ISSN 2077–9879

Eurasian Mathematical Journal

2017, Volume 8, Number 2

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia
the University of Padua

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by
the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

EURASIAN MATHEMATICAL JOURNAL

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Building no. 3
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13 Kazhymukan St
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Kazakhstan

This issue contains the first part of the collection of papers sent to the Eurasian Mathematical Journal dedicated to the 70th birthday of Professor R. Oinarov.

The first part of the collection was published in Volume 8, Number 1.

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 8, Number 2 (2017), 105 – 110

THE HARDY INEQUALITY WITH BOUNDARY OR INTERMEDIATE CONDITIONS

A. Kufner

Communicated by L.-E. Persson

Dedicated to the 70th birthday of Professor Ryskul Oinarov

Key words: Hardy's inequality, boundary conditions.

AMS Mathematics Subject Classification: 26A99, 26D10, 26D15.

Abstract. In the paper a survey is given of the necessary and sufficient conditions which guarantee the validity of Hardy inequality (1.1) for various classes of functions.

1 Introduction

We will deal with the Hardy inequality

$$\left(\int_a^b |f(x)|^q u(x)dx\right)^{1/q} \leq C\left(\int_a^b |f'(x)|^p v(x)dx\right)^{1/p}. \tag{1.1}$$

Here, $-\infty \leq a < b \leq \infty$, $1 < p \leq q < \infty$, u and v are weight functions, i.e. functions measurable and positive almost everywhere in (a, b).

We denote by

$$V^{1,p} = V^{1,p}(a, b)$$

the set of all functions f on (a, b) for which the right-hand side of (1.1) is finite.

We are interested in conditions on the weights u and v under which (1.1) holds for all $f \in V^{1,p}$. Obviously, for a constant non-zero function, which belongs to $V^{1,p}$, inequality (1.1) becomes meaningless. Therefore, we have to put on functions $f \in V^{1,p}$ some additional restrictive conditions, to exclude functions $f(x) \equiv \text{const.} \neq 0$. Hence, let us introduce the following subsets of $V^{1,p}$:

$$\begin{aligned}
V_A^{1,p} &= \{f \in V^{1,p};\ f(a) := \lim_{x\to a+} f(x) = 0\},\\
V_B^{1,p} &= \{f \in V^{1,p};\ f(b) := \lim_{x\to b-} f(x) = 0\},\\
V_C^{1,p} &= \{f \in V^{1,p};\ f(c) = 0 \text{ where } c \text{ is a fixed point of } (a, b)\},\\
V_D^{1,p} &= \{f \in V^{1,p};\ f(a) = f(b) = 0\}.
\end{aligned}$$

Furthermore, let us denote for $a \leq \alpha < \beta \leq b$

$$U(\alpha, \beta) = \int_\alpha^\beta u(t)dt;\ V(\alpha, \beta) = \int_\alpha^\beta v^{1-p'}(t)dt,\ p' = \frac{p}{p-1}. \tag{1.2}$$

We assume that the numbers $U(\alpha,\beta)$ and $V(\alpha,\beta)$ which appear in the sequel for appropriate choices of α,β are finite.

2 The case with conditions

Now, we have the following results:

(i) Hardy inequality (1.1) holds for all $f \in V_A^{1,p}$ if and only if

$$C_A := \sup_{x\in(a,b)} U^{1/q}(x,b)V^{1/p'}(a,x) < \infty. \tag{2.1}$$

(ii) Hardy inequality (1.1) holds for all $f \in V_B^{1,p}$ if and only if

$$C_B := \sup_{x\in(a,b)} U^{1/q}(a,x)V^{1/p'}(x,b) < \infty. \tag{2.2}$$

(iii) Hardy inequality (1.1) holds for all $f \in V_C^{1,p}$ if and only if

$$\begin{aligned} C_c := \max\Big[&\sup_{x\in(a,c)} U^{1/q}(a,x)V^{1/p'}(x,c);\ \sup_{x\in(c,b)} U^{1/q}(x,b)V^{1/p'}(c,x)\Big] \\ &< \infty \end{aligned} \tag{2.3}$$

(iv) Hardy inequality (1.1) holds for all $f \in V_D^{1,p}$ if and only if

$$C_D := \sup_{\substack{c,d\\ a<c<d<b}} [U^{1/q}(c,d)\cdot\min(V^{1/p'}(a,c),V^{1/p'}(d,b))] < \infty. \tag{2.4}$$

Remark 1 All foregoing results can be found in the book [4]. The constants C_A and C_B are the well-known Muckenhoupt constants. The constants $C_A,\ldots,C_D$ provide an estimate for the best possible constant C in (1.1) for functions from the corresponding subspace of $V^{1,p}$, i.e. for f from $V_A^{1,p},\ldots,V_D^{1,p}$, respectively.

Remark 2 In the paper [3], the discrete analogue of (1.1) is investigated, namely the inequality

$$\left(\sum_{n=1}^{\infty}|f_n|^q u_n\right)^{1/q} \le C\left(\sum_{n=1}^{\infty}|f_{n+1}-f_n|^p v_n\right)^{1/p}. \tag{2.5}$$

For the case which corresponds to the case (iv) above, the author gives a necessary and sufficient condition, whose "continuous" analogue reads:

$$C_{D,1} := \sup_{\substack{c,d\\ a<c<d<b}} U^{1/q}(c,d)[V^{1-p}(a,c)+V^{1-p}(d,b)]^{-1/p} < \infty. \tag{2.6}$$

It is easy to show that the expressions for C_D and for $C_{D,1}$ are equivalent.

