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ALTERNATIVE BOUNDEDNESS CHARACTERISTICS FOR THE HARDY–STEKLOV OPERATOR

E.P. Ushakova

Communicated by R. Oinarov

Dedicated to the 70^{th} *birthday of Professor Ryskul Oinarov*

Key words: Hardy–Steklov operator, weighted Lebesgue space, boundedness.

AMS Mathematics Subject Classification: 47G10, 45P05.

Abstract. Using the notions of fairway functions we give the Tomaselli and Persson–Stepanov type forms of boundedness characterizations for the Hardy–Steklov operators in Lebesgue spaces. The results are alternatives to the Muckenhoupt and Mazya–Rosin type boundedness criteria.

1 Introduction

For $s \in (0,\infty)$ let $L^s := L^s(0,\infty)$ denote the usual Lebesgue space with the (quasi–)norm $\|f\|_s := \left(\int_0^\infty |f(x)|^s\,\mathrm{d}x\right)^{1/s}$. Let v and w be non–negative weight functions (weights) on $(0,\infty)$. For some fixed real parameters $p > 1$ and $q > 0$ we consider the Hardy–Steklov operator

$$\mathcal{H}f(x) := w(x)\int_{a(x)}^{b(x)} f(y)v(y)\mathrm{d}y, \quad x \in (0,\infty) \tag{1.1}$$

with boundary functions a and b satisfying the following conditions:

$$\begin{array}{ll} \text{(i)} & a,\, b \text{ are differentiable and strictly increasing on } (0,\infty); \\ \text{(ii)} & a(0)=b(0)=0,\ a(x)<b(x) \text{ for } 0<x<\infty,\ a(\infty)=b(\infty)=\infty. \end{array} \tag{1.2}$$

The family $\mathcal{H}$ of integral transformations with both variable boundaries is applicable to many areas (e.g. differential equations, embeddings of function spaces [2, 10, 11]). The two limiting cases of $\mathcal{H}$ ($a(x) \equiv 0$ and $b(x) \equiv \infty$) are rather well–studied. In particular, the weighted Hardy integral operator

$$Hf(x) := w(x)\int_0^x f(y)v(y)\,\mathrm{d}y \tag{1.3}$$

has collected a number of results related to its boundedness properties from L^p to L^q (see e.g. [8, 7]). Systematization most of them (see e.g. [3] and [13]) led to forming two basic types of boundedness characteristics for $H : L^p \to L^q$. These are functionals (we say constants also), which depend on fixed parameters only (e.g. weights, boundaries, summation parameters p and

q, etc.) and are equivalent to $\|H\|_{L^p\to L^q}$. The Muckenhoupt A_M and the Mazya–Rozin B_{MR} functionals

$$A_M = \sup_{t>0}\left(\int_t^\infty w^q\right)^{1/q}\left(\int_0^t v^{p'}\right)^{1/p'} \qquad (1<p\le q<\infty),$$

$$B_{MR} = \left(\int_0^\infty \left[\int_t^\infty w^q\right]^{r/p}\left[\int_0^t v^{p'}\right]^{r/p'} w^q(t)\,\mathrm{d}t\right)^{1/r} \qquad (0<q<p<\infty,\ p>1),$$

where $p'=p/(p-1)$ and $1/r=1/q-1/p$, and also their duals $(A_M)^*=A_M$ and

$$(B_{MR})^* = \left(\int_0^\infty \left[\int_t^\infty w^q\right]^{r/q}\left[\int_0^t v^{p'}\right]^{r/q'} v^{p'}(t)\,\mathrm{d}t\right)^{1/r},$$

constitute the bases of the first type boundedness characteristics for $H: L^p\to L^q$. The second type is formed by alternative to A_M and B_{MR} the Tomaselli A_T and the Persson–Stepanov B_{PS} functionals

$$A_T = \sup_{t>0}\left(\int_0^t \left[\int_0^x v^{p'}\right]^q w^q(x)\,\mathrm{d}x\right)^{1/q}\left(\int_0^t v^{p'}\right)^{-1/p} \qquad (1<p\le q<\infty),$$

$$B_{PS} = \left(\int_0^\infty \left[\int_0^t \left\{\int_0^x v^{p'}\right\}^q w^q(x)\mathrm{d}x\right]^{r/p}\left[\int_0^t v^{p'}\right]^{q-r/p} w^q(t)\mathrm{d}t\right)^{1/r} \qquad (0<q<p<\infty,\ p>1),$$

and also by their duals (see [13]). The functionals A_M and B_{MR} are classical boundedness characteristics for H from L^p to L^q. They are typically used to further investigations and applications of H. The alternative to them boundedness constants A_T and B_{PS} appeared to be useful in the study of the non–linear geometric mean operator $Gf(x)=\exp\left(\frac{1}{x}\int_0^x \ln f(y)\,\mathrm{d}y\right)$ [12] and in some other problems too.

Scale of the boundedness characteristics for the Hardy–Steklov operator, analogous to that for H, started to be formed in the articles [1, 4, 6]. The most productive development, related to the characterization of (1.1) with a and b satisfying (1.2), was undertaken in [19] basing on the conception of fairway.

Definition 1 ([17, 19])**.** Given boundary functions a and b, satisfying the conditions (1.2), a number $p\in(1,\infty)$ and a weight function v such that $0<v(y)<\infty$ for almost all $y\in(0,\infty)$ and $v^{p'}$ is locally integrable on $(0,\infty)$, we define the *fairway–function* σ such that $a(x)<\sigma(x)<b(x)$ and

$$\int_{a(x)}^{\sigma(x)} v^{p'}(y)\,\mathrm{d}y = \int_{\sigma(x)}^{b(x)} v^{p'}(y)\,\mathrm{d}y \qquad \text{for all } x>0. \tag{1.4}$$

The dual fairway-function ρ appeared in [20] for deriving new forms of L^p-L^q–boundedness criteria for $\mathcal{H}$.

Definition 2 ([20, 21, 9])**.** Given boundary functions a and b on $(0,\infty)$, satisfying (1.2), a parameter $q>0$ and a weight w such that $0<w(x)<\infty$ for almost all $x\in(0,\infty)$ and $w^q(x)$ is locally integrable on $(0,\infty)$, the *dual fairway–function* ρ is defined so that $b^{-1}(y)<\rho(y)<a^{-1}(y)$ on $(0,\infty)$ and

$$\int_{b^{-1}(y)}^{\rho(y)} w^q(x)\,\mathrm{d}x = \int_{\rho(y)}^{a^{-1}(y)} w^q(x)\,\mathrm{d}x \qquad \text{for all } y>0. \tag{1.5}$$

Here a^{-1} and b^{-1} are the inverses to the boundary functions a and b, respectively.

The fairways σ and ρ are strictly increasing and differentiable functions on $(0,\infty)$ (see [16, § 2.2.1]).

The Muckenhoupt $\mathcal{A}_M$ and Mazya–Rozin $\mathcal{B}_{MR}$ type boundedness characteristics for $\mathcal{H} : L^p \to L^q$ for all $1 < p < \infty$ and $0 < q < \infty$ in terms of the fairway–function σ were first obtained in [17]:

$$\mathcal{A}_M =: \mathcal{A}_\sigma = \sup_{t>0} \left(\int_{b^{-1}(\sigma(t))}^{a^{-1}(\sigma(t))} w^q \right)^{1/q} \left(\int_{a(t)}^{b(t)} v^{p'} \right)^{1/p'} \qquad (1 < p \le q < \infty),$$

$$\mathcal{B}_{MR} =: \mathcal{B}_\sigma = \left(\int_0^\infty \left[\int_{b^{-1}(\sigma(t))}^{a^{-1}(\sigma(t))} w^q \right]^{r/p} \left[\int_{a(t)}^{b(t)} v^{p'} \right]^{r/p'} w^q(t)\mathrm{d}t \right)^{1/r} \quad (0 < q < p < \infty,\ p > 1). \quad (1.6)$$

In [20, 19], the dual to $\mathcal{A}_M$ and $\mathcal{B}_{MR}$ constants $(\mathcal{A}_\rho)^*$ and $(\mathcal{B}_\rho)^*$ were found for the parameters $p > 1$ and $q > 1$ only. Their expressions involve the notion of the dual fairway–function ρ:

$$(\mathcal{A}_\rho)^* = \sup_{t>0} \left(\int_{b^{-1}(t)}^{a^{-1}(t)} w^q \right)^{1/q} \left(\int_{a(\rho(t))}^{b(\rho(t))} v^{p'} \right)^{1/p'} \qquad (1 < p \le q < \infty),$$

$$(\mathcal{B}_\rho)^* = \left(\int_0^\infty \left[\int_{b^{-1}(t)}^{a^{-1}(t)} w^q \right]^{r/q} \left[\int_{a(\rho(t))}^{b(\rho(t))} v^{p'} \right]^{r/q'} v^{p'}(t)\,\mathrm{d}t \right)^{1/r} \qquad (1 < q < p < \infty).$$

The whole scale of Muckenhoupt and Mazya–Rosin type boundedness characteristics for $\mathcal{H} : L^p \to L^q$ was established in [9] (see also [21]) for $p > 1$ and $q > 0$, both in terms of the fairway σ and its dual function ρ. Namely, the work [9] extends $(\mathcal{B}_\rho)^*$ to all $0 < q < p < \infty$, $p > 1$ and introduces two additional couples of functionals $(\mathcal{A}_\sigma)^*$, $(\mathcal{B}_\sigma)^*$ and $\mathcal{A}_\rho$, $\mathcal{B}_\rho$, which are, similarly to $\mathcal{A}_\sigma$, $\mathcal{B}_\sigma$ and $(\mathcal{A}_\rho)^*$ and $(\mathcal{B}_\rho)^*$, dual to each other if $p, q > 1$ (see [9] or [21] for detail). This, finally, formed the basic scale of the first type boundedness constants for the Hardy–Steklov operators (1.1) from L^p to L^q. The result was applied in [9] to characterization of embeddings of a class of AC–functions in the fractional Sobolev space, and in [14, 15, 2] to descriptions of function spaces associated with a weighted Sobolev space.

First alternative characteristics of the type A_T and B_{PS} for the Hardy–Steklov operators $\mathcal{H}$ were established in [18]. Namely, it was proven that the norm $\|\mathcal{H}\|_{L^p \to L^q}$ is equivalent to the functional

$$\mathcal{A}_T =: \mathbb{A}_\sigma = \sup_{t>0} \left(\int_{\sigma^{-1}(a(t))}^{\sigma^{-1}(b(t))} \left[\int_{a(x)}^{b(x)} v^{p'} \right]^q w^q(x)\,\mathrm{d}x \right)^{1/q} \left(\int_{a(t)}^{b(t)} v^{p'} \right)^{-1/p} \qquad (1.7)$$

in the case $1 < p \le q < \infty$, and if $0 < q < p < \infty$, $p > 1$ then $\|\mathcal{H}\|_{L^p \to L^q} \approx \mathcal{B}_{PS} =: \mathbb{B}_\sigma$ with

$$\mathcal{B}_{PS} =: \mathbb{B}_\sigma = \left(\int_0^\infty \left[\int_{\sigma^{-1}(a(t))}^{\sigma^{-1}(b(t))} \left\{ \int_{a(x)}^{b(x)} v^{p'} \right\}^q w^q(x)\,\mathrm{d}x \right]^{r/p} \left[\int_{a(t)}^{b(t)} v^{p'} \right]^{q-r/p} w^q(t)\,\mathrm{d}t \right)^{1/r} \qquad (1.8)$$

(see also [19, Theorem 4.2]). $\mathbb{A}_\sigma$ and $\mathbb{B}_\sigma$ were successively applied to characterization of the geometric Steklov operator $\mathcal{G}f(x) = \exp\left(\frac{1}{b(x)-a(x)} \int_{a(x)}^{b(x)} \ln f \right)$ in [18]. By duality, the two following boundedness constants for $\mathcal{H} : L^p \to L^q$ yield the equivalences $\|\mathcal{H}\|_{L^p \to L^q} \approx \mathbb{A}_\sigma$ and

$\|\mathcal{H}\|_{L^p\to L^q} \approx \mathbb{B}_\sigma$ in the case $p, q > 1$:

$$(\mathbb{A}_\rho)^* = \sup_{t>0} \left(\int_{\rho^{-1}(b^{-1}(t))}^{\rho^{-1}(a^{-1}(t))} \left[\int_{b^{-1}(y)}^{a^{-1}(y)} w^q \right]^{p'} v^{p'}(y)\, \mathrm{d}y \right)^{1/p'} \left(\int_{b^{-1}(t)}^{a^{-1}(t)} w^q \right)^{-1/q'}, \tag{1.9}$$

$$(\mathcal{B}_{PS})^* = \left(\int_0^\infty \left[\int_{\rho^{-1}(b^{-1}(t))}^{\rho^{-1}(a^{-1}(t))} \left\{ \int_{b^{-1}(y)}^{a^{-1}(y)} w^q \right\}^{p'} v^{p'}(y)\, \mathrm{d}y \right]^{r/q'} \left[\int_{b^{-1}(t)}^{a^{-1}(t)} w^q \right]^{p'-r/q'} v^{p'}(t)\, \mathrm{d}t \right)^{1/r}. \tag{1.10}$$

