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ERLAN DAUTBEKOVICH NURSULTANOV

(to the 60th birthday)

On May 25, 2017 was the 60th birthday of Yerlan Dautbekovich Nursultanov, Doctor of Physical and Mathematical Sciences (1999), Professor (2001), Head of the Department of Mathematics and Informatics of the Kazakhstan branch of the M.V. Lomonosov Moscow State University (since 2001), member of the Editorial Board of the Eurasian Mathematical Journal.

E.D. Nursultanov was born in the city of Karaganda. He graduated from the Karaganda State University (1979) and then completed his postgraduate studies at the M.V. Lomonosov Moscow State University.

Professor Nursultanov's scientific interests are related to various areas of the theory of functions and functional analysis.

He introduced the concept of multi-parameter Lorentz spaces, network spaces and anisotropic Lorentz spaces, for which appropriate interpolation methods were developed. On the basis of the apparatus introduced by him, the questions of reiteration in the off-diagonal case for the real Lyons-Petre interpolation method, the multiplier problem for trigonometric Fourier series, the lower and upper bounds complementary to the Hardy-Littlewood inequalities for various orthonormal systems were solved. The convergence of series and Fourier transforms were studied with sufficiently general monotonicity conditions. The lower bounds for the norm of the convolution operator are obtained, and its upper bounds are improved (a stronger result than the O'Neil inequality). An exact cubature formula with explicit nodes and weights for functions belonging to spaces with a dominated mixed derivative is constructed, and a number of other problems in this area are solved.

He has published more than 50 scientific papers in high rating international journals included in the lists of Thomson Reuters and Scopus. 2 doctor of sciences, 9 candidate of sciences and 4 PhD dissertations have been defended under his supervision.

His merits and achievements are marked with badges of the Ministry of Education and Science of the Republic of Kazakhstan "For Contribution to the Development of Science" (2007), "Honored Worker of Education" (2011), "Y. Altynsarin" (2017). He is a laureate of the award named after K. Satpaev in the field of natural sciences for 2005, the grant holder "The best teacher of the university" for 2006 and 2011, the grant holder of the state scientific scholarship for outstanding contribution to the development of science and technology of the Republic of Kazakhstan for years 2007-2008, 2008 -2009. In 2017 he got the Top Springer Author award, established by Springer Nature together with JSC "National Center for Scientific and Technical Information".

The Editorial Board of the Eurasian Mathematical Journal congratulates Erlan Dautbekovich Nursultanov on the occasion of his 60th birthday and wishes him good health and successful work in mathematics and mathematical education.

JAMALBEK TUSSUPOV

(to the 60th birthday)

On April 10, 2017 was the 60th birthday of Jamalbek Tussupov, Doctor of Physical and Mathematical Sciences, Professor, Head of the Information Systems Department of the L.N. Gumilyov Eurasian National University, member of the Kazakhstan and American Mathematical Societies, member of the Association of Symbolic Logic, member of the Editorial Board of the Eurasian Mathematical Journal.

J. Tussupov was born in Taraz (Jambyl region of the Kazakh SSR). He graduated from the Karaganda State University (Kazakhstan) in 1979 and later on completed his postgraduate studies at S.L. Sobolev Institute of Mathematics of the Academy of Sciences of Russia (Novosibirsk).

Professor Tussupov's research interests are in mathematical logic, computability, computable structures, abstract data types, ontology, formal semantics. He solved the following problems of computable structures:

- the problems of S.S. Goncharov and M.S. Manasse: the problem of characterizing relative categoricity in the hyperarithmetical hierarchy given levels of complexity of Scott families, and the problem on the relationship between categoricity and relative categoricity of computable structures in the arithmetical and hyperarithmetical hierarchies;

- the problem of Yu.L. Ershov: the problem of finite algorithmic dimension in the arithmetical and hyperarithmetical hierarchies;

- the problem of C.J. Ash and A. Nerode: the problem of the interplay of relations of bounded arithmetical and hyperarithmetical complexity in computable presentations and the definability of relations by formulas of given complexity;

- the problem of S. Lempp: the problem of structures having presentations in just the degrees of all sets X such that for algebraic classes as symmetric irreflexive graphs, nilpotent groups, rings, integral domains, commutative semigroups, lattices, structure with two equivalences, bipartite graphs.

Professor Tussupov has published about 100 scientific papers, five textbooks for students and one monograph. Three PhD dissertations have been defended under his supervision.

Professor Tussupov is a fellow of "Bolashak" Scholarship, 2011 (Notre Dame University, USA), "Erasmus+", 2016 (Poitiers University, France). He was awarded the title "The Best Professor of 2012" (Kazakhstan). In 2015 Jamalbek Tussupov was also awarded for the contribution to science in the Republic of Kazakhstan.

The Editorial Board of the Eurasian Mathematical Journal congratulates Dr. Professor Jamalbek Tussupov on the occasion of his 60th aniversary and wishes him strong health, new achievements in science, inspiration for new ideas and fruitfull results.

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NET SPACES ON LATTICES, HARDY-LITTLEWOOD TYPE INEQUALITIES, AND THEIR CONVERSES

R. Akylzhanov, M. Ruzhansky

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Key words: net spaces, Lie groups, homogeneous manifolds, Hardy-Littlewood inequality.

AMS Mathematics Subject Classification: 35G10, 35L30, 46F05.

Abstract. We introduce abstract net spaces on directed sets and prove their embedding and interpolation properties. Typical examples of interest are lattices of irreducible unitary representations of compact Lie groups and of class I representations with respect to a subgroup. As an application, we prove Hardy-Littlewood type inequalities and their converses on compact Lie groups and on compact homogeneous manifolds.

1 Introduction

In [8], Hardy and Littlewood proved the following estimate on the circle $\mathbb{T}$, relating L^p-norms of a function and its Fourier coefficients:

$$\sum_{m\in\mathbb{Z}} (1+|m|)^{p-2}|\widehat{f}(m)|^p \leq C\|f\|^p_{L^p(\mathbb{T})}, \quad 1<p\leq 2. \tag{1.1}$$

They also argued this to be a suitable extension of the Plancherel's identity to the setting of L^p-spaces. In fact, they also proved that the inequality becomes an equivalence provided that the Fourier coefficients $\widehat{f}(m)$ are monotone.

By duality, we readily obtain the corresponding inequality also in the range $2\leq p<\infty$, namely, we also have the estimate

$$\|f\|^p_{L^p(\mathbb{T})} \leq C'_p \sum_{m\in\mathbb{Z}} (1+|m|)^{p-2}|\widehat{f}(m)|^p, \quad 2\leq p<\infty. \tag{1.2}$$

In this paper we are interested in inequalities of Hardy-Littlewood type. For example, let $1<p<\infty$ and $f\in L^p(\mathbb{T})$, and suppose that

$$f\sim \sum_{m\in\mathbb{Z}} \widehat{f}(m)e^{2\pi imx}.$$

Then it was shown in [13] that

$$\sum_{k=1}^{\infty} k^{p-2}\left(\sup_{\substack{e\in M_0\\ |e|\geq k}} \frac{1}{|e|}\left|\sum_{m\in e}\widehat{f}(m)\right|\right)^p \leq C\|f\|^p_{L^p(\mathbb{T})}, \quad 1<p<\infty, \tag{1.3}$$

where M_0 is the set of all finite arithmetic progressions in $\mathbb{Z}$. Especially in the range $2\leq p<\infty$ this gives a converse estimate to the Hardy-Littlewood estimate (1.2).

Net function spaces $N_{p,q}$ on $\mathbb{Z}^n$ and $\mathbb{R}^n$ were introduced in [14] as a machinery to prove the inequalities of type (1.3) for Fourier coefficients of functions on $\mathbb{T}^n$ and $\mathbb{R}^n$. Since then, they found other applications as well: we can refer to [13, 14, 12] for some applications of these spaces to problems of harmonic analysis and approximation theory.

Since the unitary dual $\widehat{\mathbb{T}^n}$ of a compact abelian Lie group $\mathbb{T}^n$ is isomorphic to $\mathbb{Z}^n$, i.e.

$$\widehat{\mathbb{T}^n} = \{e^{2\pi ik\cdot x}\}_{k\in\mathbb{Z}^n} \ni e^{2\pi ik\cdot x} \longleftrightarrow k \in \mathbb{Z}^n,$$

we can consider $N_{p,q}(\mathbb{Z}^n)$ as a net space $N_{p,q}(\widehat{\mathbb{T}^n})$ on the lattice $\mathbb{Z}^n$. It turns out that the theory of net spaces $N_{p,q}$ can be extended to arbitrary lattices provided we make certain rather natural assumptions. In this paper we develop this abstract setting to be able to use the notion of a net space on the unitary dual of a compact Lie group and on the lattice of its class I representations. In addition to a suitable definition, for our purposes we need to prove their embedding and interpolation properties. As it is common, such technique allows one to derive 'strong' estimates from 'weak' ones by interpolation.

As an application of these results, we obtain Hardy-Littlewood type inequalities on compact Lie groups and compact homogeneous manifolds, also providing the inverses to the Hardy-Littlewood inequalities that were recently obtained in [1]. In Corollary 3.1 we calculate an explicit example of such an inverse (to the Hardy-Littlewood inequality) in the case of the group SU(2). For the Hardy-Littlewood inequalities and Fourier multipliers on SU(2) and on more general compact Lie groups we refer to [2] and [3], respectively.

The obtained results also yield a noncommutative version to known estimates of the type (1.3) on a circle $\mathbb{T}$.

In Section 2 we develop the notion of net spaces on rather general lattices and prove their main properties (interpolation and embedding). In Section 3 we apply these results to obtain inverses to known Hardy-Littlewood inequalities in the settings of compact Lie groups and compact homogeneous spaces.

