_j a^j = 0$ for some reals λ_j, $j=\overline{1,m}$. Multiplying this equality by a^{m-1} we obtain $\lambda_1 a^m = 0$. Hence, $\lambda_1 = 0$. Therefore, $\sum\limits_{j=2}^{m}\lambda_j a^j = 0$. Repeating the described procedure m times we obtain $\lambda_1=\lambda_2=...=\lambda_m=0$. So, vectors a^j, $j=\overline{1,m}$, are linearly independent.

II. Let us prove that $x=0$ *is the only solution to the equation* $x^2=0$. Assume the contrary, i.e. there exists a nonzero $h\in\mathbb{R}^d$ such that $h^2=0$. Since equation (2.1) is solvable for each $y\in\mathbb{R}^d$, there exist $N\in\mathbb{N}$ and a vector $a\in\mathbb{R}^d$ such that $a^{2^N}=h$ and $2^N>d$. Since $h^2=0$, we have $a^j\neq 0$ for $j=\overline{1,2^N}$ and $a^{2^{N+1}}=0$. So, it follows from **I** that vectors a^j, $j=\overline{1,2^N}$ are linearly independent. The amount of these vectors is greater than the dimension of $\mathbb{R}^d$. This contradiction proves that $x=0$ is the only solution to the equation $x^2=0$.

III. Let us prove that $x=0$ *is the only solution to the equation $x^n=0$ for each $n\in\mathbb{N}$.* Assume the contrary. Take the minimal n such that the equation $x^n=0$ has a nonzero solution. Denote this solution by h. It follows from **II** that $n>2$. Consider two cases. If $n=2m$ for some $m\in\mathbb{N}$ then we have $(h^m)^2=h^n=0$ and $h^m\neq 0$ since $m<n$. If $n=2m+1$ for some $m\in\mathbb{N}$ then we have $(h^{m+1})^2=h\cdot h^n=0$ and $h^{m+1}\neq 0$ since $m+1<n$. In both cases we obtain a contradiction with **II**. So, $x=0$ is the only solution to the equation $x^n=0$ for each $n\in\mathbb{N}$.

IV. Let us prove that *equation (1.1) has a solution $x\in\mathbb{R}^d$ for every right-hand side $y\in\mathbb{R}^d$, for every odd $n\in\mathbb{N}$.* Let $\|\cdot\|$ be a Euclidean norm[1] at $\mathbb{R}^d$, $S^{d-1}\subset\mathbb{R}^d$ be a unit sphere centered at zero. Consider the mapping

$$F: S^{d-1}\to S^{d-1}, \quad F(x)=\frac{x^n}{\|x^n\|} \quad \forall\, x\in S^{d-1}.$$

This mapping is well defined, since $\|x^n\|\neq 0$ for every $x\in S^{d-1}$ in virtue of **III**. Moreover, it is continuously differentiable, since the mapping $x\mapsto x^n$ is polynomial, i.e. each coordinate of the vector x^n is a homogeneous polynomial of degree n of d real variables $x=(x_1,...,x_d)$. Finally, F is odd, i.e. $F(-x)=-F(x)$, since n is odd. Thus, the Lyusternik-Schnirelmann-Borsuk theorem (see, for example, [2, Chapter II, Theorem 2.4]) implies that the topological degree of F is odd. Therefore, the topological degree of F is nonzero. Hence, F is surjective. So, for arbitrary $y\in\mathbb{R}^d\setminus\{0\}$ there exists $z\in\mathbb{R}^d$ such that $F(z)=y/\|y\|$. Set

$$\lambda := \sqrt[n]{\frac{\|y\|}{\|z^n\|}}, \quad x:=\lambda z.$$

Then

$$x^n = \frac{\|y\|}{\|z^n\|} z^n = \|y\| F(z) = y.$$

[1]Everywhere in this paper we do not assume that the considered Euclidean norm is related with the multiplication by the equality $\|ab\|=\|a\|\,\|b\|$.