Together with $\mathbb{A}_\sigma$ and $\mathbb{B}_\sigma$, the $(\mathbb{A}_\rho)^*$ and $(\mathcal{B}_{PS})^*$ partially form the related scale of alternative boundedness characteristics for $\mathcal{H}$ from L^p to L^q, but it is not complete. The purpose of this work is formation the whole basic scale of alternative boundedness characteristics for the operator $\mathcal{H} : L^p \to L^q$ for all $p > 1$ and $q > 0$. Following this goal, we supplement Tomaselli and Persson–Stepanov type constants (1.7) and (1.8) by their dual analogies in terms of σ. Moreover, by using the notion of ρ, we find ρ–analogies of $\mathbb{A}_\sigma$ and $\mathbb{B}_\sigma$, and set up boundedness characteristics of types (1.9) and (1.10) for the case $0 < q < 1 < p$. Except $\mathbb{A}_\sigma$, $\mathbb{B}_\sigma$, $(\mathbb{A}_\rho)^*$ and $(\mathbb{B}_{PS})^*$, the complete set of related functionals includes the following quantities:

$$(\mathbb{A}_\sigma)^* = \sup_{t>0} \left(\int_{\sigma^{-1}(a(\sigma^{-1}(t)))}^{\sigma^{-1}(b(\sigma^{-1}(t)))} V^q w^q \right)^{1/q} \left(\int_{a(\sigma^{-1}(t))}^{b(\sigma^{-1}(t))} v^{p'} \right)^{-1/p},$$

$$(\mathbb{B}_\sigma)^* = \left(\int_0^\infty \left[\int_{\sigma^{-1}(a(\sigma^{-1}(t)))}^{\sigma^{-1}(b(\sigma^{-1}(t)))} V^q w^q \right]^{r/q} \left[\int_{a(\sigma^{-1}(t))}^{b(\sigma^{-1}(t))} v^{p'} \right]^{-r/q} v^{p'}(t)\, \mathrm{d}t \right)^{1/r},$$

$$\mathbb{A}_\rho = \sup_{t>0} \left(\int_{\rho^{-1}(b^{-1}(\rho^{-1}(t)))}^{\rho^{-1}(a^{-1}(\rho^{-1}(t)))} W^{p'} v^{p'} \right)^{1/p'} \left(\int_{b^{-1}(\rho^{-1}(t))}^{a^{-1}(\rho^{-1}(t))} w^q \right)^{-1/q'},$$

$$\mathbb{B}_\rho = \begin{cases} \mathbb{B}_{q>1} \ll \mathbb{B}_{q<1}, & q > 1 \\ \mathbb{B}_{q<1} \ll \mathbb{B}_{q>1}, & q < 1 \end{cases},$$

$$\mathbb{B}_{q>1} = \left(\int_0^\infty \left[\int_{\rho^{-1}(b^{-1}(\rho^{-1}(t)))}^{\rho^{-1}(a^{-1}(\rho^{-1}(t)))} W^{p'} v^{p'} \right]^{r/p'} \left[\int_{b^{-1}(\rho^{-1}(t))}^{a^{-1}(\rho^{-1}(t))} w^q \right]^{-r/p'} w^q(t)\, \mathrm{d}t \right)^{1/r},$$

$$\mathbb{B}_{q<1} = \left(\int_0^\infty \left[\int_{a(t)}^{b(t)} W^{p'} v^{p'} \right]^{r/p'} \left[\int_{b^{-1}(\rho^{-1}(t))}^{a^{-1}(\rho^{-1}(t))} w^q \right]^{-r/p'} w^q(t)\, \mathrm{d}t \right)^{1/r},$$

where $V(x) := \int_{a(x)}^{b(x)} v^{p'}$ and $W(y) := \int_{b^{-1}(y)}^{a^{-1}(y)} w^q$, and

$$(\mathbb{B}_\rho)^* = \begin{cases} (\mathbb{B}^-_{q>1})^* + (\mathbb{B}^+_{q>1})^*, & q > 1 \\ (\mathbb{B}^-_{q>1})^* + (\mathbb{B}^+_{q>1})^* + (\mathbb{B}^-_{q<1})^* + (\mathbb{B}^+_{q<1})^*, & q < 1 \end{cases},$$

$$(\mathbb{B}^-_{q>1})^* = \left(\int_0^\infty \left[\int_{\rho^{-1}(b^{-1}(t))}^{t} W^{p'} v^{p'} \right]^{r/q'} [W(t)]^{p'-r/q'} v^{p'}(t)\, \mathrm{d}t \right)^{1/r},$$

$$(\mathbb{B}^+_{q>1})^* = \left(\int_0^\infty \left[\int_{t}^{\rho^{-1}(a^{-1}(t))} W^{p'} v^{p'} \right]^{r/q'} [W(t)]^{p'-r/q'} v^{p'}(t)\, \mathrm{d}t \right)^{1/r},$$

$$(\mathbb{B}^-_{q<1})^* = \left(\int_0^\infty \left[\int_{a(\rho(t))}^t W^{p'} v^{p'}\right]^{r/q'} \left[W(t)\right]^{p'-r/q'} v^{p'}(t)\,\mathrm{d}t\right)^{1/r},$$

$$(\mathbb{B}^+_{q<1})^* = \left(\int_0^\infty \left[\int_t^{b(\rho(t))} W^{p'} v^{p'}\right]^{r/q'} \left[W(t)\right]^{p'-r/q'} v^{p'}(t)\,\mathrm{d}t\right)^{1/r},$$

where $(\mathbb{B}_\rho)^* \approx (\mathcal{B}_{PS})^*$ if $q > 1$.

The cumulative outcome of this work, together with the results from [18] (see also [19, Theorem 4.2]), comprises all basic boundedness characteristics (of the second type) for the Hardy–Steklov operator $\mathcal{H}$ from L^p to L^q expressed in terms of the fairway σ and its dual function ρ too. These results can be used in the study of other properties (e.g. characteristic numbers etc.) of the operators $\mathcal{H}$ and their related transforms, as well as in many applications of the Hardy–Steklov operators (e.g. Sobolev spaces etc.).

Throughout the paper the products of the form $0\cdot\infty$ are taken to be equal to 0. Relations of the type $A \ll B$ mean that $A \le cB$ with some constant c depending, possibly, on parameters p and q only. We write $A \approx B$ instead of $A \ll B \ll A$ or $A = cB$. We use $\mathbb{Z}$ and $\mathbb{N}$ for integers and natural numbers, respectively. χ_E stands for the characteristic function (indicator) of a subset $E \subset (0,\infty)$. We make use of the attribution signs $:=$ and $=:$ for introducing new quantities and denote $p' := p/(p-1)$ for a parameter $0 < p < \infty$ and $r := pq/(p-q)$ if $0 < q < p < \infty$. We assume weight functions to be non–negative and locally integrable to appropriate powers. In order to shorten big formulae, we use $\int h$ instead of $\int h(t)\,\mathrm{d}t$ for a one–variable integrand $h(t)$, where it makes sence.

2 The result

Theorem 2.1. *Let $p > 1$, $q > 0$, $q \neq 1$ and the operator $\mathcal{H}$ be defined by* (1.1) *with a, b satisfying conditions* (1.2)*. Suppose that σ is the fairway–function on $(0,\infty)$ satisfying* (1.4) *and ρ is the dual fairway–function on $(0,\infty)$ satisfying* (1.5)*. Then*

$$\|\mathcal{H}\|_{L^p\to L^q} \approx \mathbb{A}_\sigma \approx (\mathbb{A}_\sigma)^* \approx \mathbb{A}_\rho \approx (\mathbb{A}_\rho)^* \tag{2.1}$$

if $1 < p \le q < \infty$. For the case $0 < q < p < \infty$, $p > 1$ we have:

$$\|\mathcal{H}\|_{L^p\to L^q} \approx \mathbb{B}_\sigma \approx (\mathbb{B}_\sigma)^* \approx \mathbb{B}_\rho \approx (\mathbb{B}_\rho)^*, \tag{2.2}$$

where $(\mathbb{B}_\rho)^* \approx (\mathcal{B}_{PS})^*$ if $q > 1$.

Proof. Let $1 < p \le q < \infty$. The estimate $\|\mathcal{H}\|_{L^p\to L^q} \approx \mathbb{A}_\sigma$ was proven in [19, Theorem 4.2]. If we apply this result to the dual to $\mathcal{H}$ operator $\mathcal{H}^*$ of the form

$$\mathcal{H}^* g(y) := v(y) \int_{b^{-1}(y)}^{a^{-1}(y)} g(x) w(x)\,\mathrm{d}x \qquad (y > 0) \tag{2.3}$$

from $L^{q'}$ to $L^{p'}$, then we obtain the equivalence $\|\mathcal{H}\|_{L^p\to L^q} \approx (\mathbb{A}_\rho)^*$. The rest statements of the theorem in the case $1 < p \le q < \infty$ — those are $\|\mathcal{H}\|_{L^p\to L^q} \approx (\mathbb{A}_\sigma)^*$ and $\|\mathcal{H}\|_{L^p\to L^q} \approx \mathbb{A}_\rho$ — follow from the two previous estimates by changing variable $t = \varsigma^{-1}(s)$ with $\varsigma = \sigma$ in $\mathbb{A}_\sigma$ and $\varsigma = \rho^{-1}$ in $(\mathbb{A}_\rho)^*$, recpectively.

Consider the case $0 < q < p < \infty$, $p > 1$. The estimate $\|\mathcal{H}\|_{L^p\to L^q} \approx \mathbb{B}_\sigma$ was established in [19, Theorem 4.2]. To prove $\|\mathcal{H}\|_{L^p\to L^q} \ll (\mathbb{B}_\sigma)^*$ we define a sequence $\{\xi_k\}_{k\in\mathbb{Z}} \subset (0,\infty)$ so that

$$\xi_0 = 1, \qquad \xi_k = (a^{-1}\circ b)^k(1), \qquad k \in \mathbb{Z}. \tag{2.4}$$

Denote $\xi_k^- := \sigma^{-1}(a(\xi_k))$, $\xi_k^+ := \sigma^{-1}(b(\xi_k))$, $\Delta_k := [\xi_k^-, \xi_k^+] = \Delta_k^- \cup \Delta_k^+$, $\Delta_k^- := [\xi_k^-, \xi_k]$, $\Delta_k^+ := [\xi_k, \xi_k^+]$ and form intervals $\varkappa_j^k$, $j = 1, \ldots, N_k$, $N_k = j_k^- + j_k^+$, on Δ_k, $k \in \mathbb{Z}$, similar to those in [19, Lemmas 2.7, 2.8]:

$(1_\Delta^\pm)$ (1_Δ^-) if $\sigma(\xi_k) \le b(\xi_k^-)$ then $j_k^- = 1$ and $\Delta_k^- = \varkappa_1^k$;

(1_Δ^+) if $\sigma(\xi_k) \ge a(\xi_k^+)$ then $j_k^+ = 1$ and $\Delta_k^+ = \varkappa_{j_k^-+1}^k$;

$(2_\Delta^\pm)$ (2_Δ^-) if $b(\xi_k^-) < \sigma(\xi_k) \le b(\sigma^{-1}(b(\xi_k^-)))$ then $j_k^- = 2$ and $\Delta_k^- = \cup_{j=1}^2 \varkappa_j^k$, where $\varkappa_1^k = [\xi_k^-, \sigma^{-1}(b(\xi_k^-))]$ and $\varkappa_2^k = [b^{-1}(\sigma(\xi_k)), \xi_k]$;

(2_Δ^+) if $a(\sigma^{-1}(a(\xi_k^+))) \le \sigma(\xi_k) < a(\xi_k^+)$ then $j_k^+ = 2$ and $\Delta_k^+ = \cup_{j=1}^2 \varkappa_{j_k^-+j}^k$, where $\varkappa_{j_k^-+1}^k = [\xi_k, a^{-1}(\sigma(\xi_k))]$ and $\varkappa_{j_k^-+2}^k = [\sigma^{-1}(a(\xi_k^+)), \xi_k^+]$;

$(3_\Delta^\pm)$ (3_Δ^-) if $b(\sigma^{-1}(b(\xi_k^-))) < \sigma(\xi_k) \le b(\sigma^{-1}(b(\sigma^{-1}(b(\xi_k^-)))))$ then $j_k^- = 3$ and $\Delta_k^- = \cup_{j=1}^3 \varkappa_j^k$, where $\varkappa_1^k = [\xi_k^-, \sigma^{-1}(b(\xi_k^-))]$, $\varkappa_2^k = [\sigma^{-1}(b(\xi_k^-)), \sigma^{-1}(b(\sigma^{-1}(b(\xi_k^-))))]$ and $\varkappa_3^k = [b^{-1}(\sigma(\xi_k)), \xi_k]$;