2 Net spaces on lattices

Let Γ be a discrete set. We assume that there exists such partial order $\prec$ on Γ that every two elements in Γ are comparable under $\prec$ in Γ. In addition, we suppose that Γ is bounded from below. In other words, there exists an element $1 \in \Gamma$ such that $1 \prec \pi$ for all $\pi \in \Gamma$. The partial order $\prec$ on Γ makes it possible to define the notion of a net which was first introduced by Moore and Smith in [9]. This 'net' is different from 'net' in net spaces $N_{p,q}$. Let T be a topological space. A net a in T is a function from Γ to T, i.e.

$$a = \{a_\pi\}_{\pi\in\Gamma} \colon \Gamma \ni \pi \mapsto a_\pi \in T.$$

We consider two nets $\delta = \{\delta_\pi\}_{\pi\in\Gamma}$ and $\kappa = \{\kappa_\pi\}_{\pi\in\Gamma}$ in $T = \mathbb{N}$, i.e.

$$\Gamma \ni \pi \mapsto \delta_\pi \in \mathbb{N},$$
$$\Gamma \ni \pi \mapsto \kappa_\pi \in \mathbb{N}.$$

We turn Γ into a σ-finite measure space by introducing a measure

$$\nu_\Gamma(Q) := \sum_{\theta\in Q} \delta_\theta \kappa_\theta, \tag{2.1}$$

where Q is an arbitrary subset of Γ. We denote this measure space by (Γ, ν_Γ). We denote by Σ the space of matrix-valued sequences on Γ that will be realised via

$$\Sigma := \left\{h = \{h(\pi)\}_{\pi\in\Gamma}, h(\pi) \in \mathbb{C}^{\kappa_\pi\times\delta_\pi}\right\}.$$

The ℓ^p spaces on Σ can be defined, for example, motivated by the Fourier analysis on compact homogeneous spaces (see [15]), in the form

$$\|h\|_{\ell^p(\Gamma,\nu_\Gamma,\Sigma)} := \left(\sum_{\pi\in\Gamma} \delta_\pi \kappa_\pi^{p(\frac{1}{p}-\frac{1}{2})} \|h(\pi)\|_{\mathtt{HS}}^p\right)^{\frac{1}{p}}, \quad h\in\Sigma.$$

Sometimes, we can abbreviate this by writing $\ell^p(\Gamma,\nu_\Gamma), \ell^p(\Gamma)$ or ℓ^p.

If we put $\Gamma=\widehat{G}$, where $\widehat{G}$ is the unitary dual of a compact Lie group G, then Fourier transform can be regarded as an operator mapping a function $f\in L^p(G)$ to the matrix-valued sequence $\widehat{f}=\{\widehat{f}(\pi)\}_{\pi\in\widehat{G}}$ of the Fourier coefficients $\widehat{f}(\pi)\in\mathbb{C}^{d_\pi\times d_\pi}$ given by $\widehat{f}(\pi)=\int\limits_G f(u)\pi(u)^*\,du$. Let us denote by $\widehat{G}_0$ the subset of $\widehat{G}$ of representations that are class I with respect to some subgroup of G. For $\Gamma=\widehat{G}_0$ we put $\delta_\pi=d_\pi$ and $\kappa_\pi=k_\pi$, these spaces thus coincide with the $\ell^p(\widehat{G}_0)$ spaces introduced in [15]. See [11] for the definition of d_π and k_π but these notations will also be explained in detail in Section 3. See also [10] for the group setting.

It can be easily verified that the following formula holds true.

Remark 1. Let $1<p<\infty$. For $\ell^p(\Gamma,\nu_\Gamma,\Sigma)$, we have

$$\|h\|_{\ell^p(\Gamma,\nu_\Gamma)} = \sup_{\substack{g\in\ell^{p'}(\Gamma,\nu_\Gamma)\\ g\neq 0}} \frac{\left|\sum\limits_{\pi\in\Gamma}\delta_\pi \operatorname{Tr}[h(\pi)g(\pi)^*]\right|}{\|g\|_{\ell^{p'}(\Gamma,\nu_\Gamma)}}, \tag{2.2}$$

where $\frac{1}{p'}+\frac{1}{p}=1$, and $h,g\in\Sigma$. The matrix $g(\pi)^*$ is the Hermitian conjugate of $g(\pi)$ and Tr is the matrix trace.

We now give a definition of net spaces.

Definition 1. Let $\lambda=\{\lambda_\pi\}_{\pi\in\Gamma}$ be an arbitrary positive sequence over Γ. Denote by $\mathcal{M}$ a fixed arbitrary collection of finite subsets of Γ. Given a family of complex matrices $F=\{F(\pi)\}_{\pi\in\Gamma}$, $F(\pi)\in\mathbb{C}^{\kappa_\pi\times\delta_\pi}$ and $1\le p<\infty$, $1\le q\le\infty$, define

$$\|F\|_{N_{p,q}(\Gamma,\mathcal{M})} := \begin{cases} \left(\sum\limits_{\pi\in\Gamma}\left(\lambda_\pi^{\frac{1}{p}}\overline{F}[\lambda_\pi,\mathcal{M}]\right)^q \frac{\delta_\pi\kappa_\pi}{\lambda_\pi}\right)^{\frac{1}{q}}, & \text{if} \quad q<\infty,\\ \sup\limits_{\pi\in\Gamma}\lambda_\pi^{\frac{1}{p}}\overline{F}[\lambda_\pi,\mathcal{M}], & \text{if} \quad q=\infty, \end{cases}$$

where

$$\overline{F}[\lambda_\pi,\mathcal{M}] := \sup_{\substack{Q\in\mathcal{M}\\ \nu_\Gamma(Q)\ge\lambda_\pi}} \frac{1}{\nu_\Gamma(Q)}\left|\sum_{\theta\in Q} d_\theta \operatorname{Tr} F(\theta)\right|, \tag{2.3}$$

and $\operatorname{Tr}F(\theta)=\sum\limits_{j=1}^{\min(\kappa_\theta,\delta_\theta)} F(\theta)_{jj}$.

We call $\overline{F}[\lambda,\mathcal{M}]=\{\overline{F}[\lambda_\pi,\mathcal{M}]\}_{\pi\in\Gamma}$ defined by (2.3) *the averaging of* $F=\{F(\pi)\}_{\pi\in\Gamma}$ *with respect to* $\mathcal{M}$. Sometimes we may drop writing $\mathcal{M}$ to simplify the notation. In comparison to the well-known maximal function, the averaging function allows one to capture the oscillation properties of sequences/functions/nets.

In general, different partial orders $\prec_1$ and $\prec_2$ on Γ will give different $N_{p,q}$ spaces on Γ.

3 Hardy-Littlewood type inequalities

In this section we apply net spaces $N_{p,q}(\Gamma)$ on an ordered lattice Γ to establish new inequalities relating functions on G and their Fourier coefficients. In [13] the following theorem was established. In the sequel $L^{p,q}$ denotes the Lorenz space.

Theorem 3.1 ([13]). *Let $1 \leq p < \infty, 1 \leq q \leq \infty$ and $f \in L^{p,q}(\mathbb{T})$. Suppose that*

$$f \sim \sum_{m\in\mathbb{Z}} \widehat{f}(m) e^{2\pi i m x}.$$

Then we have

$$\sum_{k\in\mathbb{N}} \left(k^{\frac{1}{p'}} \sup_{\substack{e\in\mathcal{M}_0 \\ |e|\geq k}} \frac{1}{|e|} \left| \sum_{m\in e} \widehat{f}(m) \right| \right)^q \frac{1}{k} \leq C \|f\|^p_{L^{p,q}(\mathbb{T})}, \tag{3.1}$$

or equivalently in terms of net spaces $N_{p,q}(\mathbb{Z}, \mathcal{M}_0)$

$$\|\widehat{f}\|_{N_{p',q}(\mathbb{Z},\mathcal{M}_0)} \lesssim \|f\|_{L^{p,q}(\mathbb{T})}, \tag{3.2}$$

where $\mathcal{M}_0$ is the set of all finite arithmetic progressions in $\mathbb{Z}$, with the constant in (3.2) independent of f.

Remark 2. Since $N_{p,q}(\mathbb{Z}, \mathcal{M}_0)$ are interpolation spaces ([13, Theorem 1, p.88]), in order to establish (3.2) it is sufficient to establish a 'weak' inequality

$$\|\widehat{f}\|_{N_{p',\infty}(\mathbb{Z},\mathcal{M}_0)} \lesssim \|f\|_{L^{p}(\mathbb{T})}. \tag{3.3}$$

In [13, Proposition 1] it has been established that if the class $\mathcal{M}$ contains all finite subsets then $N_{p,q}$ coincides, up to constant, with the Lorenz space $L^{p,q}$:

Theorem 3.2 ([13]). *Let $1 < p < +\infty, 1 \leq q \leq \infty$, and let $\mathcal{M}_1$ be the set of all finite subsets of $\mathbb{Z}$. Then we have*

$$N_{p,q}(\mathbb{Z}, \mathcal{M}_1) \cong L^{p,q}(\mathbb{Z}). \tag{3.4}$$

For $1 < p < 2$, inequality (3.2) can be refined:

Remark 3. Let $1 < p < 2$. We have by Theorem 4.1 below and Theorem 3.2, that

$$\|\widehat{f}\|_{N_{p',p}(\mathbb{Z},\mathcal{M}_0)} \lesssim \|\widehat{f}\|_{N_{p',p}(\mathbb{Z},\mathcal{M}_1)} \cong \|\widehat{f}\|_{L^{p',p}(\mathbb{Z})} \lesssim \|f\|_{L^{p}(\mathbb{T})}. \tag{3.5}$$

The last inequality in (3.5) is essentially the Hardy-Littlewood inequality for Fourier coefficients, see [13] for the details. The application of $N_{p,q}(\Gamma)$ with $\Gamma = \widehat{G}_0$ yields the extension of Theorem 3.2 to the setting of compact homogeneous manifolds G/K. In addition, in Theorem 3.3 we characterise those classes $\mathcal{M}$ for which inequality

$$\|\widehat{f}\|_{N_{p',\infty}(\widehat{G}_0,\mathcal{M})} \leq C\|f\|_{L^{p}(G/K)}, \quad f \in L^p(G/K) \tag{3.6}$$

holds. The characterisation is given in terms of of the behaviour of Dirichlet kernel's norms $\|D_Q\|_{L^p(G/K)}$, $Q \in \mathcal{M}$.