Another equivalent condition reads

$$C_{D,2} := \sup_{\substack{c,d\\ a<c<d<b}} U^{1/q}(c,d)[V^{-q/p'}(a,c)+V^{-q/p'}(d,b)]^{-1/q}. \tag{2.7}$$

3 The case without conditions

Now, we assume that the weight function u belongs to $L^1(a,b)$, i.e., that

$$U(a,b) = \int_a^b u(t)dt < \infty \tag{3.1}$$

If we wish to consider a variant of inequality (1.1) which holds for all $f \in V^{1,p}$ (without any additional condition), then we can – instead of (1.1) – investigate on the space $V^{1,p}$ the inequality

$$\inf_{c\in\mathbb{R}} \left(\int_a^b |f(x)-c|^q u(x)dx \right)^{1/q} \le C \left(\int_a^b |f'(x)|^p v(x)dx \right)^{1/p}. \tag{3.2}$$

If we denote

$$M_f := \frac{\int_a^b f(t)u(t)dt}{\int_a^b u(t)dt} \tag{3.3}$$

then we have that

$$\begin{aligned} \frac{1}{2}\int_a^b |f(t)-M_f|^q u(t)dt &\le \inf_{c\in\mathbb{R}} \int_a^b |f(t)-c|^q u(t)dt \\ &\le \int_a^b |f(t)-M_f|^q u(t)dt. \end{aligned} \tag{3.4}$$

Indeed, the first inequality follows by the Hölder inequality, using definition (3.3) of M_f, the second one follows by the definition of infimum.

Hence, instead of Hardy inequality (1.1) without any additional condition on f, but with condition (3.1) for u, we can investigate on $V^{1,p}$ the inequality

$$\left(\int_a^b |f(x)-M_f|^q u(x)dx \right)^{1/q} \le C \left(\int_a^b |f'(x)|^p v(x)dx \right)^{1/p}. \tag{3.5}$$

The discrete analogue of (3.5) is again investigated in [3]. Two equivalent estimates for the best constant C in (2.5) are derived, whose "continuous" analogues for (3.5) read:

$$C_{E,1} := \sup_{\substack{c,d \\ a<c<d<b}} V^{1/p'}(c,d)[U^{1-q'}(a,c)+U^{1-q'}(d,b)]^{-1/q'} < \infty \tag{3.6}$$

and

$$C_{E,2} := \sup_{\substack{c,d \\ a<c<d<b}} V^{1/p'}(c,d)[U^{-p'/q}(a,c)+U^{-p'/1}(d,b)]^{-1/p'} < \infty. \tag{3.7}$$

Consequently, we have the following result.

(v) Inequality (3.5) [and consequently, Hardy inequality (3.2)] holds for all $f \in V^{1,p}$ if and only if

$$C_E := \sup_{\substack{c,d \\ a<c,d<b}} [V^{1/p'}(c,d)\cdot \min(U^{1/q}(a,c), U^{1/q}(d,b))] < \infty. \tag{3.8}$$

The equivalence relations $C_E \approx C_{E,1} \approx C_{E,2}$ hold similarly to the relations $C_D \approx C_{D,1} \approx C_{D,2}$ in Remark 2.

Remark 3 If we consider inequality (1.1) as a one-dimensional "weighted" analogue of the *Friedrichs inequality* (for $p = q = 2$ and $u(x) = v(x) \equiv 1$) or of the *Sobolev inequality* (for $1 < p < q < \infty$ and $u(x) = v(x) \equiv 1$), then inequality (3.5) is the corresponding analogue of the *Poincaré inequality.*

4 Concluding remarks

Remark 4 Some complementary information can be found in the book [1] and in its new edition which will appear, with a new co-author N. Samko, in 2017. For example, in Chapter 4 of this book higher order Hardy inequalities, i.e. inequalities of type (1.1) where f' is replaced by $f^{(n)}$, $n = 2, 3, 4$ etc. with relevant boundary conditions, are investigated. Many interesting problems remain to be solved including the case with intermediate boundary conditions as described here for $n = 1$.

Remark 5 It is well known that the Muckenhoupt conditions $C_A < \infty$ and $C_B < \infty$ in Remark 1 can be replaced by other (equivalent) conditions, even by scales of conditions (see, e.g., the review paper [2] and the references therein, both for historical development and the most recent results). In particular, this means that the characterizations given in this paper can be formulated in many alternative but equivalent ways. This can be an advantage for applications because in a concrete situation, one condition may be easier to verify then another one.

Remark 6 By using [1] and the new 2017 edition, the results of this paper can be used for further developents of this type of inequalities (of Hardy, Friedrichs, Sobolev, Poincaré type), interesting for applications e.g. in the theory of partial differential equations.

References

[1] A. Kufner, L.-E. Persson, *Weighted inequalities of Hardy type*, World Scientific, New Jersey-London-Singapore-Hong Kong, 2003.

[2] A. Kufner, L.-E. Persson, N. Samko, *Some non scales of weighted Hardy type inequalities*, Oper. Theory Adv. Appl. 228, Birkhäuser, Basel AG, 2013, 261–274.

[3] Z.W. Liao, *Discrete weighted Hardy inequalities with different kinds of boundary conditions*, arXiv:1508.04601v1 [math.FA] 19 Aug 2015, 24 pages.

[4] B. Opic, A. Kufner, *Hardy-type inequalities*, Longman, Harlow, 1990.

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Received: 26.10.2016