(3_Δ^+) if $a(\sigma^{-1}(a(\sigma^{-1}(a(\xi_k^+))))) \le \sigma(\xi_k) < a(\sigma^{-1}(a(\xi_k^+)))$ then $j_k^+ = 3$ and $\Delta_k^+ = \cup_{j=1}^3 \varkappa_{j_k^-+j}^k$, where $\varkappa_{j_k^-+1}^k = [\xi_k, a^{-1}(\sigma(\xi_k))]$, $\varkappa_{j_k^-+2}^k = [a^{-1}(\sigma(\xi_k)), a^{-1}(\sigma(a^{-1}(\sigma(\xi_k))))]$ and $\varkappa_{j_k^-+3}^k = [\sigma^{-1}(a(\xi_k^+)), \xi_k^+]$;

$\ldots \quad \ldots \quad \ldots$

Finally, we obtain that $\Delta_k^- = \cup_{j=1}^{j_k^-} \varkappa_j^k$ and $\Delta_k^+ = \cup_{j=1}^{j_k^+} \varkappa_{j_k^-+j}^k$, where for $\varkappa_j^k \subseteq \Delta_k^-$

$$\varkappa_j^k = \begin{cases} \big[(\sigma^{-1} \circ b)^{(j-1)}(\xi_k^-), (\sigma^{-1} \circ b)^{(j)}(\xi_k^-)\big], & \text{for } j = 1, \ldots, j_k^- - 1 \text{ if } j_k^- > 1, \\ \big[\max\{\xi_k^-, b^{-1}(\sigma(\xi_k))\}, \xi_k\big], & \text{for } j = j_k^-; \end{cases}$$

and if $\varkappa_j^k \subseteq \Delta_k^+$ then

$$\varkappa_j^k = \begin{cases} \big[(a^{-1} \circ \sigma)^{(j-1)}(\xi_k), (a^{-1} \circ \sigma)^{(j)}(\xi_k)\big], & \text{for } j = j_k^- + 1, \ldots, N_k^- - 1 \text{ if } j_k^+ > 1, \\ \big[\max\{\xi_k, \sigma^{-1}(a(\xi_k^+))\}, \xi_k^+\big], & \text{for } j = N_k. \end{cases}$$

Denote by l_j^k the left end-points of the $\varkappa_j^k$ and by r_j^k the right end-points of the intervals $\varkappa_j^k$.

Without loss of generality we assume that $f \ge 0$ a.e. on $(0, \infty)$. Breaking the semiaxis $(0, \infty)$ by the points ξ_k we represent the operator $\mathcal{H}$ as the sum of two sequences of operators

$$\begin{aligned} \mathcal{H}f(x) =& \sum_k \Big(\chi_{\Delta_k^-}(x)\mathcal{H}(f\chi_{\delta_\sigma^-(\xi_k^-)})(x) + \chi_{\Delta_k^-}(x)\mathcal{H}(f\chi_{\delta_\sigma(\xi_k)})(x) \\ &+ \chi_{\Delta_k^+}(x)\mathcal{H}(f\chi_{\delta_\sigma(\xi_k)})(x) + \chi_{\Delta_k^+}(x)\mathcal{H}(f\chi_{\delta_\sigma^+(\xi_k^+)})(x)\Big) \\ \le& \sum_k \left(S_k^\sigma f(x) + T_k^\sigma f(x)\right), \end{aligned} \tag{2.5}$$

where $\delta_\sigma(t) := [a(t), b(t)] = \delta_\sigma^-(t) \cup \delta_\sigma^+(t)$, $\delta_\sigma^-(t) := [a(t), \sigma(t)]$, $\delta_\sigma^+(t) := [\sigma(t), b(t)]$ and

$$S_k^\sigma f(x) := w(x) \int_{a(\xi_k^-)}^{b(x)} f(y)v(y)\,\mathrm{d}y \qquad (x \in \Delta_k^-), \tag{2.6}$$

$$T_k^\sigma f(x) := w(x) \int_{a(x)}^{b(\xi_k^+)} f(y)v(y)\,\mathrm{d}y \qquad (x \in \Delta_k^+). \tag{2.7}$$

By virtue of [19, (2.44) from Lemma 2.3 and (2.52) from Lemma 2.4] we obtain

$$\|S_k^\sigma\|^r \approx \int_{\Delta_k^-}\biggl(\int_{\xi_k^-}^t \biggl[\int_{a(\xi_k^-)}^{b(x)} v^{p'}\biggr]^q w^q(x)\,\mathrm{d}x\biggr)^{r/p}\biggl[\int_{a(\xi_k^-)}^{b(t)} v^{p'}\biggr]^{q-r/p} w^q(t)\,\mathrm{d}t \tag{2.8}$$

and

$$\|T_k^\sigma\|^r \approx \int_{\Delta_k^+}\biggl(\int_t^{\xi_k^+} \biggl[\int_{a(x)}^{b(\xi_k^+)} v^{p'}\biggr]^q w^q(x)\,\mathrm{d}x\biggr)^{r/p}\biggl[\int_{a(t)}^{b(\xi_k^+)} v^{p'}\biggr]^{q-r/p} w^q(t)\,\mathrm{d}t. \tag{2.9}$$

By covering Δ_k^- by the intervals $\varkappa_j^k$, $j=1,\dots,j_k^-$, we write, taking into account that $r/p+1=r/q$:

$$\begin{aligned}
\|S_k^\sigma\|^r &\ll \sum_{j=1}^{j_k^-}\int_{\varkappa_j^k}\biggl(\int_{\xi_k^-}^t \biggl[\int_{a(\xi_k^-)}^{b(x)} v^{p'}\biggr]^q w^q(x)\,\mathrm{d}x\biggr)^{r/p}\biggl[\int_{a(\xi_k^-)}^{b(t)} v^{p'}\biggr]^{q-r/p} w^q(t)\,\mathrm{d}t\\
&\le \sum_{j=1}^{j_k^-}\biggl(\int_{\xi_k^-}^{r_j^k} \biggl[\int_{a(\xi_k^-)}^{b(x)} v^{p'}\biggr]^q w^q(x)\,\mathrm{d}x\biggr)^{r/p}\biggl[\int_{a(\xi_k^-)}^{b(l_j^k)} v^{p'}\biggr]^{-r/p}\int_{\varkappa_j^k}\biggl[\int_{a(\xi_k^-)}^{b(t)} v^{p'}\biggr]^q w^q(t)\,\mathrm{d}t\\
&\le \sum_{j=1}^{j_k^-}\biggl(\int_{\xi_k^-}^{r_j^k} \biggl[\int_{a(\xi_k^-)}^{b(x)} v^{p'}\biggr]^q w^q(x)\,\mathrm{d}x\biggr)^{r/q}\biggl[\int_{a(\xi_k^-)}^{b(l_j^k)} v^{p'}\biggr]^{-r/p} =: \Sigma_k^-.
\end{aligned} \tag{2.10}$$

If $j_k^-=1$, we have, by (1.4),

$$\begin{aligned}
\Sigma_k^- &=\biggl(\int_{\xi_k^-}^{\xi_k} \biggl[\int_{a(\xi_k^-)}^{b(x)} v^{p'}\biggr]^q w^q(x)\,\mathrm{d}x\biggr)^{r/q}\biggl[\int_{a(\xi_k^-)}^{b(\xi_k^-)} v^{p'}\biggr]^{-r/p}\\
&\le 2\biggl(\int_{\xi_k^-}^{\xi_k} \biggl[\int_{a(\xi_k^-)}^{b(x)} v^{p'}\biggr]^q w^q(x)\,\mathrm{d}x\biggr)^{r/q}\biggl[\int_{a(\xi_k^-)}^{b(\xi_k^-)} v^{p'}\biggr]^{-r/q}\int_{\sigma(\xi_k^-)}^{b(\xi_k)} v^{p'}\\
&\le 4\biggl(\int_{\xi_k^-}^{\xi_k} \biggl[\int_{a(\xi_k^-)}^{b(x)} v^{p'}\biggr]^q w^q(x)\,\mathrm{d}x\biggr)^{r/q}\biggl[\int_{a(\xi_k^-)}^{\sigma(\xi_k)} v^{p'}\biggr]^{-r/q}\int_{\sigma(\xi_k^-)}^{\sigma(\xi_k)} v^{p'}(t)\,\mathrm{d}t\\
&\le 4\biggl(\int_{\xi_k^-}^{\xi_k} \biggl[\int_{a(\xi_k^-)}^{b(x)} v^{p'}\biggr]^q w^q(x)\,\mathrm{d}x\biggr)^{r/q}\int_{\sigma(\xi_k^-)}^{\sigma(\xi_k)}\biggl[\int_{a(\sigma^{-1}(t))}^{t} v^{p'}\biggr]^{-r/q} v^{p'}(t)\,\mathrm{d}t
\end{aligned}$$

since $a(\xi_k)=\sigma(\xi_k^-)$, and $b(\xi_k^-)\ge\sigma(\xi_k)$, by the construction of the $\varkappa_{j_k^-}^k=[\xi_k^-,\xi_k]$ in the case $j_k^-=1$. Notice that, by (1.4),

$$\int_{a(\sigma^{-1}(t))}^{t} v^{p'}=\int_{a(\sigma^{-1}(t))}^{\sigma(\sigma^{-1}(t))} v^{p'}=\int_{\sigma(\sigma^{-1}(t))}^{b(\sigma^{-1}(t))} v^{p'}=\frac{1}{2}\int_{a(\sigma^{-1}(t))}^{b(\sigma^{-1}(t))} v^{p'}. \tag{2.11}$$

Moreover, since $a(\xi_k^-)\le a(x)\le a(\xi_k)$ for $x\in\Delta_k^-$, then

$$\int_{a(\xi_k^-)}^{b(x)} v^{p'}=2\int_{a(\xi_k)}^{b(\xi_k^-)} v^{p'}+\int_{b(\xi_k^-)}^{b(x)} v^{p'}\le 2\int_{a(\xi_k)}^{b(x)} v^{p'}\le 2\int_{a(x)}^{b(x)} v^{p'}\le 2\int_{a(\xi_k^-)}^{b(x)} v^{p'}. \tag{2.12}$$

Therefore, we obtain for the case $j_k^- = 1$:

$$\begin{aligned}\Sigma_k^- \leq& 2^{r/q+2+r}\bigg(\int_{\xi_k^-}^{\xi_k}\bigg[\int_{a(x)}^{b(x)} v^{p'}\bigg]^q w^q(x)\,\mathrm{d}x\bigg)^{r/q}\int_{\sigma(\xi_k^-)}^{\sigma(\xi_k)}\bigg[\int_{a(\sigma^{-1}(t))}^{b(\sigma^{-1}(t))} v^{p'}\bigg]^{-r/q} v^{p'}(t)\,\mathrm{d}t\\ \leq& 2^{r/q+2+r}\bigg(\int_{\sigma^{-1}(a(\xi_k))}^{\sigma^{-1}(b(\xi_k^-))} V^q w^q\bigg)^{r/q}\int_{\sigma(\xi_k^-)}^{\sigma(\xi_k)}\bigg[\int_{a(\sigma^{-1}(t))}^{b(\sigma^{-1}(t))} v^{p'}\bigg]^{-r/q} v^{p'}(t)\,\mathrm{d}t\\ \leq& 2^{r/q+2+r}\int_{a(\xi_k)}^{\sigma(\xi_k)}\bigg(\int_{\sigma^{-1}(a(\sigma^{-1}(t)))}^{\sigma^{-1}(b(\sigma^{-1}(t)))} V^q w^q\bigg)^{r/q}\bigg[\int_{a(\sigma^{-1}(t))}^{b(\sigma^{-1}(t))} v^{p'}\bigg]^{-r/q} v^{p'}(t)\,\mathrm{d}t. \end{aligned} \tag{2.13}$$

If $j_k^- > 1$, then, by virtue of (2.12) and in view of (1.4),

$$\begin{aligned}\Sigma_k^- =& \sum_{j=1}^{j_k^-}\bigg(\int_{\xi_k^-}^{r_j^k}\bigg[\int_{a(\xi_k^-)}^{b(x)} v^{p'}\bigg]^q w^q(x)\,\mathrm{d}x\bigg)^{r/q}\bigg[\int_{a(\xi_k^-)}^{b(l_j^k)} v^{p'}\bigg]^{-r/p}\\ \leq& 2^r\sum_{j=1}^{j_k^-}\bigg(\int_{\xi_k^-}^{r_j^k}\bigg[\int_{a(x)}^{b(x)} v^{p'}\bigg]^q w^q(x)\,\mathrm{d}x\bigg)^{r/q}\bigg[\int_{a(l_j^k)}^{b(l_j^k)} v^{p'}\bigg]^{-r/q}\int_{a(l_j^k)}^{b(l_j^k)} v^{p'}\\ =& 2^{r+1}\sum_{j=1}^{j_k^-}\bigg(\int_{\xi_k^-}^{r_j^k} V^q w^q\bigg)^{r/q}\bigg[\int_{a(l_j^k)}^{b(l_j^k)} v^{p'}\bigg]^{-r/q}\int_{\sigma(l_j^k)}^{b(l_j^k)} v^{p'}(t)\,\mathrm{d}t\\ \leq& 2^{r+1}\sum_{j=1}^{j_k^-}\bigg(\int_{\xi_k^-}^{r_j^k} V^q w^q\bigg)^{r/q}\int_{\sigma(l_j^k)}^{b(l_j^k)}\bigg[\int_{a(\sigma^{-1}(t))}^{t} v^{p'}\bigg]^{-r/q} v^{p'}(t)\,\mathrm{d}t. \end{aligned}$$