To motivate the formulation, we start with a compact Lie group G. Identifying a representation π with its equivalence class and choosing some bases in the representation spaces of degree

d_π, we can think of $\pi \in \widehat{G}$ as a mapping $\pi : G \to \mathbb{C}^{d_\pi \times d_\pi}$. For $f \in L^1(G)$, we define its Fourier transform at $\pi \in \widehat{G}$ by

$$(\mathscr{F}_G f)(\pi) \equiv \widehat{f}(\pi) := \int_G f(u)\pi(u)^* du,$$

where du is the normalised Haar measure on G. This definition can be extended to distributions $f \in \mathcal{D}'(G)$, and the Fourier series takes the form

$$f(u) = \sum_{\pi \in \widehat{G}} d_\pi \operatorname{Tr}\left(\pi(u)\widehat{f}(\pi)\right). \tag{3.7}$$

The Plancherel identity on G is given by

$$\|f\|^2_{L^2(G)} = \sum_{\pi \in \widehat{G}} d_\pi \|\widehat{f}(\pi)\|^2_{\mathtt{HS}} =: \|\widehat{f}\|^2_{\ell^2(\widehat{G})}, \tag{3.8}$$

yielding the Hilbert space $\ell^2(\widehat{G})$. The Fourier coefficients of functions and distributions on G take values in the space

$$\Sigma = \left\{\sigma = (\sigma(\pi))_{\pi \in \widehat{G}} : \sigma(\pi) \in \mathbb{C}^{d_\pi \times d_\pi}\right\}. \tag{3.9}$$

The ℓ^p-spaces on the unitary dual $\widehat{G}$ have been developed in [15] based on fixing the Hilbert-Schmidt norms. Namely, for $1 \le p < \infty$, we define the space $\ell^p(\widehat{G})$ by the norm

$$\|\sigma\|_{\ell^p(\widehat{G})} := \left(\sum_{\pi \in \widehat{G}} d_\pi^{p\left(\frac{2}{p}-\frac{1}{2}\right)} \|\sigma(\pi)\|^p_{\mathtt{HS}}\right)^{1/p}, \ \sigma \in \Sigma, \ 1 \le p < \infty, \tag{3.10}$$

where $\|\cdot\|_{\mathtt{HS}}$ denotes the Hilbert-Schmidt matrix norm i.e.

$$\|\sigma(\pi)\|_{\mathtt{HS}} := \left(\operatorname{Tr}(\sigma(\pi)\sigma(\pi)^*)\right)^{\frac{1}{2}}.$$

It was shown in [15, Section 10.3] that, among other things, these are interpolation spaces, and that the Fourier transform $\mathscr{F}_G$ and its inverse $\mathscr{F}_G^{-1}$ satisfy the Hausdorff-Young inequalities in these spaces. We can also refer to [16] for pseudo-differential extensions of the Fourier analysis on both compact Lie groups and homogeneous manifolds.

We now describe the setting of Fourier coefficients on a compact homogeneous manifold M following [6] or [11], and referring for further details with proofs to Vilenkin [18] or to Vilenkin and Klimyk [19].

Let G be a compact motion group of M and let K be the stationary subgroup of some point. Alternatively, we can start with a compact Lie group G with a closed subgroup K, and identify $M = G/K$ as an analytic manifold in a canonical way. We normalise measures so that the measure on K is a probability one. Typical examples are the spheres $\mathbb{S}^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$ or complex spheres $\mathbb{CS}^n = \mathrm{SU}(n+1)/\mathrm{SU}(n)$.

Let us denote by $\widehat{G}_0$ the subset of $\widehat{G}$ of representations that are class I with respect to the subgroup K. This means that $\pi \in \widehat{G}_0$ if π has at least one non-zero invariant vector a with respect to K, i.e. that

$$\pi(h)a = a \ \text{ for all } \ h \in K.$$

Let $\mathcal{B}_\pi$ denote the space of these invariant vectors and let

$$k_\pi := \dim \mathcal{B}_\pi.$$

Let us fix an orthonormal basis in the representation space of π so that its first k_π vectors are the basis of B_π. The matrix elements $\pi(x)_{ij}$, $1 \leq j \leq k_\pi$, are invariant under the right shifts by K.

We note that if $K = \{e\}$ so that $M = G/K = G$ is the Lie group, we have $\widehat{G} = \widehat{G}_0$ and $k_\pi = d_\pi$ for all π. As the other extreme, if K is a massive subgroup of G, i.e., if for every π there is precisely one invariant vector with respect to K, we have $k_\pi = 1$ for all $\pi \in \widehat{G}_0$. This is, for example, the case for the spheres $M = \mathbb{S}^n$. Other examples can be found in Vilenkin [18].

We can now identify functions on $M = G/K$ with functions on G which are constant on left cosets with respect to K. Then, for a function $f \in C^\infty(M)$ we can recover it by the Fourier series of its canonical lifting $\widetilde{f}(g) := f(gK)$ to G, $\widetilde{f} \in C^\infty(G)$, and the Fourier coefficients satisfy $\widehat{\widetilde{f}}(\pi) = 0$ for all representations with $\pi \notin \widehat{G}_0$. Also, for class I representations $\pi \in \widehat{G}_0$ we have $\widehat{\widetilde{f}}(\pi)_{ij} = 0$ for $i > k_\pi$.

With this, we can write the Fourier series of f (or of $\widetilde{f}$, but we identify these) in terms of the spherical functions π_{ij} of the representations $\pi \in \widehat{G}_0$, with respect to the subgroup K. Namely, the Fourier series (3.7) becomes

$$f(x) = \sum_{\pi \in \widehat{G}_0} d_\pi \sum_{i=1}^{d_\pi} \sum_{j=1}^{k_\pi} \widehat{f}(\pi)_{ji} \pi(x)_{ij} = \sum_{\pi \in \widehat{G}_0} d_\pi \operatorname{Tr}(\widehat{f}(\pi)\pi(x)), \tag{3.11}$$

where, in order to have the last equality, we adopt the convention of setting $\pi(x)_{ij} := 0$ for all $j > k_\pi$, for all $\pi \in \widehat{G}_0$. With this convention the matrix $\pi(x)\pi(x)^*$ is diagonal with the first k_π diagonal entries equal to one and others equal to zero, so that we have

$$\|\pi(x)\|_{\mathtt{HS}} = \sqrt{k_\pi} \text{ for all } \pi \in \widehat{G}_0,\ x \in G/K. \tag{3.12}$$

Following [6], we will say that *the collection of Fourier coefficients* $\{\widehat{f}(\pi)_{ij} : \pi \in \widehat{G}, 1 \leq i, j \leq d_\pi\}$ *is of class I with respect to* K if $\widehat{f}(\pi)_{ij} = 0$ whenever $\pi \notin \widehat{G}_0$ or $i > k_\pi$. By the above discussion, if the collection of Fourier coefficients is of class I with respect to K, then the expressions (3.7) and (3.11) coincide and yield a function f such that $f(xh) = f(h)$ for all $h \in K$, so that this function becomes a function on the homogeneous space G/K.

For the space of Fourier coefficients of class I we define the analogue of the set Σ in (3.9) by

$$\Sigma(G/K) := \{\sigma : \pi \mapsto \sigma(\pi) :\ \pi \in \widehat{G}_0,\ \sigma(\pi) \in \mathbb{C}^{d_\pi \times d_\pi},\ \sigma(\pi)_{ij} = 0 \text{ for } i > k_\pi\}. \tag{3.13}$$

In analogy to (3.10), we can define the Lebesgue spaces $\ell^p(\widehat{G}_0)$ by the following norms which we will apply to Fourier coefficients $\widehat{f} \in \Sigma(G/K)$ of $f \in \mathcal{D}'(G/K)$. Thus, for $\sigma \in \Sigma(G/K)$ we set

$$\|\sigma\|_{\ell^p(\widehat{G}_0)} := \left(\sum_{\pi \in \widehat{G}_0} d_\pi k_\pi^{p(\frac{1}{p} - \frac{1}{2})} \|\sigma(\pi)\|_{\mathtt{HS}}^p \right)^{1/p},\ 1 \leq p < \infty. \tag{3.14}$$

In the case $K = \{e\}$, so that $G/K = G$, these spaces coincide with those defined by (3.10) since $k_\pi = d_\pi$ in this case. Again, by the same argument as that in [15], these spaces are interpolation spaces and the Hausdorff-Young inequality holds for them. We refer to [11] for some more details on these spaces.

Let $\mathcal{M}$ be an arbitrary collection of finite subsets of $\widehat{G}_0$. Denote by $\mathcal{M}_1$ the collection of all finite subsets of $\widehat{G}_0$. For $Q \subset \mathcal{M}_1$, the measure ν_Γ with $\Gamma = \widehat{G}_0$ in (2.1) is finite and is given by

$$\nu_\Gamma(Q) = \sum_{\theta \in Q} d_\theta k_\theta, \quad Q \in \mathcal{M}_1 \subset \widehat{G}_0. \tag{3.15}$$

Let $Q \in \mathcal{M}$ and write

$$D_Q(x) := \sum_{\pi \in Q} d_\pi \operatorname{Tr}[\pi(x)].$$

Denote by $D(\mathcal{M})$ the set of all Dirichlet kernels with their spectrum embedded in some $Q \in \mathcal{M}$, i.e.

$$D(\mathcal{M}) := \{D_Q(x), Q \in \mathcal{M}\}.$$

Now, we can characterise those $\mathcal{M}$ for which

$$\|\widehat{f}\|_{N_{p',\infty}(\widehat{G}_0,\mathcal{M})} \leq C\|f\|_{L^p(G/K)}, \quad f \in L^p(G/K),$$

via a certain condition on the size of the $\|D_Q\|_{L^{p'}(G/K)}$ norm. In the theorem below we have $\Gamma = \widehat{G}_0$ and the measure $\nu_\Gamma = \nu_{\widehat{G}_0}$ is given by (3.15). In the sequel we can use both notations in the case of homogeneous manifolds G/K.