V. Let us prove that *equation (1.1) has a solution $x \in \mathbb{R}^d$ for every right-hand side $y \in \mathbb{R}^d$, for every even $n \in \mathbb{N}$.* Obviously there exist positive integers m and N such that $n = m2^N$ and m is odd. Since equation (2.1) is solvable for each right-hand side, there exist vectors $x_j \in \mathbb{R}^d$, $j = \overline{1,N}$ such that $x_1^2 = y$, $x_2^2 = x_1$, ..., $x_N^2 = x_{N-1}$. In virtue of **IV** there exists $x \in \mathbb{R}^d$ such that $x^m = x_N$. Thus,

$$x^n = (x^m)^{2^N} = x_N^{2^N} = (x_N^2)^{2^{N-1}} = x_{N-1}^{2^{N-1}} = ... = x_1^2 = y.$$

□

Let us discuss the question on solvability of polynomial equations in power-associative $\mathbb{R}$-algebras.

In order to give a definition of polynomial function it is usually assumed that the multiplication is associative (i.e. $a(bc) \equiv (ab)c$), commutative and $\mathbb{R}^d$ contains a unit element. Given $a \in \mathbb{R}^d$ and a nonnegative integer n, a monomial of degree n is a mapping $\mathbb{R}^d \to \mathbb{R}^d$ defined by formula $x \mapsto ax^n$, and a polynomial function is a finite sum of monomials. For this type of polynomial functions the classical example of solvability theorem provides the fundamental theorem of algebra (FTA), that states that *every non-constant polynomial function with complex coefficients has at least one complex root.*

If the multiplication is associative but not commutative, the definition of polynomial can be stated as follows. Given nonnegative integer n and vectors $a_0, ..., a_n \in \mathbb{R}^d$, a monomial of degree n is a mapping $x \mapsto a_0 x a_1 x ... a_{n-1} x a_n$. Polynomial functions can be defined standardly as a finite sum of monomials. In [1, Theorem 1], the following analogue of FTA was obtained for this type of polynomial functions in the algebra of real quaternions[2] $\mathbb{H}$. *Let $n \geq 1$, a_j be nonzero elements of $\mathbb{H}$, $j = \overline{0,n}$, $f : \mathbb{H} \to \mathbb{H}$ be a polynomial function,*

$$f(x) \equiv a_0 x a_1 x ... a_{n-1} x a_n + g(x),$$

where g is a finite sum of monomials, whose degree is less or equal than $n-1$. Then there exists $x \in \mathbb{H}$ such that $f(x) = 0$.

Comparing this result with FTA we observe the following. FTA provides necessary and sufficient conditions for a polynomial equation to have a solution, whereas the result from [1] is a sufficient but not necessary condition. Indeed, set $f(x) := \mathbf{i}x + x\mathbf{i} + g$, where g is a constant. This function does not satisfy the assumptions of theorem from [1]. However, it is a straightforward task to ensure that the equation $f(x) = 0$ has a solution if $g = \mathbf{1}$, and has no solutions if $g = \mathbf{j}$.

Let us consider now the most general case, when the multiplication can be neither commutative nor associative, and the algebra $(\mathbb{R}^d, \cdot)$ contains no identity element. A definition of a polynomial function can be stated as follows. A monomial of degree 0 is a constant mapping $\mathbb{R}^d \to \mathbb{R}^d$ defined by formula $x \mapsto a$; a monomial of degree 1 is a mapping $\mathbb{R}^d \to \mathbb{R}^d$ defined by formula $x \mapsto x$; a monomial of degree $n \geq 1$ is a product of two monomials of degrees $n_1 \geq 0$ and $n_2 \geq 0$ such that $n_1 + n_2 = n$; a polynomial function is a finite sum of monomials.

The question of polynomial equation solvability for the case when multiplication is not commutative was also considered in [1]. It was stated that an analog of Theorem 1 from [1] is valid in the Cayley algebra $\mathbb{O}$.

Below we consider a more general case that the one in [1]. We investigate an arbitrary power-associative algebra $(\mathbb{R}^d, \cdot)$ Note that the situation is complicated by the fact that the algebra may contain zero devisors, i.e. nonzero vectors a, b such that $ab = 0$. It is a straightforward task to ensure that if $ab = 0$ for some nonzero vectors a and b, then there exists a vector y such that the equation

$$ax = y$$

[2] Recall that the algebra of real quaternions $\mathbb{H}$ is a 4-dimensional vector space with a fixed basis $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k} \in \mathbb{H}$ and an associative multiplication such that $\mathbf{1}$ is an identity element and $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -\mathbf{1}$.

has no solutions. The above mentioned algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ contain no zero devisor. The examples of a power-associative $\mathbb{R}$-algebras that contain zero devisors are the ring of sedenions $\mathbb{S}$ and the ring of real square $k \times k$ matrices, $k \geq 2$. Note that it is natural for $\mathbb{R}$-algebras to have zero devisors. In [3], it was shown that if $d \notin \{1, 2, 4, 8\}$ then every $\mathbb{R}$-algebra $(\mathbb{R}^d, \cdot)$ contains zero devisors.