Further, we obtain by (2.11), taking into account $\xi_k^- \geq \sigma^{-1}(a(r_j^k))$ and $b(l_j^k) = \sigma(r_j^k)$, and following the construction of the $\varkappa_j^k$, $j = 1, \ldots, j_k^-$ in the case $j_k^- > 1$, that

$$\begin{aligned}\Sigma_k^- \leq& 2^{r/q+1+r}\sum_{j=1}^{j_k^-}\bigg(\int_{\xi_k^-}^{r_j^k} V^q w^q\bigg)^{r/q}\int_{\sigma(l_j^k)}^{b(l_j^k)}\bigg[\int_{a(\sigma^{-1}(t))}^{b(\sigma^{-1}(t))} v^{p'}\bigg]^{-r/q} v^{p'}(t)\,\mathrm{d}t\\ \leq& 2^{r/q+1+r}\sum_{j=1}^{j_k^-}\bigg(\int_{\sigma^{-1}(a(r_j^k))}^{\sigma^{-1}(b(l_j^k))} V^q w^q\bigg)^{r/q}\int_{\sigma(l_j^k)}^{\sigma(r_j^k)}\bigg[\int_{a(\sigma^{-1}(t))}^{b(\sigma^{-1}(t))} v^{p'}\bigg]^{-r/q} v^{p'}(t)\,\mathrm{d}t\\ \leq& 2^{r/q+1+r}\sum_{j=1}^{j_k^-}\int_{\sigma(l_j^k)}^{b(l_j^k)}\bigg(\int_{\sigma^{-1}(a(\sigma^{-1}(t)))}^{\sigma^{-1}(b(\sigma^{-1}(t)))} V^q w^q\bigg)^{r/q}\bigg[\int_{a(\sigma^{-1}(t))}^{b(\sigma^{-1}(t))} v^{p'}\bigg]^{-r/q} v^{p'}(t)\,\mathrm{d}t\\ \leq& 2^{r/q+2+r}\int_{a(\xi_k)}^{\sigma(\xi_k)}\bigg(\int_{\sigma^{-1}(a(\sigma^{-1}(t)))}^{\sigma^{-1}(b(\sigma^{-1}(t)))} V^q w^q\bigg)^{r/q}\bigg[\int_{a(\sigma^{-1}(t))}^{b(\sigma^{-1}(t))} v^{p'}\bigg]^{-r/q} v^{p'}(t)\,\mathrm{d}t \end{aligned} \tag{2.14}$$

because no point of $[a(\xi_k), \sigma(\xi_k)]$ lies in more than two of the $[\sigma(l_j^k), b(l_j^k)]$. Combining (2.8), (2.10) with (2.13) and (2.14) leads to

$$\|S_k^\sigma\|^r \ll \int_{a(\xi_k)}^{\sigma(\xi_k)}\bigg(\int_{\sigma^{-1}(a(\sigma^{-1}(t)))}^{\sigma^{-1}(b(\sigma^{-1}(t)))} V^q w^q\bigg)^{r/q}\bigg[\int_{a(\sigma^{-1}(t))}^{b(\sigma^{-1}(t))} v^{p'}\bigg]^{-r/q} v^{p'}(t)\,\mathrm{d}t. \tag{2.15}$$

Analogously, by covering Δ_k^+ by the intervals $\varkappa_j^k$, where $j = j_k^- + 1, \dots, N_k$, we obtain from (2.9) that

$$\|T_k^\sigma\|^r \ll \int_{\sigma(\xi_k)}^{b(\xi_k)} \Big(\int_{\sigma^{-1}(a(\sigma^{-1}(t)))}^{\sigma^{-1}(b(\sigma^{-1}(t)))} V^q w^q\Big)^{r/q} \Big[\int_{a(\sigma^{-1}(t))}^{b(\sigma^{-1}(t))} v^{p'}\Big]^{-r/q} v^{p'}(t)\,\mathrm{d}t. \tag{2.16}$$

Now, from (2.5), (2.15) and(2.16), we have by virtue of [19, Lemma 3.1] and by Hölder's inequality with the powers r/q and p/q,

$$\begin{aligned}
\|\mathcal{H}f\|_q^q &\le \sum_{k\in\mathbb{Z}} \big\{\|S_k^\sigma f\|_q^q + \|T_k^\sigma f\|_q^q\big\} \\
&\ll \sum_{k\in\mathbb{Z}} \bigg\{ \bigg(\int_{a(\xi_k)}^{\sigma(\xi_k)} \Big[\int_{\sigma^{-1}(a(\sigma^{-1}(t)))}^{\sigma^{-1}(b(\sigma^{-1}(t)))} V^q w^q\Big]^{r/q} \Big[\int_{a(\sigma^{-1}(t))}^{b(\sigma^{-1}(t))} v^{p'}\Big]^{-r/q} v^{p'}(t)\,\mathrm{d}t\bigg)^{q/r} \big\|f\chi_{[a(\xi_k^-),b(\xi_k)]}\big\|_p^q \\
&\quad + \bigg(\int_{\sigma(\xi_k)}^{b(\xi_k)} \Big[\int_{\sigma^{-1}(a(\sigma^{-1}(t)))}^{\sigma^{-1}(b(\sigma^{-1}(t)))} V^q w^q\Big]^{r/q} \Big[\int_{a(\sigma^{-1}(t))}^{b(\sigma^{-1}(t))} v^{p'}\Big]^{-r/q} v^{p'}(t)\,\mathrm{d}t\bigg)^{q/r} \big\|f\chi_{[a(\xi_k),b(\xi_k^+)]}\big\|_p^q \bigg\} \\
&\ll \sum_{k\in\mathbb{Z}} \bigg(\int_{a(\xi_k)}^{b(\xi_k)} \Big[\int_{\sigma^{-1}(a(\sigma^{-1}(t)))}^{\sigma^{-1}(b(\sigma^{-1}(t)))} V^q w^q\Big]^{r/q} \Big[\int_{a(\sigma^{-1}(t))}^{b(\sigma^{-1}(t))} v^{p'}\Big]^{-r/q} v^{p'}(t)\,\mathrm{d}t\bigg)^{q/r} \big\|f\chi_{[a(\xi_k^-),b(\xi_k^+)]}\big\|_p^q \\
&\le \bigg(\sum_{k\in\mathbb{Z}} \int_{a(\xi_k)}^{b(\xi_k)} \int_{\sigma^{-1}(a(\sigma^{-1}(t)))}^{\sigma^{-1}(b(\sigma^{-1}(t)))} V^q w^q\Big]^{r/q} \Big[\int_{a(\sigma^{-1}(t))}^{b(\sigma^{-1}(t))} v^{p'}\Big]^{-r/q} v^{p'}(t)\,\mathrm{d}t\bigg)^{q/r} \\
&\qquad \times \bigg(\sum_{k\in\mathbb{Z}} \big\|f\chi_{[a(\xi_k^-),b(\xi_k^+)]}\big\|_p^p\bigg)^{q/p} \ll \big[(\mathbb{B}_\sigma)^*\big]^q \|f\|_p^q,
\end{aligned}$$

since no point of $(0,\infty)$ lies in more than two of the $[a(\xi_k), b(\xi_k)]$ and in more than three of the $[a(\xi_k^-), b(\xi_k^+)]$. From here the estimate $\|\mathcal{H}\| \ll (\mathbb{B}_\sigma)^*$ follows immediately.

The inequality $\|\mathcal{H}\| \gg (\mathbb{B}_\sigma)^*$ can be obtained analogously to the estimate $\|\mathcal{H}\| \gg \mathcal{B}_\sigma$ in [19, §4.1], where $\mathcal{B}_\sigma$ is the Mazya–Rozin type boundedness constant (1.6) for the operator $\mathcal{H}$ from L^p to L^q. Namely, changing variables in the initial inequality

$$\big\|\mathcal{H}f\big\|_q \le C\|f\|_p,$$

that reflects the boundedness of the operator $\mathcal{H}: L^p \to L^q$, we pass to the inequality

$$\bigg(\int_0^\infty \bigg|\int_{\tilde a(x)}^{\tilde b(x)} fv\bigg|^q \tilde w^q(x)\,\mathrm{d}x\bigg)^{1/q} \le C\bigg(\int_0^\infty |f|^p\bigg)^{1/p} \tag{2.17}$$

with $\tilde a(x) = a(\sigma^{-1}(x))$, $\tilde b(x) = b(\sigma^{-1}(x))$ and $\tilde w(x) = w(\sigma^{-1}(x))[(\sigma^{-1})'(x)]^{1/q}$. Notice, that, by (2.11), the function $\tilde\sigma(x) = x$ is the fairway for the boundaries $\tilde a$ and $\tilde b$ with respect to $v^{p'}$ in this case. To establish the relation $\|\mathcal{H}\| \gg (\tilde{\mathbb{B}}_\sigma^-)^*$ for the case $\tilde\sigma(x) = x$ with

$$(\tilde{\mathbb{B}}_\sigma^-)^* = \bigg(\int_0^\infty \Big[\int_{\tilde a(t)}^t V^q \tilde w^q\Big]^{r/q} \Big[\int_{\tilde a(t)}^{\tilde b(t)} v^{p'}\Big]^{-r/q} v^{p'}(t)\,\mathrm{d}t\bigg)^{1/r}$$

we employ the method for the proof of the estimate $\|\mathcal{H}\| \gg \mathcal{B}_\sigma$ in [19, §4.1, pp. 477–481], extracting the inequality

$$\|\mathcal{H}\| \gg \lim_{N\to\infty} \bigg(\sum_{|k|\le N} \lambda_k\bigg)^{1/r} \tag{2.18}$$