Theorem 3.3. *Let $1 < p \leq \infty$ and let $\mathcal{M}$ be an arbitrary collection of finite subsets of $\widehat{G}_0$. Then*

$$\|\widehat{f}\|_{N_{p',\infty}(\widehat{G}_0,\mathcal{M})} \leq C\|f\|_{L^p(G/K)}, \text{ for all } f \in L^p(G/K), \tag{3.16}$$

if and only if

$$C_{p\mathcal{M}} := \sup_{\pi \in \widehat{G}_0} \lambda_\pi^{\frac{1}{p'}} \sup_{\substack{Q \in \mathcal{M} \\ \nu_\Gamma(Q) \geq \lambda_\pi}} \frac{1}{\nu_\Gamma(Q)} \|D_Q\|_{L^{p'}(G/K)} < +\infty, \tag{3.17}$$

with $\nu_\Gamma = \nu_{\widehat{G}_0}$, and λ_π is the sequence used in the Defintion 1.

In Proposition 3.1 we will check in the sequel that condition (3.17) for all indices $1 < p < \infty$ is satisfied in the example of the tori $\mathbb{T}^n$ if we take λ_π to be the sequence of the eigenvalues of the Laplacian counted with multiplicities. In Theorem 3.4, verifying condition (3.17), we will give an unconditional version of Theorem 3.3 for the range of indices $1 < p \leq 2$ on general compact homogeneous manifolds based on the interpolation properties of net spaces to be established in the next section. In Corollary 3.1 we give an example on the group $\mathrm{SU}(2)$, again if we take λ_π to be the sequence of the eigenvalues of the Laplacian counted with multiplicities, yielding an inverse to the Hardy-Littlewood inequality there.

Remark 4. It follows from the proof that $C_{p\mathcal{M}} \leq C$ and inequality (3.16) holds true for $C = C_{p\mathcal{M}}$.

The interpolation properties of $N_{p,q}$ spaces allow us to formulate and prove a version of the Hardy-Littlewood inequality in terms of $N_{p,q}$ spaces.

Theorem 3.4. *Let $1 < p \leq 2$, $1 \leq q \leq \infty$, and let $\mathcal{M}_1$ be the collection of all finite subsets of $\widehat{G}_0$. Then we have*

$$\|\widehat{f}\|_{N_{p',q}(\widehat{G}_0,\mathcal{M}_1)} \leq C_{p,q}\|f\|_{L^{p,q}(G/K)}. \tag{3.18}$$

We give a corollary of Theorem 3.4 on $\mathrm{SU}(2)$. In this case, we simplify general notation. It can be shown that the unitary dual $\widehat{\mathrm{SU}(2)}$ can be 'labelled' by the set of half-integers $\frac{1}{2}\mathbb{N}_0$. Thus, we write $\widehat{f}(l)$ for the Fourier coefficient with respect to the element $t^l \in \widehat{\mathrm{SU}(2)}$ associated with $l \in \frac{1}{2}\mathbb{N}_0$. Here $d_{t^l} = 2l+1$ so that $\widehat{f}(l) \in \mathbb{C}^{(2l+1)\times(2l+1)}$, $l \in \frac{1}{2}\mathbb{N}_0$.

Corollary 3.1. *Let $1 < p \le 2$. Then we have*

$$\sum_{\xi\in\frac{1}{2}\mathbb{N}_0}(2\xi+1)^{\frac{5p}{2}-4}\left(\sup_{\substack{k\in\frac{1}{2}\mathbb{N}_0\\(2k+1)\ge(2\xi+1)}}\frac{1}{(2k+1)^3}\left|\sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0\\2l+1\le 2k+1}}(2l+1)\operatorname{Tr}\widehat{f}(l)\right|\right)^p \le C_p\|f\|^p_{L^p(\mathrm{SU}(2))}. \quad (3.19)$$

We will prove Corollary 3.1 in Section 4 after Remark 5.

Proof of Theorem 3.4. For $1 < p \le 2$ the condition (3.17) is satisfied with $\mathcal{M} = \mathcal{M}_1$. Indeed, by Hausdorff-Young inequality, with $\Gamma = \widehat{G}_0$ and $\nu_\Gamma(Q) = \sum_{\pi\in Q} d_\pi k_\pi$, $Q \subset \widehat{G}_0$, we have

$$\|D_Q\|_{L^{p'}(G/K)} \le \nu_\Gamma(Q)^{\frac{1}{p}}, \quad Q \in \mathcal{M}_1,\ 1 < p \le 2. \quad (3.20)$$

Then, we get

$$C_{p\mathcal{M}_1} = \sup_{\pi\in\widehat{G}_0}\lambda_\pi^{\frac{1}{p'}}\sup_{\substack{Q\in\mathcal{M}_1\\\nu_\Gamma(Q)\ge\lambda_\pi}}\frac{1}{\nu_\Gamma(Q)}\|D_Q\|_{L^{p'}(G/K)} \le \sup_{\pi\in\widehat{G}_0}\lambda_\pi^{\frac{1}{p'}}\sup_{\substack{Q\in\mathcal{M}_1\\\nu_\Gamma(Q)\ge\lambda_\pi}}\frac{1}{\nu_\Gamma(Q)^{\frac{1}{p'}}} = 1. \quad (3.21)$$

This proves that the condition (3.17) is satisfied. Thus, the application of Theorem 3.3 yields

$$\|\widehat{f}\|_{N_{p',\infty}(\widehat{G}_0,\mathcal{M}_1)} \le C_{p\mathcal{M}_1}\|f\|_{L^p(G/K)}, \quad 1 < p \le 2.$$

Let $1 < p_1 < p < p_2 \le 2$. Then interpolating between two inequalities

$$\|\widehat{f}\|_{N_{p_1',\infty}(\widehat{G}_0,\mathcal{M}_1)} \le C_{p_1\mathcal{M}_1}\|f\|_{L^{p_1}(G/K)},$$
$$\|\widehat{f}\|_{N_{p_2',\infty}(\widehat{G}_0,\mathcal{M}_1)} \le C_{p\mathcal{M}_1}\|f\|_{L^{p_2}(G/K)}.$$

(see Theorem 4.2 below), we obtain

$$\|\widehat{f}\|_{N_{p',q}(\widehat{G}_0,\mathcal{M}_1)} \le \|\widehat{f}\colon (N_{p_1,\infty}(\widehat{G}_0,\mathcal{M}_1), N_{p_1,\infty}(\widehat{G}_0,\mathcal{M}_1))_{\theta,q}\| \le C_{p,q}\|f\colon (L^{p_1}(G/K), L^{p_2}(G/K))_{\theta,q}\| = C_{p,q}\|f\|_{L^{p,q}(G/K)},$$

where in the last equality we used the fact that $L^{p,q}(G/K)$ are interpolation spaces. This completes the proof. □

Now, we show that Theorem 3.3 includes as a particular case Theorem 3.1.

Proposition 3.1. *Let $1 < p < \infty$, let M_a be the set of all finite arithmetic progressions in $\mathbb{Z}^n$, $G = \mathbb{T}^n$, $\lambda_{e^{2\pi im\cdot x}} = m$, $d_{e^{2\pi im\cdot x}} = k_{e^{2\pi im\cdot x}} = 1$ for $m \in \mathbb{Z}^n$, and hence $\nu_{\mathbb{Z}^n}(Q) = |Q|$, $Q \in \mathcal{M}_a$. Then Theorem 3.3 implies Theorem 3.1.*

Here $|\cdot|$ is the counting measure.

Proof of Proposition 3.1. We show that condition (3.17) holds true for this case. Indeed, using the L^p-space duality, we have

$$\|D_Q\|_{L^{p'}(\mathbb{T}^n)} = \sup_{\substack{f\in L^p(\mathbb{T}^n)\\f\ne 0}}\frac{\left|(D_Q, f)_{L^2(\mathbb{T}^n)}\right|}{\|f\|_{L^p(\mathbb{T}^n)}}. \quad (3.22)$$

By the Hardy-Littlewood rearrangement inequality (see [5, p.44 Theorem 2.2]), we obtain

$$\left|(D_Q, f)_{L^2(\mathbb{T}^n)}\right| \leq \int\limits_{\mathbb{T}^n} |D_Q(x)f(x)|\,dx \leq \int\limits_0^1 D_Q^*(t)f^*(t)\,dt. \tag{3.23}$$

In [13, p. 98 Lemma 5] it has been shown that

$$D_Q^*(t) \lesssim \frac{|Q|^{\frac{1}{p}}}{t^{\frac{1}{p'}}}, \tag{3.24}$$

where $D_Q(x) = \sum\limits_{k\in Q} e^{2\pi ik\cdot x},\ Q \subset \mathcal{M}_a$. Hence, we get

$$\left|(D_Q, f)_{L^2(\mathbb{T}^n)}\right| \leq |Q|^{\frac{1}{p}} \int\limits_0^1 t^{\frac{1}{p}} f^*(t)\frac{dt}{t} = |Q|^{\frac{1}{p}}\|f\|_{L^{p,1}(\mathbb{T}^n)}. \tag{3.25}$$

In [7, p. 220 Theorem 4.7] it has been shown that the following equality holds true

$$\|f\|_{L^{p,q}(X,\mu)} = \sup_{g\neq 0} \frac{\int\limits_X |fg|d\mu}{\|g\|_{L^{p',q'}(X,\mu)}}. \tag{3.26}$$

Using the $L^{p,q}$-space duality (3.26) and (3.25), we get

$$\|D\|_{L^{p',1}(\mathbb{T}^n)} \leq |Q|^{\frac{1}{p}}.$$

The application of this and of the embedding propeties of the Lorenz spaces (see [7, p.217 Proposition 4.2]) yield

$$\|D_Q\|_{L^{p'}(\mathbb{T}^n)} \leq \|D_Q\|_{L^{p',q}(\mathbb{T}^n)} \leq |Q|^{\frac{1}{p}}, \quad Q \in M_a. \tag{3.27}$$

Finally, using this, we obtain

$$\sup_{k\in\mathbb{Z}^n} k^{\frac{1}{p'}} \sup_{\substack{Q\in\mathcal{M}_a\\ |Q|\geq k}} \frac{1}{|Q|}\|D_Q\|_{L^{p'}(\mathbb{T}^n)} \leq \sup_{k\in\mathbb{Z}^n} k^{\frac{1}{p'}} \sup_{\substack{Q\in\mathcal{M}_a\\ |Q|\geq k}} \frac{1}{|Q|^{\frac{1}{p'}}} = 1. \tag{3.28}$$

This completes the proof. □

Proof of Theorem 3.3. We shall show that either of inequalities (3.17) and (3.16) implies each other. Let us first claim that we have

$$(f, D_Q)_{L^2(G/K)} = \sum_{\pi\in Q} d_\pi \operatorname{Tr} \widehat{f}(\pi). \tag{3.29}$$

If we assume this claim for the moment, the proof proceeds as follows.