Theorem 2.2. *Assume that an $\mathbb{R}$-algebra $(\mathbb{R}^d, \cdot)$ is power-associative, equation (2.1) is solvable for every $y \in \mathbb{R}^d$. Let n be a positive integer, $g : \mathbb{R}^d \to \mathbb{R}^d$ be a sum of a finite number of monomials, whose degrees are less than n, $a \in \mathbb{R}^d$, a polynomial $f : \mathbb{R}^d \to \mathbb{R}^d$ be defined as follows:*

$$f(x) = ax^n + g(x) \quad \forall x \in \mathbb{R}^d.$$

If n is odd and a is not a zero devisor then there exists a solution $x \in \mathbb{R}^d$ to the equation

$$f(x) = 0. \tag{2.2}$$

Proof. Consider the contrary: equation (2.2) has no solutions. Let $\|\cdot\|$ be a Euclidean norm in $\mathbb{R}^d$, $S^{d-1} \subset \mathbb{R}^d$ be a unit sphere centered at zero. Since a is not a zero devisor, it follows from **III** that $ax^n \neq 0$ for each nonzero vector $x \in \mathbb{R}^d$. Thus, there exists a number $c_1 > 0$ such that $\|ax^n\| \geq c_1\|x\|^n$ for every $x \in \mathbb{R}^d$. Since the degree of each monomial of g is less than n, there exist numbers $c_2 \geq 0$, $c_3 \geq 0$ such that $\|g(x)\| \leq c_2\|x\|^{n-1} + c_3$ for every $x \in \mathbb{R}^d$. Fix arbitrary $R > 0$ such that $c_1R^n > c_2R^{n-1} + c_3$. The inequalities above imply that $\|a(Rx)^n\| > \|g(Rx)\|$ for each $x \in S^{d-1}$.

Consider mappings $F, H : S^{d-1} \times [0, R] \to S^{d-1}$ defined as follows:

$$F(x,t) = \frac{f(tx)}{\|f(tx)\|} \quad \forall x \in S^{d-1}, \quad \forall t \in [0, R],$$

$$H(x,t) = \frac{a(Rx)^n + R^{-1}(R-t)g(Rx)}{\|a(Rx)^n + R^{-1}(R-t)g(Rx)\|} \quad \forall x \in S^{d-1}, \quad \forall t \in [0, R].$$

The mappings F and H are well-defined, since the denominators in the above formulae do not vanish. Indeed, $\|f(tx)\| \neq 0$ for every x and t, since equation (2.2) has no solutions. Moreover, $\|a(Rx)^n + (R-t)g(Rx)\| \neq 0$ for every x and t, since

$$\|a(Rx)^n + R^{-1}(R-t)g(Rx)\| \geq \|a(Rx)^n\| - \|R^{-1}(R-t)g(Rx)\| \geq$$

$$\geq \|a(Rx)^n\| - \|g(Rx)\| > 0$$

in virtue of the choice of R. Obviously, the mappings F and H are continuously differentiable. Since $F(x,R) \equiv H(x,0)$, the mappings $F(\cdot,0)$ and $H(\cdot,R)$ are smoothly homotopic. Thus, the topological degrees of $F(\cdot,0)$ and $H(\cdot,R)$ coincide. However, the topological degree of $F(\cdot,0)$ is zero, since $F(x,0) \equiv f(0)/\|f(0)\|$. At the same time the mapping $H(\cdot,R)$ is odd since n is odd and $H(x,R) \equiv ax^n/\|ax^n\|$. Thus, the Lyusternik-Schnirelmann-Borsuk theorem (see, for example, [2, Chapter II, Theorem 2.4]) implies that the topological degree of $H(\cdot,R)$ is odd. This contradiction proves the theorem. □

3 Discussions and examples

Let us discuss Theorems 2.1 and 2.2. In order to apply these results to an $\mathbb{R}$-algebra $(\mathbb{R}^d, \cdot)$ we have to verify two properties: power-associativity of multiplication and solvability of equation (2.1) for every right-hand side $y \in \mathbb{R}^d$. A proposition below allows to verify the second property for some types of $\mathbb{R}$-algebras.