with
$$\lambda_k = \int_{\eta_{k-1}}^{\eta_{k+2}} \Big(\int_t^{\eta_{k+2}} \tilde{w}^q(x)\,\mathrm{d}x\Big)^{\frac{r}{q}} \Big(\int_{\eta_{k-1}}^t v^{p'}(y)\,\mathrm{d}y\Big)^{\frac{r}{q'}} v^{p'}(t)\,\mathrm{d}t,$$
where
$$\eta_0 = 1, \quad \eta_{k+1} = \tilde{a}^{-1}(\eta_k), \quad \eta_{k-1} = \tilde{a}(\eta_k), \qquad k \in \mathbb{Z}. \tag{2.19}$$
It is possible to establish that
$$\lambda_k \gg \int_{\eta_{k-1}}^{\eta_{k+2}} \Big(\int_{\eta_{k-1}}^t \Big[\int_{\eta_{k-1}}^x v^{p'}(y)\,\mathrm{d}y\Big]^q \tilde{w}^q(x)\,\mathrm{d}x\Big)^{r/q} \Big(\int_{\eta_{k-1}}^t v^{p'}(y)\,\mathrm{d}y\Big)^{-r/q} v^{p'}(t)\,\mathrm{d}t \tag{2.20}$$
(see [19, §4.2, pp. 491–492] or [16, §2.2.3, p. 90] for details). We also have, by (1.4) and (2.19), for $t \in [\eta_{k+1}, \eta_{k+2}]$ that
$$\begin{aligned}\int_{\eta_{k-1}}^t v^{p'}(y)\,\mathrm{d}y &= \int_{\eta_{k-1}}^{\eta_k} v^{p'}(y)\,\mathrm{d}y + \int_{\eta_k}^{\eta_{k+1}} v^{p'}(y)\,\mathrm{d}y + \int_{\eta_{k+1}}^t v^{p'}(y)\,\mathrm{d}y\\ &= \int_{\eta_k}^{\tilde{b}(\eta_k)} v^{p'}(y)\,\mathrm{d}y + \int_{\eta_{k+1}}^{\tilde{b}(\eta_{k+1})} v^{p'}(y)\,\mathrm{d}y + \int_{\eta_{k+1}}^t v^{p'}(y)\,\mathrm{d}y\\ &\le \int_{\eta_k}^{\tilde{b}(\eta_{k+1})} v^{p'}(y)\,\mathrm{d}y + \int_{\eta_{k+1}}^{\tilde{b}(t)} v^{p'}(y)\,\mathrm{d}y + \int_{\eta_{k+1}}^t v^{p'}(y)\,\mathrm{d}y\\ &\qquad \le 2\int_{\eta_{k+1}}^{\tilde{b}(\eta_{k+1})} v^{p'}(y)\,\mathrm{d}y + 2\int_{\eta_{k+1}}^{\tilde{b}(t)} v^{p'}(y)\,\mathrm{d}y \le 4\int_{\tilde{a}(t)}^{\tilde{b}(t)} v^{p'}(y)\,\mathrm{d}y.\end{aligned}$$
Combining this with (2.20) we obtain, by (1.4), (2.19) and (2.11), that
$$\begin{aligned}\lambda_k &\gg \int_{\eta_{k+1}}^{\eta_{k+2}} \Big(\int_{\eta_k}^t \Big[\int_{\eta_{k-1}}^x v^{p'}(y)\,\mathrm{d}y\Big]^q \tilde{w}^q(x)\,\mathrm{d}x\Big)^{r/q} \Big(\int_{\eta_{k-1}}^t v^{p'}(y)\,\mathrm{d}y\Big)^{-r/q} v^{p'}(t)\,\mathrm{d}t\\ &\gg \int_{\eta_{k+1}}^{\eta_{k+2}} \Big(\int_{\eta_k}^t \Big[\int_{\tilde{a}(x)}^x v^{p'}(y)\,\mathrm{d}y\Big]^q \tilde{w}^q(x)\,\mathrm{d}x\Big)^{r/q} \Big(\int_{\tilde{a}(t)}^{\tilde{b}(t)} v^{p'}(y)\,\mathrm{d}y\Big)^{-r/q} v^{p'}(t)\,\mathrm{d}t\\ &\gg \int_{\eta_{k+1}}^{\eta_{k+2}} \Big(\int_{\tilde{a}(t)}^t \Big[\int_{\tilde{a}(x)}^{\tilde{b}(x)} v^{p'}(y)\,\mathrm{d}y\Big]^q \tilde{w}^q(x)\,\mathrm{d}x\Big)^{r/q} \Big(\int_{\tilde{a}(t)}^{\tilde{b}(t)} v^{p'}(y)\,\mathrm{d}y\Big)^{-r/q} v^{p'}(t)\,\mathrm{d}t,\end{aligned}$$
and, therefore, by (2.18),
$$\begin{aligned}\|\mathcal{H}\| &\gg \lim_{N\to\infty}\Big(\sum_{|k|\le N}\int_{\eta_{k+1}}^{\eta_{k+2}} \Big(\int_{\tilde{a}(t)}^t \Big[\int_{\tilde{a}(x)}^{\tilde{b}(x)} v^{p'}(y)\mathrm{d}y\Big]^q \tilde{w}^q(x)\mathrm{d}x\Big)^{r/q} \Big(\int_{\tilde{a}(t)}^{\tilde{b}(t)} v^{p'}(y)\mathrm{d}y\Big)^{-r/q} v^{p'}(t)\mathrm{d}t\Big)^{1/r}\\ &= \int_0^\infty \Big(\int_{\tilde{a}(t)}^t \Big[\int_{\tilde{a}(x)}^{\tilde{b}(x)} v^{p'}\Big]^q \tilde{w}^q(x)\,\mathrm{d}x\Big)^{r/q} \Big(\int_{\tilde{a}(t)}^{\tilde{b}(t)} v^{p'}\Big)^{-r/q} v^{p'}(t)\,\mathrm{d}t\Big)^{1/r} = (\tilde{\mathbb{B}}_\sigma^-)^*.\end{aligned} \tag{2.21}$$
The estimate $\|\mathcal{H}\| \gg (\tilde{\mathbb{B}}_\sigma^+)^*$ with
$$(\tilde{\mathbb{B}}_\sigma^+)^* = \Big(\int_0^\infty \Big[\int_t^{\tilde{b}(t)} V^q\tilde{w}^q\Big]^{r/q} \Big[\int_{\tilde{a}(t)}^{\tilde{b}(t)} v^{p'}\Big]^{-r/q} v^{p'}(t)\,\mathrm{d}t\Big)^{1/r}$$

can be proven similarly, using the intervals $[\zeta_k, \zeta_{k+1})$ formed by the boundary $b(x)$:

$$\zeta_0 = 1, \quad \zeta_{k+1} = \tilde{b}(\zeta_k), \quad \zeta_{k-1} = \tilde{b}^{-1}(\zeta_k), \qquad k \in \mathbb{Z}. \tag{2.22}$$

By using and (2.21), we obtain that

$$\|\mathcal{H}\| \gg \int_0^\infty \biggl(\int_{\tilde{a}(t)}^{\tilde{b}(t)} \biggl[\int_{\tilde{a}(x)}^{\tilde{b}(t)} v^{p'}(y)\,\mathrm{d}y \biggr]^q \tilde{w}^q(x)\,\mathrm{d}x \biggr)^{r/q} \times \biggl(\int_{\tilde{a}(t)}^{\tilde{b}(t)} v^{p'}(y)\,\mathrm{d}y \biggr)^{-r/q} v^{p'}(t)\,\mathrm{d}t \biggr)^{1/r} = (\mathbb{B}_\sigma)^*, \tag{2.23}$$

since $\tilde{a}(x) = a(\sigma^{-1}(x))$, $\tilde{b}(x) = b(\sigma^{-1}(x))$ and $\tilde{w}(x) = w(\sigma^{-1}(x))[(\sigma^{-1})'(x)]^{1/q}$.

If $q > 1$ then the estimates $\|\mathcal{H}\| \approx \mathbb{B}_\rho \approx (\mathbb{B}_\rho)^*$ follow, by duality, from the equivalences $\|\mathcal{H}\| \approx (\mathbb{B}_\sigma)^* \approx \mathbb{B}_\sigma$ applied to operator (2.3) from $L^{q'}$ to $L^{p'}$ (see [20, Theorem 2.2] for details).

To show that $\|\mathcal{H}\| \ll \mathbb{B}_\rho$ for $0 < q < 1$ we introduce sequence (2.4) and cover $\mathcal{H}$ as follows:

$$\begin{aligned} \mathcal{H}f(x) &= \sum_{k\in\mathbb{Z}} \chi_{[\xi_{k-1},\xi_k]}(x)\mathcal{H}\bigl(f\chi_{\delta_\rho^-(\xi_k)}\bigr)(x) + \chi_{[\xi_k, a^{-1}(\rho^{-1}(\xi_k))]}(x)\mathcal{H}\bigl(f\chi_{\delta_\rho^-(\xi_k)}\bigr)(x) \\ &\quad + \sum_{k\in\mathbb{Z}} \chi_{[b^{-1}(\rho^{-1}(\xi_k)),\xi_k]}(x)\mathcal{H}\bigl(f\chi_{\delta_\rho^+(\xi_k)}\bigr)(x) + \chi_{[\xi_k,\xi_{k+1}]}(x)\mathcal{H}\bigl(f\chi_{\delta_\rho^+(\xi_k)}\bigr)(x) \\ &\le \sum_{k\in\mathbb{Z}} S_k^\rho f(x) + T_k^\rho f(x), \end{aligned} \tag{2.24}$$

where $\delta_\rho(t) := [a(t), b(t)] = \delta_\rho^-(t) \cup \delta_\rho^+(t)$, $\delta_\rho^-(t) := [a(t), \rho^{-1}(t)]$, $\delta_\rho^+(t) := [\rho^{-1}(t), b(t)]$ and

$$S_k^\rho f(x) := w(x) \int_{a(\xi_k)}^{\min\{b(x),\rho^{-1}(\xi_k)\}} f(y)v(y)\,\mathrm{d}y \qquad (x \in [\xi_{k-1}, a^{-1}(\rho^{-1}(\xi_k))]), \tag{2.25}$$

$$T_k^\rho f(x) := w(x) \int_{\max\{\rho^{-1}(\xi_k),a(x)\}}^{b(\xi_k)} f(y)v(y)\,\mathrm{d}y \qquad (x \in [b^{-1}(\rho^{-1}(\xi_k)), \xi_{k+1}]). \tag{2.26}$$

Similarly to $\varkappa_j^k$, $j = 1, \dots, N_k$, on Δ_k (see the first part of the proof), we form intervals ϱ_i^k, $i = 1, \dots, M_k$, $M_k = i_k^- + i_k^+$, on $\delta_k := \delta_\rho(\xi_k) = \delta_k^- \cup \delta_k^+$, $k \in \mathbb{Z}$, where $\delta_k^\pm := \delta_\rho^\pm(\xi_k)$:

($1_\delta^\pm$) (1_δ^-) if $b^{-1}(\rho^{-1}(\xi_k)) \le \rho(a(\xi_k))$ then $i_k^- = 1$ and $\delta_k^- = \varrho_1^k$;

(1_δ^+) if $a^{-1}(\rho^{-1}(\xi_k)) \ge \rho(b(\xi_k))$ then $i_k^+ = 1$ and $\delta_k^+ = \varrho_{i_k^-+1}^k$;

($2_\delta^\pm$) (2_δ^-) if $b^{-1}(\rho^{-1}(\xi_k)) > \rho(a(\xi_k)) \ge b^{-1}(\rho^{-1}(b^{-1}(\rho^{-1}(\xi_k))))$ then $i_k^- = 2$ and $\delta_k^- = \cup_{i=1}^2 \varrho_i^k$, where $\varrho_1^k = [b(\xi_{k-1}), b(\rho(b(\xi_{k-1})))]$ and $\varrho_2^k = [\rho^{-1}(b^{-1}(\rho^{-1}(\xi_k))), \rho^{-1}(\xi_k)]$;

(2_δ^+) if $a^{-1}(\rho^{-1}(\xi_k)) < \rho(b(\xi_k)) \le a^{-1}(\rho^{-1}(a^{-1}(\rho^{-1}(\xi_k))))$ then $i_k^+ = 2$ and $\delta_k^+ = \cup_{i=1}^2 \varrho_{i_k^-+i}^k$, where $\varrho_{i_k^-+1}^k = [\rho^{-1}(\xi_k), \rho^{-1}(a^{-1}(\rho^{-1}(\xi_k)))]$ and $\varrho_{i_k^-+2}^k = [a(\rho(a(\xi_{k+1}))), a(\xi_{k+1})]$;

($3_\delta^\pm$) (3_δ^-) if $(b^{-1}\circ\rho^{-1})^{(2)}(\xi_k) > \rho(a(\xi_k)) \ge (b^{-1}\circ\rho^{-1})^{(3)}(\xi_k)$ then $i_k^+ = 3$ and $\delta_k^- = \cup_{i=1}^3 \varrho_i^k$, where $\varrho_1^k = [b(\xi_{k-1}), b(\rho(b(\xi_{k-1})))]$, $\varrho_2^k = [(\rho^{-1}\circ b^{-1})^{(2)}(\rho^{-1}(\xi_k)), (\rho^{-1}\circ b^{-1})^{(1)}(\rho^{-1}(\xi_k))]$ and $\varrho_3^k = [\rho^{-1}(b^{-1}(\rho^{-1}(\xi_k))), \rho^{-1}(\xi_k)]$;

(3_δ^+) if $(a^{-1}\circ\rho^{-1})^{(2)}(\xi_k) < \rho(b(\xi_k)) \le (a^{-1}\circ\rho^{-1})^{(3)}(\xi_k)$ then $i_k^+ = 3$ and $\delta_k^+ = \cup_{i=1}^3 \varrho_{i_k^-+i}^k$, where $\varrho_{i_k^-+1}^k = [\rho^{-1}(\xi_k), \rho^{-1}(a^{-1}(\rho^{-1}(\xi_k)))]$, $\varrho_{i_k^-+2}^k = [(\rho^{-1}\circ a^{-1})^{(1)}(\rho^{-1}(\xi_k)), (\rho^{-1}\circ a^{-1})^{(2)}(\rho^{-1}(\xi_k))]$ and $\varrho_{i_k^-+3}^k = [a(\rho(a(\xi_{k+1}))), a(\xi_{k+1})]$;

… … …

We, finally, obtain that $\delta_k^- = \cup_{i=1}^{i_k^-}\varrho_i^k$ and $\delta_k^+ = \cup_{i=1}^{i_k^+}\varrho_{i_k^-+i}^k$, where for $\varrho_i^k \subseteq \delta_k^-$

$$\varrho_i^k = \begin{cases} \big[b(\xi_{k-1}), \min\{b(\rho(b(\xi_{k-1}))), \rho^{-1}(\xi_k)\}\big], & \text{for } i=1,\\ \big[(\rho^{-1}\circ b^{-1})^{(i_k^- - i+1)}(\rho^{-1}(\xi_k)), (\rho^{-1}\circ b^{-1})^{(i_k^- - i)}(\rho^{-1}(\xi_k))\big], & \text{for } i=2,\dots,i_k^- \\ & \quad \text{and if } i_k^- > 1; \end{cases}$$

and if $\varrho_i^k \subseteq \delta_k^+$ then

$$\varrho_i^k = \begin{cases} \big[(\rho^{-1}\circ a^{-1})^{(i-i_k^- -1)}(\rho^{-1}(\xi_k)), (\rho^{-1}\circ b^{-1})^{(i-i_k^-)}(\rho^{-1}(\xi_k))\big], & \text{for } i=i_k^- +1,\dots \\ & \dots, M_k^- - 1 \\ & \quad \text{and if } i_k^+ > 1, \\ \big[\max\{\rho^{-1}(\xi_k), a(\rho(a(\xi_{k+1})))\}, a(\xi_{k+1})\big], & \text{for } i = N_k. \end{cases}$$

Denote by s_i^k the lower end-points of the ϱ_i^k and by t_i^k the upper end-points of the intervals ϱ_i^k.