⇒. By Hölder inequality, we have

$$|f * D_Q(0)| = \left|(f, D_Q)_{L^2(G/K)}\right| \leq \|f\|_{L^p(G/K)}\|D_Q\|_{L^{p'}(G/K)}. \tag{3.30}$$

We multiply the left-hand side in (3.30) by $\lambda_\pi^{\frac{1}{p'}}/\nu_{\widehat{G}_0}(Q)$ to get

$$\frac{\lambda_\pi^{\frac{1}{p'}}}{\nu_{\widehat{G}_0}(Q)}\,|(f * D_Q)(0)| \leq \frac{\lambda_\pi^{\frac{1}{p'}}}{\nu_{\widehat{G}_0}(Q)}\|D_Q\|_{L^{p'}(G/K)}\|f\|_{L^p(G/K)}. \tag{3.31}$$

Fixing $\pi \in \widehat{G}_0$ and then taking supremum over all $Q \in \mathcal{M}$ such that $\nu_{\widehat{G}_0}(Q) \geq \lambda_\pi$ in the right-hand side in (3.31), we get

$$\frac{\lambda_\pi^{\frac{1}{p'}}}{\nu_{\widehat{G}_0}(Q)} \left|(f * D_Q)(0)\right| \leq \left[\lambda_\pi^{\frac{1}{p'}} \sup_{\substack{Q\in\mathcal{M} \\ \nu_{\widehat{G}_0}(Q)\geq\lambda_\pi}} \frac{\|D_Q\|_{L^{p'}(G/K)}}{\nu_{\widehat{G}_0}(Q)}\right] \|f\|_{L^p(G/K)}. \tag{3.32}$$

Again taking supremum over all $\pi \in \widehat{G}_0$ in the right-hand side in (3.32), we finally obtain

$$\frac{\lambda_\pi^{\frac{1}{p'}}}{\nu_{\widehat{G}_0}(Q)} \left|(f * D_Q)(0)\right| \leq \left[\sup_{\pi\in\widehat{G}_0} \lambda_\pi^{\frac{1}{p'}} \sup_{\substack{Q\in\mathcal{M} \\ \nu_{\widehat{G}_0}(Q)\geq\lambda_\pi}} \frac{\|D_Q\|_{L^{p'}(G/K)}}{\nu_{\widehat{G}_0}(Q)}\right] \|f\|_{L^p(G/K)}. \tag{3.33}$$

Applying the preceding procedure of taking the supremum on the left-hand side in (3.33), we show

$$\begin{aligned}\|\widehat{f}\|_{N_{p',\infty}(\widehat{G}_0,\mathcal{M})} &= \sup_{\pi\in\widehat{G}_0} \sup_{\substack{Q\in\mathcal{M} \\ \nu_{\widehat{G}_0}(Q)\geq\lambda_\pi}} \frac{\lambda_\pi^{\frac{1}{p'}}}{\nu_{\widehat{G}_0}(Q)} \left|(f * D_Q)(0)\right| \|f\|_{L^p(G/K)} \\ &\leq C_{p\mathcal{M}} \|f\|_{L^p(G/K)}, \end{aligned} \tag{3.34}$$

where $C^p_{D\mathcal{M}}$ is the constant defined in the hypothesis of Theorem 3.3.

$\Leftarrow$.

By the definition of the $\|\cdot\|_{N_{p',\infty}(\widehat{G}_0,\mathcal{M})}$-norm, it follows from (3.16) that

$$\sup_{\pi\in\widehat{G}_0} \sup_{\substack{Q\in\mathcal{M} \\ \nu_{\widehat{G}_0}(Q)\geq\lambda_\pi}} \frac{\lambda_\pi^{\frac{1}{p'}}}{\nu_{\widehat{G}_0}(Q)} \left|(f * D_Q)(0)\right| \lesssim \|f\|_{L^p(G/K)}, \quad f \in L^p(G/K). \tag{3.35}$$

Then for any pair $(\pi, Q) \in \widehat{G}_0 \times \mathcal{M}$ such that $\nu_{\widehat{G}_0}(Q) \geq \lambda_\pi$ and $f \in L^p(G/K)$ we have

$$\frac{\lambda_\pi^{\frac{1}{p'}}}{\nu_{\widehat{G}_0}(Q)} \left|(f, D_Q)_{L^2(G/K)}\right| \leq C\|f\|_{L^p(G/K)}, \tag{3.36}$$

and we used the fact that $f * D_Q(0) = (f, D_Q)_{L^2(G/K)}$. Multiplying both sides of (3.36), we get

$$\left|(f, D_Q)_{L^2(G/K)}\right| \leq C\frac{\nu_{\widehat{G}_0}(Q)}{\lambda_\pi^{\frac{1}{p'}}}\|f\|_{L^p(G/K)}. \tag{3.37}$$

The inverse Hölder inequality then implies that

$$\|D_Q\|_{L^{p'}(G/K)} \leq C\frac{\nu_{\widehat{G}_0}(Q)}{\lambda_\pi^{\frac{1}{p'}}}. \tag{3.38}$$

Equivalently, we have

$$\frac{\lambda_\pi^{\frac{1}{p'}}}{\nu_{\widehat{G}_0}(Q)}\|D_Q\|_{L^{p'}(G/K)} \leq C. \tag{3.39}$$

Taking supremum in the left-hand side in (3.39), we obtain

$$C_{p\mathcal{M}} = \sup_{\pi\in\widehat{G}_0} \lambda_\pi^{\frac{1}{p'}} \sup_{\substack{Q\in\mathcal{M}\\ \nu_{\widehat{G}_0}(Q)\geq\lambda_\pi}} \frac{\|D_Q\|_{L^{p'}(G/K)}}{\nu_{\widehat{G}_0}(Q)} \leq C < +\infty, \tag{3.40}$$

in view of the fact that (π, Q) is any pair satisfying $\mu(Q) \geq \lambda_\pi$ and C is fixed and does not depend on Q nor on π.

Now, it remains to establish (3.29). Since the trace Tr is invariant under taking Hermitian conjugate $\pi \mapsto \pi^*$, up to complex conjugation i.e.

$$\operatorname{Tr}\pi^* = \overline{\operatorname{Tr}\pi},$$

we have

$$\begin{aligned}(g, D_Q)_{L^2(G/K)} &= \int_{G/K} g(x)\overline{D_Q(x)}\,dx = \int_{G/K} g(x)\overline{\sum_{\pi\in Q} d_\pi \operatorname{Tr}\pi(x)}\,dx \\ &= \int_{G/K} g(x)\sum_{\pi\in Q} d_\pi \overline{\operatorname{Tr}\pi(x)}\,dx = \int_{G/K} g(x)\sum_{\pi\in Q} d_\pi \operatorname{Tr}\pi^*(x)\,dx \\ &= \sum_{\pi\in Q}\int_{G/K} g(x)\,d_\pi \operatorname{Tr}\pi^*(x)\,dx, \end{aligned} \tag{3.41}$$

where $D_Q(x) = \sum\limits_{\pi\in Q} d_\pi \operatorname{Tr}\pi(x)$. Interchanging $\int\limits_{G/K}$ and Tr in the last line in (3.41), we get

$$(g, D_Q)_{L^2(G/K)} = \sum_{\pi\in Q} d_\pi \operatorname{Tr}\int_{G/K} g(x)\pi^*(x)\,dx = \sum_{\pi\in Q} d_\pi \operatorname{Tr}\widehat{g}(\pi), \tag{3.42}$$

where we used that the Fourier coefficients $\widehat{g}(\pi)$ are, by definition, equal to

$$\widehat{g}(\pi) = \int_{G/K} g(x)\pi^*(x)\,dx. \tag{3.43}$$

This proves (3.29). This completes the proof. □

4 On some properties of net spaces

We now formulate some assumptions which will allow us to establish interpolation theory of $N_{p,q}$ spaces which was needed in the proof of Theorem 3.4. In the case $\Gamma = \widehat{G}$ or $\Gamma = \widehat{G}_0$, these assumptions will be satisfied.

Assumption 4.1. *Suppose that a positive net $\{\lambda_\pi\}_{\pi\in\Gamma}$ is monotone increasing, i.e.*

$$\xi \prec \pi \quad \text{if and only if} \quad \lambda_\xi \leq \lambda_\pi. \tag{4.1}$$

Assumption 4.2. *Let $\beta \in \mathbb{R}$ with $\beta \neq -1$. Suppose that the following formulae are true*

$$\sum_{\substack{\theta\in\Gamma\\ \lambda_\theta\leq\lambda_\pi}} \lambda_\theta^\beta \kappa_\theta \delta_\theta = C_\beta \lambda_\pi^{\beta+1} \text{ for } \beta > -1, \tag{4.2}$$

$$\sum_{\substack{\theta\in\Gamma\\ \lambda_\theta\geq\lambda_\pi}} \lambda_\theta^\beta \kappa_\theta \delta_\theta = C_\beta \lambda_\pi^{\beta+1} \text{ for } \beta < -1. \tag{4.3}$$

where C_β is a constant depending on β.