Proposition 3.1. *Let an $\mathbb{R}$-algebra $(\mathbb{R}^d,\cdot)$ be unitary, $e\in\mathbb{R}^d$ be an identity element, $d\geq 2$. Assume that the equation*

$$x^2=e \tag{3.1}$$

has only two solutions $x=e$ and $x=-e$, the equation

$$x^2=0 \tag{3.2}$$

has the only solution $x=0$. Then equation (2.1) has a solution for every right-hand side $y\in\mathbb{R}^d$.

Remark 2. *Note that in this proposition we do not assume that the considered algebra is power-associative.*

Proof of Proposition 3.1. Let $\mathbb{R}P^{d-1}$ be the $(d-1)$-dimensional projective space over $\mathbb{R}$, $\|\cdot\|$ be a Euclidean norm in $\mathbb{R}^d$ such that $\|e\|=1$, $S^{d-1}\subset\mathbb{R}^d$ be a unit sphere centered at zero. Consider the mappings $Q:\mathbb{R}^d\to\mathbb{R}^d$, $F:\mathbb{R}P^{d-1}\to S^{d-1}$ defined as follows:

$$Q(x)=x^2\quad\forall\,x\in\mathbb{R}^d,\quad F(\chi)=\frac{Q(x)}{\|Q(x)\|}\quad\forall\,\chi\in\mathbb{R}P^{d-1},\quad\forall\,x\in\chi$$

(here an element χ of the projective space $\mathbb{R}P^{d-1}$ is an equivalence class, x is its representative). The mapping F is well-defined, since $Q(x)\neq 0$ for every nonzero vector $x\in\mathbb{R}^d$ and Q is positively homogeneous (i.e. $Q(\lambda x)=\lambda^2Q(x)$ for every real λ, for every $x\in\mathbb{R}^d$).

The constructed mapping F is obviously continuously differentiable. Let us prove that its topological degree modulo two equals one. Since equation (3.1) has only two solutions $x=e$ and $x=-e$, equality $F(\chi)=e$ holds only for the equivalence class $\chi=\chi_e\in\mathbb{R}P^{d-1}$ of the element e. Let us prove that e is a regular value of mapping F. Obviously it is enough to prove that e is a regular value of the mapping Q. Since

$$Q(e+\delta)=(e+\delta)^2=Q(e)+2e\delta+\delta^2\quad\forall\,\delta\in\mathbb{R}^d,$$

we have $Q'(e)=2I$, where $I:\mathbb{R}^d\to\mathbb{R}^d$ is the identity linear (over $\mathbb{R}$) mapping. Analogously, $Q'(-e)=-2I$. By assumption, the value e of the mapping Q has only two preimages e and $-e$. The corresponding derivatives $Q'(e)$ and $Q'(-e)$ do not degenerate. So, e is a regular value of Q and, therefore, χ_e is a regular value of F. Since the regular value e of the mapping F has the only preimage χ_e, the topological degree modulo two of the mapping F equals one.

The mapping F is surjective, since its topological degree modulo two is not zero. Thus, Q is surjective, since Q is positively homogeneous. Therefore, equation (2.1) has a solution for every right-hand side $y\in\mathbb{R}^d$. □

Let us now present a class of $\mathbb{R}$-algebras that satisfy assumptions of Theorems 2.1 and 2.2. For this purpose we recall the Cayley-Dickson construction (for more references see, for example, [4, 5]) .