Now, fixed $k\in\mathbb{Z}$, we cover S_k^ρ and T_k^ρ by operators $S_{k,i}^\rho f(x) := S_k^\rho(f\chi_{[s_i^k,t_i^k]})(x)$, $1\le i\le i_k^-$, and $T_{k,i}^\rho f(x) := T_k^\rho(f\chi_{[s_i^k,t_i^k]})(x)$, $i_k^- +1\le i\le M_k$. By [19, (2.50) in Lemma 2.4 and (2.42) in Lemma 2.3],

$$\begin{aligned} \|S_{k,i}^\rho\|^r &\approx \int_{b^{-1}(s_i^k)}^{a^{-1}(\rho^{-1}(\xi_k))}\left(\int_z^{a^{-1}(\rho^{-1}(\xi_k))} w^q\right)^{r/p}\left(\int_{s_i^k}^{\min\{t_i^k,b(z)\}} v^{p'}\right)^{r/p'} w^q(z)\,\mathrm{d}z,\\ &\le \int_{b^{-1}(s_i^k)}^{a^{-1}(\rho^{-1}(\xi_k))}\left(\int_z^{a^{-1}(\rho^{-1}(\xi_k))} w^q\right)^{-r/p'}\\ &\quad\times\left(\int_{s_i^k}^{\min\{t_i^k,b(z)\}}\left[\int_{b^{-1}(y)}^{a^{-1}(\rho^{-1}(\xi_k))} w^q\right]^{p'} v^{p'}(y)\,\mathrm{d}y\right)^{r/p'} w^q(z)\,\mathrm{d}z,\\ \|T_{k,i}^\rho\|^r &\approx \int_{b^{-1}(\rho^{-1}(\xi_k))}^{a^{-1}(t_i^k)}\left(\int_{b^{-1}(\rho^{-1}(\xi_k))}^{z} w^q\right)^{r/p}\left(\int_{\max\{s_i^k,a(z)\}}^{t_i^k} v^{p'}\right)^{r/p'} w^q(z)\,\mathrm{d}z\\ &\le \int_{b^{-1}(\rho^{-1}(\xi_k))}^{a^{-1}(t_i^k)}\left(\int_{b^{-1}(\rho^{-1}(\xi_k))}^{z} w^q\right)^{-r/p'}\\ &\quad\times\left(\int_{\max\{s_i^k,a(z)\}}^{t_i^k}\left[\int_{b^{-1}(\rho^{-1}(\xi_k))}^{a^{-1}(y)} w^q\right]^{p'} v^{p'}(y)\,\mathrm{d}y\right)^{r/p'} w^q(z)\,\mathrm{d}z. \end{aligned}$$

Notice that for $b(\xi_{k-1})\le y\le \rho^{-1}(\xi_k)$, by (1.5),

$$\int_{b^{-1}(y)}^{a^{-1}(\rho^{-1}(\xi_k))} w^q = \int_{b^{-1}(y)}^{\xi_k} w^q + \int_{\xi_k}^{a^{-1}(\rho^{-1}(\xi_k))} w^q = \int_{b^{-1}(y)}^{\xi_k} w^q + \int_{b^{-1}(\rho^{-1}(\xi_k))}^{\xi_k} w^q \le 2\int_{b^{-1}(y)}^{\xi_k} w^q;$$

analogously, for $\rho^{-1}(\xi_k)\le y\le a(\xi_{k+1})$

$$\int_{b^{-1}(\rho^{-1}(\xi_k))}^{a^{-1}(y)} w^q = \int_{b^{-1}(\rho^{-1}(\xi_k))}^{\xi_k} w^q + \int_{\xi_k}^{a^{-1}(y)} w^q = \int_{\xi_k}^{a^{-1}(\rho^{-1}(\xi_k))} w^q + \int_{\xi_k}^{a^{-1}(y)} w^q \le 2\int_{\xi_k}^{a^{-1}(y)} w^q.$$

Therefore, since $-r/p'+1=-r/q'$, where $r/q'<0$ for $0<q<1$, and in view of (1.5),

$$\|S^\rho_{k,i}\|^r \ll \int_{b^{-1}(s^k_i)}^{a^{-1}(\rho^{-1}(\xi_k))}\bigg(\int_z^{a^{-1}(\rho^{-1}(\xi_k))} w^q\bigg)^{-r/p'}\bigg(\int_{s^k_i}^{\min\{t^k_i,b(z)\}}\bigg[\int_{b^{-1}(y)}^{\xi_k} w^q\bigg]^{p'} v^{p'}(y)\mathrm{d}y\bigg)^{r/p'} w^q(z)\mathrm{d}z$$

$$\le \int_{b^{-1}(s^k_i)}^{a^{-1}(\rho^{-1}(\xi_k))}\bigg(\int_z^{a^{-1}(\rho^{-1}(\xi_k))} w^q\bigg)^{-r/p'}\bigg(\int_{s^k_i}^{\min\{t^k_i,b(z)\}}[W(y)]^{p'} v^{p'}(y)\,\mathrm{d}y\bigg)^{r/p'} w^q(z)\,\mathrm{d}z$$
$$\ll \bigg(\int_{s^k_i}^{t^k_i} W^{p'}v^{p'}\bigg)^{r/p'}\bigg(\int_{b^{-1}(s^k_i)}^{\xi_k} w^q\bigg)^{-r/q'}$$

and

$$\|T^\rho_{k,i}\|^r \ll \int_{b^{-1}(\rho^{-1}(\xi_k))}^{a^{-1}(t^k_i)}\bigg(\int_{b^{-1}(\rho^{-1}(\xi_k))}^{z} w^q\bigg)^{-r/p'}\bigg(\int_{\max\{s^k_i,a(z)\}}^{t^k_i}\bigg[\int_{\xi_k}^{a^{-1}(y)} w^q\bigg]^{p'} v^{p'}(y)\,\mathrm{d}y\bigg)^{r/p'} w^q(z)\,\mathrm{d}z$$
$$\ll \int_{b^{-1}(\rho^{-1}(\xi_k))}^{a^{-1}(t^k_i)}\bigg(\int_{b^{-1}(\rho^{-1}(\xi_k))}^{z} w^q\bigg)^{-r/p'}\bigg(\int_{\max\{s^k_i,a(z)\}}^{t^k_i}[W(y)]^{p'} v^{p'}(y)\,\mathrm{d}y\bigg)^{r/p'} w^q(z)\,\mathrm{d}z$$
$$\ll \bigg(\int_{s^k_i}^{t^k_i} W^{p'}v^{p'}\bigg)^{r/p'}\bigg(\int_{\xi_k}^{a^{-1}(t^k_i)} w^q\bigg)^{-r/q'}.$$

Thus, since $0<q<1$,

$$\|S^\rho_k f\|^q_q \le \sum_{i=1}^{i^-_k}\|S^\rho_{k,i}f\|^q_q \ll \sum_{i=1}^{i^-_k}\bigg[\bigg(\int_{s^k_i}^{t^k_i} W^{p'}v^{p'}\bigg)^{r/p'}\bigg(\int_{b^{-1}(s^k_i)}^{\xi_k} w^q\bigg)^{-r/q'}\bigg]^{q/r}\|f\chi_{\varrho^k_i}\|^q_p \tag{2.27}$$

and

$$\|T^\rho_k f\|^q_q \le \sum_{i=1}^{i^+_k}\|T^\rho_{k,i^-_k+i}f\|^q_q \ll \sum_{i=i^-_k+1}^{M_k}\bigg[\bigg(\int_{s^k_i}^{t^k_i} W^{p'}v^{p'}\bigg)^{r/p'}\bigg(\int_{\xi_k}^{a^{-1}(t^k_i)} w^q\bigg)^{-r/q'}\bigg]^{q/r}\|f\chi_{\varrho^k_i}\|^q_p. \tag{2.28}$$

Consider estimates (2.27) and (2.28). If $i^-_k=1$, then $b^{-1}(\rho^{-1}(\xi_k))\le\rho(a(\xi_k))$, and we have by (1.5):

$$\|S^\rho_k\|^r=\|S^\rho_{k,i^-_k}\|^r=\|S^\rho_{k,1}\|^r \ll \bigg(\int_{b(\xi_{k-1})}^{\rho^{-1}(\xi_k)} W^{p'}v^{p'}\bigg)^{r/p'}\bigg(\int_{\xi_{k-1}}^{\xi_k} w^q\bigg)^{-r/q'}$$
$$=2\bigg(\int_{b(\xi_{k-1})}^{\rho^{-1}(\xi_k)} W^{p'}v^{p'}\bigg)^{r/p'}\bigg(\int_{\xi_{k-1}}^{\xi_k} w^q\bigg)^{-r/p'}\int_{\rho(a(\xi_k))}^{\xi_k} w^q(t)\,\mathrm{d}t.$$

Since $a(t)\le a(\xi_k)=b(\xi_{k-1})$ for $t\le\xi_k$, and $\rho^{-1}(\xi_k)\le b(t)$ for $b^{-1}(\rho^{-1}(\xi_k))\le\rho(a(\xi_k))\le t$, then

$$\|S^\rho_k\|^r=\|S^\rho_{k,1}\|^r \ll \bigg(\int_{\xi_{k-1}}^{\xi_k} w^q\bigg)^{-r/p'}\int_{\rho(a(\xi_k))}^{\xi_k}\bigg(\int_{a(t)}^{b(t)} W^{p'}v^{p'}\bigg)^{r/p'} w^q(t)\,\mathrm{d}t.$$

Since $\xi_{k-1} \le b^{-1}(\rho^{-1}(t))$ for $t \ge \rho(b(\xi_{k-1}) = \rho(a(\xi_k))$, we obtain, in view of (1.5), that

$$\begin{aligned}\|S_k^\rho\|^r =& \|S_{k,1}^\rho\|^r \ll \int_{\rho(a(\xi_k))}^{\xi_k} \Big(\int_{a(t)}^{b(t)} W^{p'} v^{p'}\Big)^{r/p'} \Big(\int_{b^{-1}(\rho^{-1}(t))}^{t} w^q\Big)^{-r/p'} w^q(t)\,\mathrm{d}t\\ =& 2^{r/p'} \int_{\rho(a(\xi_k))}^{\xi_k} \Big(\int_{a(t)}^{b(t)} W^{p'} v^{p'}\Big)^{r/p'} \Big(\int_{b^{-1}(\rho^{-1}(t))}^{a^{-1}(\rho^{-1}(t))} w^q\Big)^{-r/p'} w^q(t)\,\mathrm{d}t.\end{aligned}$$

Analogously, if $i_k^+ = 1$, then

$$\|T_k^\rho\|^r = \|T_{k,1}^\rho\|^r \ll \int_{\xi_k}^{\rho(b(\xi_k))} \Big(\int_{a(t)}^{b(t)} W^{p'} v^{p'}\Big)^{r/p'} \Big(\int_{b^{-1}(\rho^{-1}(t))}^{a^{-1}(\rho^{-1}(t))} w^q\Big)^{-r/p'} w^q(t)\,\mathrm{d}t.$$

Let $i_k^- > 1$ and/or $i_k^+ > 1$. By the construction of the ϱ_i^k,

$$b^{-1}(t_i^k) = \rho(s_i^k) \quad (1 \le i \le i_k^-,\quad i_k^- > 1); \quad a^{-1}(s_i^k) = \rho(t_i^k) \quad (i_k^- + 1 \le i \le M_k,\quad i_k^+ > 1).$$

Then

$$\begin{aligned}&\int_{b^{-1}(s_i^k)}^{\xi_k} w^q \le \int_{b^{-1}(s_i^k)}^{a^{-1}(s_i^k)} w^q = 2\int_{\rho(s_i^k)}^{a^{-1}(s_i^k)} w^q \le 2\int_{b^{-1}(t_i^k)}^{a^{-1}(t_i^k)} w^q = 4\int_{b^{-1}(t_i^k)}^{\rho(t_i^k)} w^q = 4\int_{\rho(s_i^k)}^{\rho(t_i^k)} w^q,\\ &\int_{\xi_k}^{a^{-1}(t_i^k)} w^q \le \int_{b^{-1}(t_i^k)}^{a^{-1}(t_i^k)} w^q = 2\int_{b^{-1}(t_i^k)}^{\rho(t_i^k)} w^q \le 2\int_{b^{-1}(s_i^k)}^{a^{-1}(s_i^k)} w^q = 4\int_{\rho(s_i^k)}^{a^{-1}(s_i^k)} w^q = 4\int_{\rho(s_i^k)}^{\rho(t_i^k)} w^q;\end{aligned}$$

and for $\rho(s_i^k) \le t \le \rho(t_i^k)$

$$\begin{aligned}t_i^k \le b(t),\ a(t) \le a(\rho(t_i^k)) \le a(\xi_k) \le s_i^k \quad (1 \le i \le i_k^-,\quad i_k^- > 1);\\ a(t) \le s_i^k,\ t_i^k \le b(\xi_k) \le b(\rho(s_i^k)) \le b(t) \quad (i_k^- + 1 \le i \le M_k,\quad i_k^+ > 1).\end{aligned}$$