Remark 5. Assumption 4.2 is satisfied for $\Gamma = \widehat{G}_0, \lambda_\theta = \langle\theta\rangle^n, \delta_\theta = d_\theta, \kappa_\theta = k_\theta$, $\theta \in \widehat{G}_0$, where $\langle\theta\rangle$ are the eigenvalues of the first-order elliptic pseudo-differential operator $(I - \Delta_{G/K})^{\frac{1}{2}}$ on the compact manifold G/K of dimension n, namely, we have

$$\sum_{\substack{\theta\in\widehat{G}_0\\ \langle\theta\rangle\le\langle\pi\rangle}} \langle\theta\rangle^{\beta n} k_\theta d_\theta \simeq \langle\pi\rangle^{(\beta+1)n} \text{ for } \beta > -1, \tag{4.4}$$

$$\sum_{\substack{\theta\in\widehat{G}_0\\ \langle\theta\rangle\ge\langle\pi\rangle}} \langle\theta\rangle^{\beta n} k_\theta d_\theta \simeq \langle\pi\rangle^{(\beta+1)n} \text{ for } \beta < -1. \tag{4.5}$$

This will be proved in Section 5. We can also recall that if $K = \{e\}$ and hence $G/K = G$ is a compact Lie group, then $\widehat{G}_0 = \widehat{G}$ and $k_\theta = d_\theta$ is the dimension of the representation $[\theta] \in \widehat{G}$.

Now we can give a proof of Corollary 3.1.

Proof of Corollary 3.1. For $G = \mathrm{SU}(2)$ and $\lambda_l = (2l+1)^3$, it is straightforward to check that Assumption 4.2 holds true. Then by Theorem 3.4, we get

$$\|\widehat{f}\|_{N_{p',p}(\frac{1}{2}\mathbb{N}_0,\mathcal{M}_1)} \le C_p \|f\|_{L^p(\mathrm{SU}(2))}, \quad 1 < p \le 2. \tag{4.6}$$

By Definition 1 with $\Gamma = \widehat{\mathrm{SU}(2)}, \delta_l = \kappa_l = (2l+1)$, the left-hand side in (4.6) is equal to

$$\sum_{\xi\in\frac{1}{2}\mathbb{N}_0} (2\xi+1)^{\frac{5p}{2}-4} \left(\sup_{\substack{k\in\frac{1}{2}\mathbb{N}_0\\ (2k+1)\ge(2\xi+1)}} \frac{1}{(2k+1)^3} \left| \sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0\\ 2l+1\le 2k+1}} (2l+1)\,\mathrm{Tr}\,\widehat{f}(l) \right| \right)^p. \tag{4.7}$$

Thus, we have established inequality (3.19). This completes the proof. □

Lemma 4.1. *Suppose that Assumption 4.1 holds true. Then the averaging $\overline{F}(\lambda_\pi)$ of F is a monotone decreasing net.*

Proof. Let $\pi, \xi, \theta \in \Gamma$ and $\pi \succ \xi$. We will show that $\overline{F}(\lambda_\pi) \le \overline{F}(\lambda_\xi)$. Since $\lambda = \{\lambda_\pi\}_{\pi\in\Gamma}$ is a monotone increasing net, we have

$$\{Q \in \mathcal{M} \colon \nu_\Gamma(Q) \ge \lambda_\pi\} \subset \{Q \in \mathcal{M} \colon \nu_\Gamma(Q) \ge \lambda_\xi\}.$$

Therefore, we get

$$\overline{F}(\lambda_\pi) = \sup_{\substack{Q\in\mathcal{M}\\ \nu_\Gamma(Q)\ge\lambda_\pi}} \frac{1}{\nu_\Gamma(Q)} \left| \sum_{\theta\in Q} d_\theta \,\mathrm{Tr}\, F(\theta) \right| \le \sup_{\substack{Q\in\mathcal{M}\\ \nu_\Gamma(Q)\ge\lambda_\xi}} \frac{1}{\nu_\Gamma(Q)} \left| \sum_{\theta\in Q} d_\theta \,\mathrm{Tr}\, F(\theta) \right| = \overline{F}(\lambda_\xi).$$

This completes the proof. □

Theorem 4.1. *1. Let $1 \le p < \infty$, $1 < q \le \infty$ and let $\mathcal{M}_1 \subset \mathcal{M}_2$ be two arbitrary fixed collections of finite subsets of Γ, then we have*

$$N_{p,q}(\mathcal{M}_2) \hookrightarrow N_{p,q}(\mathcal{M}_1). \tag{4.8}$$

2. *Let $1 \leq p < \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$. Suppose that Assumptions 4.1 and 4.2 hold true and $\mathcal{M}$ is a fixed arbitrary collection of finite subsets of Γ. Then we have the following embedding*

$$N_{p,q_1}(\Gamma,\mathcal{M}) \hookrightarrow N_{p,q_2}(\Gamma,\mathcal{M}). \tag{4.9}$$

Proof of Theorem 4.1. First, we notice that (4.8) follows directly from the definition. Therefore, we concentrate on proving the second part of the Theorem.

Let first $1 \leq q_1 \leq q_2 < \infty$. By definition, we have

$$\|F\|_{N_{p,q_2}(\Gamma,\mathcal{M})} = \left(\sum_{\pi\in\Gamma}\left(\lambda_\pi^{\frac{1}{p}}\overline{F}[\lambda_\pi]\right)^{q_2}\frac{\delta_\pi\kappa_\pi}{\lambda_\pi}\right)^{\frac{1}{q_2}}. \tag{4.10}$$

Using formula (4.2) from Assumption 4.2 with $\beta = q_1(\frac{1}{p}+\varepsilon)-1$, we get, with a sufficiently large $\varepsilon > 0$,

$$\lambda_\pi^{\frac{q_2}{p}} = \lambda_\pi^{-\varepsilon q_2}\lambda_\pi^{q_2(\frac{1}{p}+\varepsilon)} = C_\beta^{\frac{q_2}{q_1}}\lambda_\pi^{-\varepsilon q_2}\left(\sum_{\substack{\theta\in\Gamma\\ \lambda_\theta\leq\lambda_\pi}}\lambda_\theta^{q_1(\frac{1}{p}+\varepsilon)-1}\delta_\theta\kappa_\theta\right)^{\frac{q_2}{q_1}}.$$

Thus, up to a constant, the expression in (4.10) equals to

$$\left(\sum_{\pi\in\Gamma}\lambda_\pi^{-\varepsilon q_2}\left(\overline{F}[\lambda_\pi]\right)^{q_2}\left(\sum_{\substack{\theta\in\Gamma\\ \lambda_\theta\leq\lambda_\pi}}\lambda_\theta^{q_1(\frac{1}{p}+\varepsilon)}\frac{\delta_\theta\kappa_\theta}{\lambda_\theta}\right)^{\frac{q_2}{q_1}}\frac{\delta_\pi\kappa_\pi}{\lambda_\pi}\right)^{\frac{1}{q_2}}. \tag{4.11}$$

In view of Lemma 4.1 the averaging function $\overline{F}(\lambda_\pi)$ is a monotone decreasing net. Therefore, (4.11) does not exceed

$$\left[\left(\sum_{\pi\in\Gamma}\lambda_\pi^{-\varepsilon q_2}\left(\sum_{\substack{\theta\in\Gamma\\ \lambda_\theta\leq\lambda_\pi}}\lambda_\theta^{q_1(\frac{1}{p}+\varepsilon)}\overline{F}[\lambda_\theta]^{q_1}\frac{\delta_\theta\kappa_\theta}{\lambda_\theta}\right)^{\alpha}\frac{\delta_\pi\kappa_\pi}{\lambda_\pi}\right)^{\frac{1}{\alpha}}\right]^{\frac{1}{q_1}},$$

where $\alpha = \frac{q_2}{q_1}$. We have thus proved that

$$\|F\|_{N_{p,q_2}(\Gamma,\mathcal{M})} \leq C_\beta^{\frac{1}{q_1}}\left[\left(\sum_{\pi\in\Gamma}\lambda_\pi^{-\varepsilon q_2}\left(\sum_{\substack{\theta\in\Gamma\\ \lambda_\theta\leq\lambda_\pi}}\lambda_\theta^{q_1(\frac{1}{p}+\varepsilon)}\overline{F}[\lambda_\theta]^{q_1}\frac{\delta_\theta\kappa_\theta}{\lambda_\theta}\right)^{\alpha}\frac{\delta_\pi\kappa_\pi}{\lambda_\pi}\right)^{\frac{1}{\alpha}}\right]^{\frac{1}{q_1}}. \tag{4.12}$$

Now, we consider an $\ell^\alpha(\Gamma,\omega)$ space with the measure ω defined as follows

$$\omega_\Gamma(\pi) = \lambda_\pi^{-\varepsilon q_2-1}\delta_\pi\kappa_\pi.$$

Denote by I the right-hand side in (4.12). Then using the $\ell^\alpha(\Gamma,\omega_\Gamma)$-space duality, we have

$$I^{q_1} = \sup_{\substack{b\in\ell^{\alpha'}(\Gamma,\omega_\Gamma)\\ \|b\|_{\ell^{\alpha'}}=1}}\left|\sum_{\pi\in\Gamma}\left(\sum_{\substack{\theta\in\Gamma\\ \lambda_\theta\leq\lambda_\theta}}\overline{F}[\lambda_\theta]^{q_1}\lambda_\theta^{q_1(\frac{1}{p}+\varepsilon)-1}\delta_\theta\kappa_\theta\right)\overline{b_\pi}\omega_\Gamma(\pi)\right|,$$

where α' denotes the exponent conjugate to α so that $\frac{1}{\alpha}+\frac{1}{\alpha'}=1$. Then using Fubini theorem and Hölder inequality, we have