Let $A=(\mathbb{R}^d,\cdot)$ be a unitary algebra with an identity element e. Denote

$$\mathbb{R}^1=\{\lambda e:\ \lambda\in\mathbb{R}\},\quad \mathbb{R}^1_+=\{\lambda e:\ \lambda\geq 0\}.$$

Recall that an *involution* is a linear (over $\mathbb{R}$) mapping $\mathbb{R}^d\to\mathbb{R}^d$, $x\mapsto\overline{x}$ such that

$$\overline{ab}=\overline{b}\overline{a},\quad \overline{\overline{a}}=a\quad\forall\,a,b\in\mathbb{R}^d.$$

Let in $(\mathbb{R}^d,\cdot)$ be defined an involution such that

$$x+\overline{x}\in\mathbb{R}^1,\quad x\overline{x}\in\mathbb{R}^1_+\quad\forall\,x\in\mathbb{R}^d,\quad x\overline{x}=0\Leftrightarrow x=0.$$

In the linear space $\mathbb{R}^{2d}$ define a multiplication by formula

$$(a,b)\cdot(c,d) = (ac - \overline{d}b, da + b\overline{c}) \quad \forall\, a,b,c,d \in \mathbb{R}^d.$$

It is a straightforward task to ensure that the linear space $\mathbb{R}^{2d}$ equipped with this multiplication is a unitary $\mathbb{R}$-algebra, the vector $(e,0)$ is the identity element, the mapping

$$(a,b) \mapsto \overline{(a,b)} := (\overline{a}, -b) \quad \forall\,(a,b) \in \mathbb{R}^{2d}$$

is an involution, this involution satisfies the relations

$$(a,b) + \overline{(a,b)} \in \mathbb{R}^1, \quad (a,b)\cdot\overline{(a,b)} \in \mathbb{R}^1_+ \quad \forall\,(a,b) \in \mathbb{R}^{2d},$$

$$(a,b)\cdot\overline{(a,b)} = 0 \Leftrightarrow (a,b) = (0,0)$$

(here $\mathbb{R}^1 = \{(\lambda e, 0) : \lambda \in \mathbb{R}\}$, $\mathbb{R}^1_+ = \{(\lambda e, 0) : \lambda \geq 0\}$). Denote the obtained $\mathbb{R}$-algebra by $\mathcal{CD}(A)$.

The described procedure that allows to obtain a $2d$-dimensional unitary $\mathbb{R}$-algebra with involution from a d-dimensional unitary $\mathbb{R}$-algebra with involution is called *the Cayley-Dickson construction.* The Cayley-Dickson construction can be carried on ad infinitum. Denote

$$\mathcal{A}_1 := \mathbb{R}, \quad \mathcal{A}_{m+1} = \mathcal{CD}(\mathcal{A}_m) \quad \forall\, n \geq 1.$$

In this notation, the above mentioned algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, and $\mathbb{S}$ can be written as follows

$$\mathbb{R} = \mathcal{A}_1, \quad \mathbb{C} = \mathcal{A}_2, \quad \mathbb{H} = \mathcal{A}_3, \quad \mathbb{O} = \mathcal{A}_4, \quad \mathbb{S} = \mathcal{A}_5.$$

It turns out that all of the algebras $\mathcal{A}_m$ (except for $\mathcal{A}_1$) satisfy the assumptions of Theorems 2.1 and 2.2. Namely, the following assertion takes place.

Proposition 3.2. *For the above constructed $\mathbb{R}$-algebras, the following assertions take place.*

(i) *For every $m \geq 1$, the algebra $\mathcal{A}_m$ is power-associative (see [4]).*

(ii) *For every $m \geq 2$, for every $y \in \mathcal{A}_m$, equation (2.1) has a solution $x \in \mathcal{A}_m$.*

Proof. Assertion (i) was proved in [4]. Let us prove assertion (ii).

Let us prove by induction that for every $m \geq 2$ equation (3.2) in $\mathcal{A}_m$ has the zero solution only. For $m = 2$ it is obvious. Assume that this fact is valid for some $m \geq 2$. Consider equation (3.2) in the algebra $\mathcal{A}_{m+1}$. This equation can be introduced as a system of equations

$$\begin{cases} x_1^2 - \overline{x}_2 x_2 = 0, \\ x_2 x_1 + x_2 \overline{x}_1 = 0. \end{cases}$$

with unknown $x_1, x_2 \in \mathcal{A}_m$. Since $(x_1 + \overline{x}_1) \in \mathbb{R}^1$, for every solution (x_1, x_2) to this equation either $x_2 = 0$ or $x_1 + \overline{x}_1 = 0$. In the first case, the assumption of the induction implies $x_1 = 0$. In the second case, we obtain $x_1^2 = -\overline{x}_1 x_1$, so $x_1^2 - \overline{x}_2 x_2 = 0$ iff $x_1 = x_2 = 0$. Hence, equation (3.2) in the algebra $\mathcal{A}_{m+1}$ has the trivial solution only.