Moreover, if $\rho(s_i^k) \le t \le \rho(t_i^k)$,

$$\begin{aligned}b^{-1}(\rho^{-1}(t)) \ge b^{-1}(s_i^k),\ \rho(t_i^k) \le \xi_k \quad (1 \le i \le i_k^-,\quad i_k^- > 1),\\ a^{-1}(\rho^{-1}(t)) \le b^{-1}(t_i^k),\ \xi_k \ge \rho(s_i^k) \quad (i_k^- + 1 \le i \le M_k,\quad i_k^+ > 1).\end{aligned}$$

Thus, by taking into account (1.5), we obtain for the components in the right hand sides of (2.27) and (2.28) if $i_k^- > 1$ and $i_k^+ > 1$:

$$\begin{aligned}&\Big(\int_{s_i^k}^{t_i^k} W^{p'} v^{p'}\Big)^{r/p'} \Big(\int_{b^{-1}(s_i^k)}^{\xi_k} w^q\Big)^{-r/q'}\\ &\qquad \le 4\Big(\int_{s_i^k}^{t_i^k} W^{p'} v^{p'}\Big)^{r/p'} \Big(\int_{b^{-1}(s_i^k)}^{\xi_k} w^q\Big)^{-r/p'} \int_{\rho(s_i^k)}^{\rho(t_i^k)} w^q(t)\,\mathrm{d}t\\ &\qquad \le 4\int_{\rho(s_i^k)}^{\rho(t_i^k)} \Big(\int_{a(t)}^{b(t)} W^{p'} v^{p'}\Big)^{r/p'} \Big(\int_{b^{-1}(\rho^{-1}(t))}^{t} w^q\Big)^{-r/p'} w^q(t)\,\mathrm{d}t\\ &\qquad = 2^{2+r/p'} \int_{\rho(s_i^k)}^{\rho(t_i^k)} \Big(\int_{a(t)}^{b(t)} W^{p'} v^{p'}\Big)^{r/p'} \Big(\int_{b^{-1}(\rho^{-1}(t))}^{a^{-1}(\rho^{-1}(t))} w^q\Big)^{-r/p'} w^q(t)\,\mathrm{d}t,\end{aligned}$$

$$\begin{aligned}
&\bigg(\int_{s_i^k}^{t_i^k} W^{p'}v^{p'}\bigg)^{r/p'}\bigg(\int_{\xi_k}^{a^{-1}(t_i^k)} w^q\bigg)^{-r/q'}\\
&\qquad\le 4\bigg(\int_{s_i^k}^{t_i^k} W^{p'}v^{p'}\bigg)^{r/p'}\bigg(\int_{\xi_k}^{a^{-1}(t_i^k)} w^q\bigg)^{-r/p'}\int_{\rho(s_i^k)}^{\rho(t_i^k)} w^q(t)\,\mathrm{d}t\\
&\qquad\le 4\int_{\rho(s_i^k)}^{\rho(t_i^k)}\bigg(\int_{a(t)}^{b(t)} W^{p'}v^{p'}\bigg)^{r/p'}\bigg(\int_{t}^{a^{-1}(\rho^{-1}(t))} w^q\bigg)^{-r/p'} w^q(t)\,\mathrm{d}t\\
&\qquad= 2^{2+r/p'}\int_{\rho(s_i^k)}^{\rho(t_i^k)}\bigg(\int_{a(t)}^{b(t)} W^{p'}v^{p'}\bigg)^{r/p'}\bigg(\int_{b^{-1}(\rho^{-1}(t))}^{a^{-1}(\rho^{-1}(t))} w^q\bigg)^{-r/p'} w^q(t)\,\mathrm{d}t.
\end{aligned}$$

This yields, in particular, (2.27), by Hölder's inequality with powers r/q and p/q,

$$\begin{aligned}
\|S_k^\rho f\|_q^q &\ll \sum_{i=1}^{i_k^-}\bigg[\int_{\rho(s_i^k)}^{\rho(t_i^k)}\bigg(\int_{a(t)}^{b(t)} W^{p'}v^{p'}\bigg)^{r/p'}\bigg(\int_{b^{-1}(\rho^{-1}(t))}^{a^{-1}(\rho^{-1}(t))} w^q\bigg)^{-r/p'} w^q(t)\,\mathrm{d}t\bigg]^{q/r}\|f\chi_{\varrho_i^k}\|_p^q\\
&\le \bigg(\sum_{i=1}^{i_k^-}\int_{\rho(s_i^k)}^{\rho(t_i^k)}\bigg(\int_{a(t)}^{b(t)} W^{p'}v^{p'}\bigg)^{r/p'}\bigg(\int_{b^{-1}(\rho^{-1}(t))}^{a^{-1}(\rho^{-1}(t))} w^q\bigg)^{-r/p'} w^q(t)\,\mathrm{d}t\bigg)^{q/r}\bigg(\sum_{i=1}^{i_k^-}\|f\chi_{\varrho_i^k}\|_p^p\bigg)^{q/p}\\
&\ll \bigg(\int_{\rho(a(\xi_k))}^{\xi_k}\bigg(\int_{a(t)}^{b(t)} W^{p'}v^{p'}\bigg)^{r/p'}\bigg(\int_{b^{-1}(\rho^{-1}(t))}^{a^{-1}(\rho^{-1}(t))} w^q\bigg)^{-r/p'} w^q(t)\,\mathrm{d}t\bigg)^{q/r}\|f\chi_{[a(\xi_k),\rho^{-1}(\xi_k)]}\|_p^q
\end{aligned}$$

since no point of $[\rho(a(\xi_k)),\xi_k]$ lies in more than two of the $[\rho(s_i^k),\rho(t_i^k)]$, and no point of $[a(\xi_k),\rho^{-1}(\xi_k)]$ lies in more than two of the $[s_i^k,t_i^k]$, where

$$[\rho(a(\xi_k)),\xi_k]=\bigcup_{i=1}^{i_k^-}[\rho(s_i^k),\rho(t_i^k)]\quad\text{and}\quad[a(\xi_k),\rho^{-1}(\xi_k)]=\bigcup_{i=1}^{i_k^-}[s_i^k,t_i^k].$$

Analogously, we obtain the following estimate for $\|T_k^\rho f\|$:

$$\|T_k^\rho f\|_q^q \ll \bigg(\int_{\xi_k}^{\rho(b(\xi_k))}\bigg(\int_{a(t)}^{b(t)} W^{p'}v^{p'}\bigg)^{r/p'}\bigg(\int_{b^{-1}(\rho^{-1}(t))}^{a^{-1}(\rho^{-1}(t))} w^q\bigg)^{-r/p'} w^q(t)\,\mathrm{d}t\bigg)^{q/r}\|f\chi_{[\rho^{-1}(\xi_k),b(\xi_k)]}\|_p^q.$$

Now, we have, by Hölder's inequality with the powers r/q and p/q, by virtue of [19, Lemma 3.1], that

$$\begin{aligned}
\|\mathcal{H}f\|_q^q &\ll \sum_{k\in\mathbb{Z}}\|S_k^\rho f\|_q^q+\|T_k^\rho f\|_q^q\\
&\ll \sum_{k\in\mathbb{Z}}\bigg(\int_{\rho(a(\xi_k))}^{\xi_k}\bigg(\int_{a(t)}^{b(t)} W^{p'}v^{p'}\bigg)^{r/p'}\big[W(\rho^{-1}(t))\big]^{-r/p'} w^q(t)\,\mathrm{d}t\bigg)^{q/r}\|f\chi_{[a(\xi_k),\rho^{-1}(\xi_k)]}\|_p^q\\
&\quad+\bigg(\int_{\xi_k}^{\rho(b(\xi_k))}\bigg(\int_{a(t)}^{b(t)} W^{p'}v^{p'}\bigg)^{r/p'}\big[W(\rho^{-1}(t))\big]^{-r/p'} w^q(t)\,\mathrm{d}t\bigg)^{q/r}\|f\chi_{[\rho^{-1}(\xi_k),b(\xi_k)]}\|_p^q\\
&\ll \sum_{k\in\mathbb{Z}}\bigg(\int_{\rho(a(\xi_k))}^{\rho(b(\xi_k))}\bigg(\int_{a(t)}^{b(t)} W^{p'}v^{p'}\bigg)^{r/p'}\big[W(\rho^{-1}(t))\big]^{-r/p'} w^q(t)\,\mathrm{d}t\bigg)^{q/r}\|f\chi_{[a(\xi_k),b(\xi_k)]}\|_p^q\\
&\le \bigg(\sum_{k\in\mathbb{Z}}\int_{\rho(a(\xi_k))}^{\rho(b(\xi_k))}\bigg(\int_{a(t)}^{b(t)} W^{p'}v^{p'}\bigg)^{r/p'}\big[W(\rho^{-1}(t))\big]^{-r/p'} w^q(t)\,\mathrm{d}t\bigg)^{q/r}
\end{aligned}$$

$$\times\biggl(\sum_{k\in\mathbb{Z}}\|f\chi_{[a(\xi_k),b(\xi_k)]}\|_p^p\biggr)^{q/p}\ll\mathbb{B}_\rho^q\|f\|_p^q,$$

and the estimate $\|\mathcal{H}\|\ll\mathbb{B}_\rho$ is proven.

To establish the estimate $\|\mathcal{H}\|\ll(\mathbb{B}_\rho)^*$ in the case $0<q<1$ we cover $S_k^\rho+T_k^\rho$, $k\in\mathbb{Z}$, as follows

$$\begin{aligned}\|S_k^\rho f+T_k^\rho f\|_q^q&\le\sum_{i=1}^{M_k-1}\|\mathcal{H}_{k,i}^\rho f+\mathcal{H}_{k,i+1}^\rho f\|_q^q\\&\le\sum_{i=1}^{M_k-1}\biggl[\biggl(\int_{s_i^k}^{t_{i+1}^k}v^{p'}\biggr)^{r/p'}\biggl(\int_{b^{-1}(s_i^k)}^{a^{-1}(t_{i+1}^k)}w^q\biggr)^{r/q}\biggr]^{q/r}\|f\chi_{\rho_i^k\cup\rho_{i+1}^k}\|_p^q.\end{aligned}\tag{2.29}$$

Here $\mathcal{H}_{k,i}^\rho$ stands for $S_{k,i}^\rho$ if $1\le i\le i_k^-$ and for $T_{k,i}^\rho$ in the case $i_k^-+1\le i\le M_k$.

For any $i\in\{1,\dots,M_k-1\}$ we compare $V_i^k:=\int_{\varrho_i^k}v^{p'}$ and V_{i+1}^k and denote

$$\tilde\varrho_i^k:=\begin{cases}\varrho_i^k, & V_i^k\ge V_{i+1}^k,\\ \varrho_{i+1}^k, & V_i^k<V_{i+1}^k.\end{cases}$$

Notice that $W(s_i^k)\approx W(t_i^k)\approx W(s_{i+1}^k)\approx W(t_{i+1}^k)\approx W(x)$, $x\in\varrho_i^k\cup\varrho_{i+1}^k$, by the construction of $\{\varrho_i^k\}$.