$$\begin{aligned}
&\left|\sum_{\pi\in\Gamma}\left(\sum_{\substack{\theta\in\Gamma\\ \lambda_\theta\le\lambda_\pi}}\overline{F}[\lambda_\theta]^{q_1}\lambda_\theta^{q_1(\frac{1}{p}+\varepsilon)-1}\delta_\theta\kappa_\theta\right)\overline{b_\pi}\omega_\Gamma(\pi)\right| \\
&\qquad\le\sum_{\pi\in\Gamma}\left(\sum_{\substack{\theta\in\Gamma\\ \lambda_\theta\le\lambda_\pi}}\overline{F}[\lambda_\theta]^{q_1}\lambda_\theta^{q_1(\frac{1}{p}+\varepsilon)-1}\delta_\theta\kappa_\theta\right)|b_\pi|\,\omega_\Gamma(\pi) \\
&\qquad=\sum_{\theta\in\Gamma}\overline{F}[\lambda_\theta]^{q_1}\lambda_\theta^{q_1(\frac{1}{p}+\varepsilon)-1}\delta_\theta\kappa_\theta\sum_{\substack{\pi\in\Gamma\\ \lambda_\pi\ge\lambda_\theta}}|b_\pi|\,\omega_\Gamma(\pi) \\
&\le\sum_{\theta\in\Gamma}\overline{F}[\lambda_\theta]^{q_1}\lambda_\theta^{q_1(\frac{1}{p}+\varepsilon)-1}\delta_\theta\kappa_\theta\left(\sum_{\substack{\pi\in\Gamma\\ \lambda_\pi\ge\lambda_\theta}}\omega_\Gamma(\pi)\right)^{\frac{1}{\alpha}}\left(\sum_{\substack{\pi\in\Gamma\\ \lambda_\pi\ge\lambda_\theta}}|b_\pi|^{\alpha'}\,\omega_\Gamma(\pi)\right)^{\frac{1}{\alpha'}} \\
&\qquad\le\left[\sum_{\theta\in\Gamma}\overline{F}[\lambda_\theta]^{q_1}\left(\sum_{\substack{\pi\in\Gamma\\ \lambda_\pi\ge\lambda_\theta}}\omega_\Gamma(\pi)\right)^{\frac{1}{\alpha}}\lambda_\theta^{q_1(\frac{1}{p}+\varepsilon)-1}\delta_\theta\kappa_\theta\right]\cdot\|b\|_{\ell^{\alpha'}(\Gamma,\omega_\Gamma)}.
\end{aligned}$$

Thus, using this and taking supremum over all $b\in\ell^{\alpha'}(\Gamma,\omega_\Gamma)$, we get from (4.12)

$$\|F\|_{N_{p,q_2}(\Gamma,\mathcal{M})}\le C_\beta^{\frac{1}{q_1}}\,(I^{q_1})^{\frac{1}{q_1}}\le C_\beta^{\frac{1}{q_1}}\left[\sum_{\theta\in\Gamma}\overline{F}[\lambda_\theta]^{q_1}\left(\sum_{\substack{\pi\in\Gamma\\ \lambda_\pi\ge\lambda_\theta}}\omega_\Gamma(\pi)\right)^{\frac{1}{\alpha}}\lambda_\theta^{q_1(\frac{1}{p}+\varepsilon)-1}\delta_\theta\kappa_\theta\right]^{\frac{1}{q_1}}.$$

Again, using formula (4.5) with $\beta=-\varepsilon q_2-1$ from Assumption 4.2 and recalling that $\alpha=\frac{q_2}{q_1}$, we get

$$\left(\sum_{\substack{\pi\in\Gamma\\ \lambda_\pi\ge\lambda_\theta}}\omega_\Gamma(\pi)\right)^{\frac{1}{\alpha}}=\left(\sum_{\substack{\pi\in\Gamma\\ \lambda_\pi\ge\lambda_\theta}}\lambda_\pi^{-\varepsilon q_2-1}\delta_\pi\kappa_\pi\right)^{\frac{q_1}{q_2}}=C_\beta^{\frac{q_1}{q_2}}\lambda_\theta^{-\varepsilon q_1},$$

for sufficiently large ε. Finally, we have

$$\begin{aligned}
\|F\|_{N_{p,q_2}(\Gamma,\mathcal{M})}&\le C_\beta^{\frac{2}{q_1}}\left[\sum_{\theta\in\Gamma}\overline{F}[\lambda_\theta]^{q_1}\lambda_\theta^{-\varepsilon q_1}\lambda^{q_1(\frac{1}{p}+\varepsilon)-1}\delta_\theta\kappa_\theta\right]^{\frac{1}{q_1}} \\
&=C_\beta^{\frac{2}{q_1}}\left[\sum_{\theta\in\Gamma}\left(\lambda_\pi^{\frac{1}{p}}\overline{F}[\lambda_\theta]\right)^{q_1}\frac{\delta_\theta\kappa_\theta}{\lambda_\theta}\right]^{\frac{1}{q_1}}=C_\beta^{\frac{2}{q_1}}\|F\|_{N_{p,q_1}(\Gamma,\mathcal{M})}.
\end{aligned}$$

For $q_2=\infty$ and $q_1<q_2=\infty$, using Assumption 4.2, we have

$$\|F\|_{N_{p,\infty}(\Gamma,\mathcal{M})}=\sup_{\pi\in\Gamma}\lambda_\pi^{\frac{1}{p}}\overline{F}[\lambda_\pi]=C_{\frac{q_1}{p}-1}^{\frac{1}{q_1}}\sup_{\pi\in\Gamma}\left(\sum_{\substack{\theta\in\Gamma\\ \lambda_\theta\leq\lambda_\pi}}\lambda_\theta^{\frac{q_1}{p}}\overline{F}[\lambda_\pi]^{q_1}\frac{\kappa_\theta\delta_\theta}{\lambda_\theta}\right)^{\frac{1}{q_1}}$$

$$\lesssim\sup_{\pi\in\Gamma}\left(\sum_{\substack{\theta\in\Gamma\\ \lambda_\theta\leq\lambda_\pi}}\lambda_\theta^{\frac{q_1}{p}}\overline{F}[\lambda_\theta]^{q_1}\frac{\kappa_\theta\delta_\theta}{\lambda_\theta}\right)^{\frac{1}{q_1}}\leq\left(\sum_{\theta\in\Gamma}\lambda_\theta^{\frac{q_1}{p}}\overline{F}[\lambda_\theta]^{q_1}\frac{\kappa_\theta\delta_\theta}{\lambda_\theta}\right)^{\frac{1}{q_1}}=\|F\|_{N_{p,q_1}(\Gamma,\mathcal{M})},$$

where in the first inequality we used the monotonicity of the averaging function $\overline{F}$ established in Lemma 4.1. This completes the proof. □

We now establish interpolation properties of net spaces. Let (A_0,A_1) be a compatible pair of Banach spaces (cf. [4]) and let

$$K(t,F;A_0,A_1):=\inf_{F=F_0+F_1}\left(\|F_0\|_{A_0}+t\|F_1\|_{A_1}\right),\quad F\in A_0+A_1,$$

be the Peetre functional.

Theorem 4.2. *Let $1\leq p_1<p_2<\infty$, $1\leq q_1,q_2,q\leq\infty$, $0<\theta<1$. Suppose that Assumptions 4.1 and 4.2 hold true. Then we have*

$$(N_{p_1,q_1}(\Gamma,\mathcal{M}),N_{p_2,q_2}(\Gamma,\mathcal{M}))_{\theta,q}\hookrightarrow N_{p,q}(\Gamma,\mathcal{M}),\tag{4.13}$$

where $\frac{1}{p}=\frac{1-\theta}{p_1}+\frac{\theta}{p_2}$.

Proof of Theorem 4.2. Since by Theorem 4.1 we have $N_{p_i,q_i}\hookrightarrow N_{p_i,\infty}$, $i=1,2$, it is sufficient to prove

$$(N_{p_1,\infty},N_{p_2,\infty})_{\theta,q}\hookrightarrow N_{p,q}.$$

Assume first that $q<+\infty$. Let $\pi\in\Gamma, F=F_1+F_2, F_1\in N_{p_1,\infty}$ and $F_2\in N_{p_2,\infty}$. It is clear that

$$\overline{F}[\lambda_\pi]\leq\overline{F_1}[\lambda_\pi]+\overline{F_2}[\lambda_\pi].$$

Denote $v(t):=t^{\frac{1}{\frac{1}{p_1}-\frac{1}{p_2}}}$, $t>0$. It is obvious that

$$\sup_{\substack{\xi\in\Gamma\\ \lambda_\xi\leq v(t)}}\lambda_\xi^{\frac{1}{p_1}}\overline{F}[\lambda_\xi]\leq\sup_{\xi\in\Gamma}\lambda_\xi^{\frac{1}{p_1}}\overline{F_1}[\lambda_\xi]+\sup_{\substack{\xi\in\Gamma\\ \lambda_\xi\leq v(t)}}\lambda_\xi^{\frac{1}{p_1}-\frac{1}{p_2}+\frac{1}{p_2}}\overline{F_2}[\lambda_\xi]$$

$$\leq\sup_{\xi\in\Gamma}\lambda_\xi^{\frac{1}{p_1}}\overline{F_1}[\lambda_\xi]+t\sup_{\xi\in\Gamma}\lambda_\xi^{\frac{1}{p_2}}\overline{F_2}[\lambda_\xi].$$

Taking the infimum over all possible reprsentations $F=F_1+F_2$, we obtain

$$\sup_{\substack{\xi\in\Gamma\\ \lambda_\xi\leq v(t)}}\lambda_\xi^{\frac{1}{p_1}}\overline{F}[\lambda_\xi]\lesssim K(t,F;N_{p_1,\infty},N_{p_2,\infty}).\tag{4.14}$$

Thus, making a substitution $t\to t^{\frac{1}{p_1}-\frac{1}{p_2}}$, we have

$$\int_0^{+\infty}\left(t^{-\theta}K(t,F)\right)^q\frac{dt}{t}\geq\int_0^{+\infty}\left(t^{-\theta}\sup_{\substack{\pi\in\Gamma\\ \lambda_\pi\leq v(t)}}\lambda_\pi^{\frac{1}{p_1}}\overline{F}[\lambda_\pi]\right)^q\frac{dt}{t}$$

$$\cong\int_0^{+\infty}\left(t^{-\theta(\frac{1}{p_1}-\frac{1}{p_2})}\sup_{\substack{\pi\in\Gamma\\ \lambda_\pi\leq t}}\lambda_\pi^{\frac{1}{p_1}}\overline{F}[\lambda_\pi]\right)^q\frac{dt}{t}.\tag{4.15}$$