Let us prove by induction that for every $m \geq 2$ equation (3.1) in $\mathcal{A}_m$ has only two solutions $x = e$ and $x = -e$. For $m = 2$ it is obvious. Assume that this fact is valid for some $m \geq 2$. Consider equation (3.1) in the algebra $\mathcal{A}_{m+1}$. This equation can be introduced as a system of equations

$$\begin{cases} x_1^2 - \overline{x}_2 x_2 = e, \\ x_2 x_1 + x_2 \overline{x}_1 = 0. \end{cases}$$

with unknown $x_1, x_2 \in \mathcal{A}_m$ (here e is an identity element of $\mathcal{A}_m$). Since $(x_1 + \overline{x}_1) \in \mathbb{R}^1$, for every solution (x_1, x_2) to this equation either $x_2 = 0$ or $x_1 + \overline{x}_1 = 0$. In the first case, the assumption of the induction implies $x_1 = e$ or $x_1 = -e$. In the second case, we obtain $x_1^2 = -\overline{x}_1 x_1$, so $x_1^2 - \overline{x}_2 x_2 = e$ which has no solutions in $\mathcal{A}_m$. Hence, equation (3.1) in the algebra $\mathcal{A}_{m+1}$ has only two solutions $x = e$ and $x = -e$.

It follows from Proposition 3.1 that equation (2.1) in each algebra $\mathcal{A}_m$, $m \geq 2$, has a solution for every right-hand side $y \in \mathcal{A}_m$. □

Proposition 3.1 directly imply the following assertion.

Theorem 3.1. *Let $\mathcal{A}_m$ be the above defined Cayley-Dickson algebra. If $m \geq 2$ then*

(i) *equation (1.1) has a solution for every $n \geq 1$, $y \in \mathcal{A}_m$;*

(ii) *equation (2.2) has a solution for polynomial mappings $f : \mathcal{A}_m \to \mathcal{A}_m$ such that*

$$f(x) = ax^n + g(x) \quad \forall\, x \in \mathcal{A}_m,$$

n is odd, $a \in \mathcal{A}_m$ is not a zero devisor, $g : \mathbb{R}^d \to \mathbb{R}^d$ is the sum of a finite number of monomials, whose degrees are less than n.

This result provides an example of application of Theorems 2.1 and 2.2 to a class of $\mathbb{R}$-algebras.

Acknowledgments

The investigation was supported by the grant of the President of the Russian Federation (projects no. MK-2085.2017.1, MK-1938.2017.1) and by the “RUDN University Program 5-100”. The results of Section 2 are due to the second author, who was supported by the grant of the Russian Science Foundation (project no. 17-11-01168).

References

[1] S. Eilenberg, I. Niven, *The fundamental theorem of algebra for quaternions.* Bull. Am. Math. Soc. 509 (1944), 246–248.

[2] M.A. Krasnosel'skii, *Topological methods in the theory of nonlinear integral equations.* International series of monographs in pure and applied mathematics, Pergamon Press, 1968.

[3] J. Milnor, *Some consequences of a theorem of Bott.* Annals of Math. 68 (1958), no. 2., 444–449.

[4] R.D. Schafer, *On the algebras formed by the Cayley-Dickson process.* American J. of Math. 76 (1954), no. 2., 435–446.

[5] R.D. Schafer, *An introduction to nonassociative algebras.* Pure and Appl. Math. 22. N.Y. Academic Press, 1966.

Andronick Aramovich Arutyunov
Department of Higher Mathematics
Moscow Institute of Physics and Technology
Inststitutskii per., 9
141700, Dolgoprudny, Moscow region, Russian Federation
and
S.M. Nikol'skii Mathematical Institute
Peoples' Friendship University of Russia (RUDN University)
6 Miklukho-Maklaya Street
117198, Moscow, Russian Federation
E-mail: andronick.arutyunov@gmail.com

Sergey Evgen'evich Zhukovskiy
S.M. Nikol'skii Mathematical Institute
Department of Nonlinear Analysis
Peoples' Friendship University of Russia (RUDN University)
6 Miklukho-Maklaya Street
117198, Moscow, Russian Federation
E-mail: s-e-zhuk@yandex.ru

Received: 01.04.2017