Decomposing

$$(0,+\infty)=\bigsqcup_{s\in\mathbb{Z}}[2^s,2^{s+1}),$$

we have

$$\begin{aligned}\int_0^{+\infty}\left(t^{-\theta(\frac{1}{p_1}-\frac{1}{p_2})}\sup_{\substack{\pi\in\Gamma\\ \lambda_\pi\le t}}\lambda_\pi^{\frac{1}{p_1}}\overline{F}[\lambda_\pi]\right)^q\frac{dt}{t}&=\sum_{s\in\mathbb{Z}}\int_{2^s}^{2^{s+1}}\left(t^{-\theta(\frac{1}{p_1}-\frac{1}{p_2})}\sup_{\substack{\pi\in\Gamma\\ \lambda_\pi\le t}}\lambda_\pi^{\frac{1}{p_1}}\overline{F}[\lambda_\pi]\right)^q\frac{dt}{t}\\ &\cong\sum_{s\in\mathbb{Z}}\left(2^{-s\theta(\frac{1}{p_1}-\frac{1}{p_2})}\sup_{\substack{\pi\in\Gamma\\ \lambda_\pi\le 2^s}}\lambda_\pi^{\frac{1}{p_1}}\overline{F}[\lambda_\pi]\right)^q\int_{2^s}^{2^{s+1}}\frac{dt}{t}\cong\sum_{s\in\mathbb{Z}}\left(2^{-s\theta(\frac{1}{p_1}-\frac{1}{p_2})}\sup_{\substack{\pi\in\Gamma\\ \lambda_\pi\le 2^s}}\lambda_\pi^{\frac{1}{p_1}}\overline{F}[\lambda_\pi]\right)^q\\ &\ge\sum_{s\in\mathbb{Z}}\left(2^{-s\theta(\frac{1}{p_1}-\frac{1}{p_2})}\sup_{\substack{\pi\in\Gamma\\ 2^{s-1}\le\lambda_\pi\le 2^s}}\lambda_\pi^{\frac{1}{p_1}}\overline{F}[\lambda_\pi]\right)^q\ge\sum_{s\in\mathbb{Z}}\left(2^{-s\theta(\frac{1}{p_1}-\frac{1}{p_2})}2^{(s-1)\frac{1}{p_1}}\overline{F}[2^{s-1}]\right)^q.\end{aligned}\tag{4.16}$$

Using formulae (4.2) with $\beta=0$ from Assumption 4.2, we get

$$\sum_{\substack{\theta\in\Gamma\\ 2^s\le\lambda_\theta\le 2^{s+1}}}\frac{\delta_\theta\kappa_\theta}{\lambda_\theta}\cong\frac{1}{2^s}\left(\sum_{\substack{\theta\in\Gamma\\ \lambda_\theta\le 2^{s+1}}}\delta_\theta\kappa_\theta-\sum_{\substack{\theta\in\Gamma\\ \lambda_\theta\le 2^s}}\delta_\theta\kappa_\theta\right)\cong\frac{2^{s+1}-2^s}{2^s}=1.\tag{4.17}$$

Therefore recalling that $\frac{1}{p}=\frac{1-\theta}{p_1}+\frac{\theta}{p_2}$ and combining formulae (4.15), (4.16), (4.17), we have

$$\begin{aligned}\int_0^{+\infty}\left(t^{-\theta}K(t,F)\right)^q\frac{dt}{t}&\ge\sum_{s\in\mathbb{Z}}\left(2^{\frac{s}{p}}\overline{F}[2^s]\right)^q\sum_{\substack{\theta\in\Gamma\\ 2^s\le\lambda_\theta\le 2^{s+1}}}\frac{\delta_\theta\kappa_\theta}{\lambda_\theta}\\ &\cong\sum_{s\in\mathbb{Z}}\left(2^{\frac{s+1}{p}}\overline{F}[2^s]\right)^q\sum_{\substack{\theta\in\Gamma\\ 2^s\le\lambda_\theta\le 2^{s+1}}}\frac{\delta_\theta\kappa_\theta}{\lambda_\theta}\ge\sum_{s\in\mathbb{Z}}\sum_{\substack{\theta\in\Gamma\\ 2^s\le\lambda_\theta\le 2^{s+1}}}\left(\lambda_\theta^{\frac{1}{p}}\overline{F}[\lambda_\theta]\right)^q\frac{\delta_\theta\kappa_\theta}{\lambda_\theta}\\ &=\sum_{\theta\in\Gamma}\left(\lambda_\theta^{\frac{1}{p}}\overline{F}[\lambda_\theta]\right)^q\frac{\delta_\theta\kappa_\theta}{\lambda_\theta}=\|F\|_{N_{p,q}(\Gamma,\mathcal{M})}.\end{aligned}\tag{4.18}$$

where in the last inequality we used that $\overline{F}(t)$ is a decreasing function of t. This proves (4.13) for $q<+\infty$. For $q=\infty$, making again substitution $t\mapsto t^{\frac{1}{p_1}-\frac{1}{p_2}}$, we have

$$\sup_{t>0}t^{-\theta}K(t,F)=\sup_{t>0}t^{-\theta(\frac{1}{p_1}-\frac{1}{p_2})}K(t^{\frac{1}{p_1}-\frac{1}{p_2}},F).\tag{4.19}$$

Then using formula (4.14), we continue inequality (4.19) as

$$\begin{aligned}\ge\sup_{t>0}t^{-\theta(\frac{1}{p_1}-\frac{1}{p_2})}\sup_{\substack{\pi\in\Gamma\\ \lambda_\pi\le t}}\lambda_\pi^{\frac{1}{p_1}}\overline{F}[\lambda_\pi]\gtrsim\lambda_\xi^{-\theta(\frac{1}{p_1}-\frac{1}{p_2})}\sup_{\substack{\pi\in\Gamma\\ \lambda_\pi\le\lambda_\xi}}\lambda_\pi^{\frac{1}{p_1}}\overline{F}[\lambda_\pi]\\ \ge\lambda_\xi^{-\theta(\frac{1}{p_1}-\frac{1}{p_2})}\lambda_\xi^{\frac{1}{p_1}}\overline{F}[\lambda_\xi]=\lambda_\xi^{\frac{1}{p}}\overline{F}[\lambda_\xi],\end{aligned}\tag{4.20}$$

here λ_ξ is an arbitrary element of the net $\lambda=\{\lambda_\xi\}_{\xi\in\Gamma}$. Taking thus supremum over all $\xi\in\Gamma$, we finally obtain

$$\sup_{t>0}t^{-\theta}K(t,F)\gtrsim\sup_{\xi\in\Gamma}\lambda_\xi^{\frac{1}{p}}\overline{F}[\lambda_\xi]=\|F\|_{N_{p,\infty}(\Gamma,\mathcal{M})}.\tag{4.21}$$

We have established embedding (4.13) also for $q=\infty$. This completes the proof. □

5 Proof of Remark 5

Here we prove Remark 5.

We denote by s_k and m_k the enumerated eigenvalues of $(1-\Delta_{G/K})^{\frac{1}{2}}$ and $(1-\Delta_{G/K})^{\frac{n}{2}}$ respectively. We assume that they are ordered, i.e.

$$\begin{aligned} s_1 \leq s_2 \leq \ldots \leq s_k \leq s_{k+1} \leq \ldots, \\ m_1 \leq m_2 \leq \ldots \leq m_k \leq m_{k+1} \leq \ldots, \end{aligned} \tag{5.1}$$

multiplicities taken into account. Let us also denote by $N_k(L)$ the eigenvalue counting function for the eigenvalues $\lambda^k = \{\lambda_i^k\}_{i\in\mathbb{N}}$ of the k-th order operator $(I-\Delta_{G/K})^{\frac{k}{2}}$, i.e.

$$N_k(L) = \sum_{\substack{i\in\mathbb{N} \\ \lambda_i^k \leq L}} 1.$$

First, we show that

$$m_k \cong k. \tag{5.2}$$

We use the Weyl counting function asymptotics for the first order elliptic pseudo-differential operator $(1-\Delta_{G/K})^{\frac{1}{2}}$ on the compact homogeneous manifold G/K, to get that the eigenvalue counting function $N_1(L)$ of eigenvalues of $(1-\Delta_{G/K})^{\frac{1}{2}}$ counted with multiplicities (see e.g. [17]), as

$$N_1(L) = \sum_{i:\ s_i \leq L} 1 \cong L^n.$$

Then we get the following asymptotic for the n-th order elliptic pseudo-differential operator $A = (I-\Delta_{G/K})^{\frac{n}{2}}$

$$N_n(L) = \sum_{i:\ m_i \leq L} 1 \cong L. \tag{5.3}$$

Now, we fix an arbitrary eigenvalue m_k and set $L = m_k$ in (5.3) to get

$$N_n(m_k) \cong m_k. \tag{5.4}$$

Since (5.1), we have

$$N_n(m_k) = k. \tag{5.5}$$

Combining (5.4) and (5.5), we get (5.2). Further, we fix an arbitrary $\pi \in \widehat{G}_0$ coresponding to a μ_{k_0} such that $\langle\pi\rangle^n = \mu_{k_0}$. Then using (5.2), we get

$$\sum_{\substack{\xi\in\widehat{G}_0 \\ \langle\xi\rangle\leq\langle\pi\rangle}} \langle\xi\rangle^{n\beta} = \sum_{k:\ \mu_k\leq\mu_{k_0}} \mu_k^\beta \cong \sum_{k=1}^{k_0} k^\beta \cong k_0^{\beta+1} \cong \mu_{k_0}^{\beta+1} = \langle\pi\rangle^{n(\beta+1)}.$$

This proves (4.2) with $\Gamma = \widehat{G}_0, \lambda_\theta = \langle\theta\rangle^n, \delta_\theta = d_\theta, \kappa_\theta = k_\theta$, $\theta \in \widehat{G}_0$.

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