We write, in view of $r/q=r-r/q'=rp'/q'+p'-r/q'$,

$$\begin{aligned}\|\mathcal{H}_{k,i}^\rho+\mathcal{H}_{k,i+1}^\rho\|^r:=&\biggl(\int_{s_i^k}^{t_{i+1}^k}v^{p'}\biggr)^{r/p'}\biggl(\int_{b^{-1}(s_i^k)}^{a^{-1}(t_{i+1}^k)}w^q\biggr)^{r/q}\\\approx&\bigl[W(s_i^k)\bigr]^{r/q}\biggl(\int_{s_i^k}^{t_{i+1}^k}v^{p'}\biggr)^{r/q'}\int_{s_i^k}^{t_{i+1}^k}v^{p'}(t)\,\mathrm{d}t\\\le&2\biggl(\int_{s_i^k}^{t_{i+1}^k}W^{p'}v^{p'}\biggr)^{r/q'}\int_{\tilde\varrho_i^k}[W(t)]^{p'-r/q'}v^{p'}(t)\,\mathrm{d}t=:\mathbb{B}_i^k.\end{aligned}$$

If $\tilde\varrho_i^k=\varrho_i^k$ then, since $q'<0$,

$$\begin{aligned}\mathbb{B}_i^k=&\biggl(\int_{s_i^k}^{t_{i+1}^k}W^{p'}v^{p'}\biggr)^{r/q'}\int_{s_i^k}^{t_i^k}[W(t)]^{p'-r/q'}v^{p'}(t)\,\mathrm{d}t\\\le&\int_{s_i^k}^{t_i^k}\biggl(\int_t^{t_{i+1}^k}W^{p'}v^{p'}\biggr)^{r/q'}[W(t)]^{p'-r/q'}v^{p'}(t)\,\mathrm{d}t\\\le&\int_{s_i^k}^{t_i^k}\biggl(\int_t^{\min\{\rho^{-1}(a^{-1}(t)),b(\rho(t))\}}W^{p'}v^{p'}\biggr)^{r/q'}[W(t)]^{p'-r/q'}v^{p'}(t)\,\mathrm{d}t\\\le&\int_{s_i^k}^{t_i^k}\biggl(\int_t^{\rho^{-1}(a^{-1}(t))}W^{p'}v^{p'}\biggr)^{r/q'}[W(t)]^{p'-r/q'}v^{p'}(t)\,\mathrm{d}t\\&+\int_{s_i^k}^{t_i^k}\biggl(\int_t^{b(\rho(t))}W^{p'}v^{p'}\biggr)^{r/q'}[W(t)]^{p'-r/q'}v^{p'}(t)\,\mathrm{d}t.\end{aligned}$$

Analogously, if $\tilde{\varrho}_i^k = \varrho_{i+1}^k$ then

$$\begin{aligned}\mathbb{B}_i^k =&\Big(\int_{s_i^k}^{t_{i+1}^k} W^{p'}v^{p'}\Big)^{r/q'}\int_{s_{i+1}^k}^{t_{i+1}^k}[W(t)]^{p'-r/q'}v^{p'}(t)\,\mathrm{d}t\\ \le&\int_{s_{i+1}^k}^{t_{i+1}^k}\Big(\int_{s_i^k}^{t} W^{p'}v^{p'}\Big)^{r/q'}[W(t)]^{p'-r/q'}v^{p'}(t)\,\mathrm{d}t\\ \le&\int_{s_{i+1}^k}^{t_{i+1}^k}\Big(\int_{\max\{\rho^{-1}(b^{-1}(t)),a(\rho(t))\}}^{t} W^{p'}v^{p'}\Big)^{r/q'}[W(t)]^{p'-r/q'}v^{p'}(t)\,\mathrm{d}t\\ \le&\int_{s_{i+1}^k}^{t_{i+1}^k}\Big(\int_{\rho^{-1}(b^{-1}(t))}^{t} W^{p'}v^{p'}\Big)^{r/q'}[W(t)]^{p'-r/q'}v^{p'}(t)\,\mathrm{d}t\\ &+\int_{s_{i+1}^k}^{t_{i+1}^k}\Big(\int_{a(\rho(t))}^{t} W^{p'}v^{p'}\Big)^{r/q'}[W(t)]^{p'-r/q'}v^{p'}(t)\,\mathrm{d}t.\end{aligned}$$

We, finally, obtain from (2.29), by Hölder's inequality with the powers r/q and p/q:

$$\begin{aligned}\|\mathcal{H}f\|_q^q \ll&\sum_{k\in\mathbb{Z}}\|S_k^\rho f+T_k^\rho f\|_q^q\\ \le&\sum_{k\in\mathbb{Z}}\sum_{i=1}^{M_k-1}\|\mathcal{H}_{k,i}^\rho f+\mathcal{H}_{k,i+1}^\rho f\|_q^q \ll \sum_{k\in\mathbb{Z}}\sum_{i=1}^{M_k-1}\big[\mathbb{B}_i^k\big]^{q/r}\|f\chi_{\varrho_i^k\cup\varrho_{i+1}^k}\|_p^q\\ \le&\Big(\sum_{k\in\mathbb{Z}}\sum_{i=1}^{M_k-1}\mathbb{B}_i^k\Big)^{q/r}\Big(\sum_{k\in\mathbb{Z}}\sum_{i=1}^{M_k-1}\|f\chi_{\varrho_i^k\cup\varrho_{i+1}^k}\|_p^p\Big)^{q/p}\\ \le&2^{q/p}\Big(\sum_{k\in\mathbb{Z}}\sum_{i=1}^{M_k-1}\mathbb{B}_i^k\Big)^{q/r}\Big(\sum_{k\in\mathbb{Z}}\|f\chi_{[a(\xi_k),b(\xi_k)]}\|_p^p\Big)^{q/p}\\ \le&2^{q/p+q/r}\big[(\mathbb{B}_\rho)^*\big]^q\Big(\sum_{k\in\mathbb{Z}}\|f\chi_{[a(\xi_k),b(\xi_k)]}\|_p^p\Big)^{q/p}\le 2\big[(\mathbb{B}_\rho)^*\big]^q\|f\|_p^q.\end{aligned}$$

For proving the lower estimates $\|\mathcal{H}\|\gg\mathbb{B}_\rho$, $\|\mathcal{H}\|\gg(\mathbb{B}_\rho)^*$ in the case $0<q<1$ we assume, first, that $\rho(y)=y$. The claims will be established under this condition on ρ if we show that

$$\|\mathcal{H}\|\gg\mathbb{B}_{q<1}^\pm,\qquad \|\mathcal{H}\|\gg(\mathbb{B}_{q>1}^\pm)^*,\qquad \|\mathcal{H}\|\gg(\mathbb{B}_{q<1}^\pm)^*,\tag{2.30}$$

where

$$\mathbb{B}_{q<1}^\pm:=\Big(\int_0^\infty\Big[\int_{\delta_\rho^-(t)}W^{p'}v^{p'}\Big]^{r/p'}W(\rho^{-1}(t))^{-r/p'}w^q(t)\,\mathrm{d}t\Big)^{1/r}.$$

Let us prove inequalities (2.30) with $\mathbb{B}_{q<1}^-$, $(\mathbb{B}_{q>1}^-)^*$ and $(\mathbb{B}_{q<1}^-)^*$. The arguments for $\mathbb{B}_{q<1}^+$, $(\mathbb{B}_{q>1}^+)^*$ and $(\mathbb{B}_{q<1}^+)^*$ are similar.

Splitting $(0,\infty)$ by points (2.19) (with $\tilde{a}=a$) we form the sequence of intervals $\delta_\rho^-(\eta_k)$, taking into account that $\rho(y)=y$. After this, we cover each $\delta_\rho^-(\eta_{k+1})$ by $[m_{n-1}^k,m_n^k]$, $n=1,\dots,n_k^-$, constructed as follows. For fixed $k\in\mathbb{Z}$ we denote $[\mathcal{N}_k^-]$ the integer part of the number

$$\mathcal{N}_k^-:=\log_2\frac{\int_{b^{-1}(\eta_k)}^{\eta_{k+1}}w^q}{\int_{b^{-1}(\eta_{k+1})}^{\eta_{k+1}}w^q}.$$

Then we put $m_0=\eta_k$, $m_{n_k^-}=\eta_{k+1}$ and choose m_n, $n=0,\dots,n_k^-$, as follows:

1. if $[\mathcal{N}_k^-] \le 1$ then $n_k^- = 1$;

2. if $\mathcal{N}_k^- > 1$ then $n_k^- = \begin{cases} [\mathcal{N}_k^-], & \mathcal{N}_k^- = [\mathcal{N}_k^-] \\ [\mathcal{N}_k^-] + 1, & \mathcal{N}_k^- > [\mathcal{N}_k^-] \end{cases}$, and we choose m_n for $1 \le n \le [\mathcal{N}_k^-]$ so that
$$\int_{b^{-1}(m_{n-1})}^{\eta_{k+1}} w^q = 2\int_{b^{-1}(m_n)}^{\eta_{k+1}} w^q. \tag{2.31}$$

Fixed $k \in \mathbb{Z}$ we denote $\gamma_n^k := \int_{b^{-1}(m_n)}^{\eta_{k+1}} w^q$ the elements of the strongly decreasing sequence γ_n^k with $0 \le n \le [\mathcal{N}_k^-]$ (see [5, Definition 2.2(a)] for details), because $\gamma_{n-1}^k \ge 2\gamma_n^k$ by the construction. Moreover,
$$\int_{b^{-1}(m_{n-1}^k)}^{\eta_{k+1}} w^q \approx \int_{b^{-1}(m_n^k)}^{\eta_{k+1}} w^q \approx W(y), \qquad y \in [m_{n-1}^k, m_n^k]. \tag{2.32}$$

Let
$$l_{k,n} := \left(\int_{b^{-1}(m_{n-1}^k)}^{\eta_{k+1}} w^q(x)\,\mathrm{d}x\right)^{-r/(pq')+p'-1} \left(\int_{m_{n-1}^k}^{m_n^k} W^{p'}(y) v^{p'}(y)\,\mathrm{d}y\right)^{r/(pq')}$$
and define
$$f_a(t) := \sum_{k=-K}^{K} \sum_{n=1}^{n_k^-} v(t)^{p'-1} \chi_{[m_{n-1}^k, m_n^k]}(t)\ l_{k,n} \qquad K \in \mathbb{N}.$$

Since (m_{n-1}^k, m_n^k) are mutually disjoint for all $k \in \mathbb{Z}$ and $n = 1, \ldots, n_k^-$, and since $r/p' = 1 + r/q'$,
$$\begin{aligned} \|f_a\|_p^p &= \sum_{k\in\mathbb{Z}} \sum_{n=1}^{n_k^-} \int_{m_{n-1}^k}^{m_n^k} f_a^p(y)\,\mathrm{d}y = \sum_{k=-K}^{K} \sum_{n=1}^{n_k^-} l_{k,n}^p \int_{m_{n-1}^k}^{m_n^k} v^{p'}(y)\,\mathrm{d}y \\ &\approx \sum_{k=-K}^{K} \sum_{n=1}^{n_k^-} \left(\int_{m_{n-1}^k}^{m_n^k} W(y)^{p'} v^{p'}(y)\,\mathrm{d}y\right)^{r/p'} \left(\int_{b^{-1}(m_{n-1}^k)}^{\eta_{k+1}} w^q\right)^{-r/q'}. \end{aligned} \tag{2.33}$$

Since $a(m_n^k) \le m_0^k \le m_{n-1}^k$ then $[m_{n-1}^k, m_n^k] \subset [a(x), b(x)]$ for $x \in [b^{-1}(m_n^k), m_n^k]$, $n = 1, \ldots, n_k^-$. Moreover,
$$b(m_{n-1}^k) \ge m_n^k, \tag{2.34}$$
where $b^{-1}(m_1^k) = m_0^k$ if $n_k^- > 1$. If we assume the contrary to (2.34) for some $1 \le n \le n_k^-$ then, by the construction,
$$\int_{b^{-1}(m_{n-1}^k)}^{\eta_{k+1}} w^q \le 2\int_{b^{-1}(m_n^k)}^{\eta_{k+1}} w^q < 2\int_{m_{n-1}^k}^{\eta_{k+1}} w^q,$$
that contradicts (1.5).

Now we cover $[\eta_k, \eta_{k+1}]$ by $\{\tau_n^k\}_{n=1}^{n_k^-}$ as follows. If $n_k^- = 1$ then $\tau_1^k = [\eta_k, \eta_{k+1}]$. In the case $n_k^- > 1$ we put $\tau_n^k = [b^{-1}(m_n^k), m_n^k]$ for $n = n_k^- - 1$ and $n = n_k^-$, and let $\tau_n^k = [b^{-1}(m_n^k), b^{-1}(m_{n+1}^k)]$

for $1 \le n < n_k^- - 1$ if $n_k^- > 2$. Since $r/(pq') + 1 = r/(p'q)$ and $-r/p' + 1 = -r/q'$, then

$$
\begin{aligned}
\|(\mathcal{H} f_a) w\|_q^q &\ge \frac{1}{2} \sum_{k=-K}^{K} \sum_{n=1}^{n_k^-} \int_{\tau_n^k} \left(\int_{a(x)}^{b(x)} f_a(y) v(y)\,\mathrm{d}y \right)^q w^q(x)\,\mathrm{d}x \\
&\ge \sum_{k=-K}^{K} \sum_{n=1}^{n_k^-} \int_{\tau_n^k} \left(\int_{a(x)}^{b(x)} l_{k,n}\, v^{p'}(y) \chi_{[m_{n-1}^k, m_n^k]}(y)\,\mathrm{d}y \right)^q w^q(x)\,\mathrm{d}x \\
&\gg \sum_{k=-K}^{K} \sum_{n=1}^{n_k^-} \left(\int_{m_{n-1}^k}^{m_n^k} W^{p'} v^{p'} \right)^q \left(\int_{b^{-1}(m_{n-1}^k)}^{\eta_{k+1}} w^q \right)^{-rq/(pq')-q+1} \left(\int_{m_{n-1}^k}^{m_n^k} W^{p'} v^{p'} \right)^{rq/(pq')} \\
&\approx \sum_{k=-K}^{K} \sum_{n=1}^{n_k^-} \left(\int_{m_{n-1}^k}^{m_n^k} [W(y)]^{p'} v^{p'}(y)\,\mathrm{d}y \right)^{\frac{r}{p'}} \left(\int_{b^{-1}(m_{n-1}^k)}^{\eta_{k+1}} w^q \right)^{-\frac{r}{q'}},
\end{aligned}
\tag{2.35}
$$

because no point of $[\eta_k, \eta_{k+1}]$ lies in more than two of the τ_n^k. In combination with (2.33) and under assumption $\|\mathcal{H}\| < \infty$ this yields the estimate

$$
\|\mathcal{H}\|^r \gg \lim_{K \to \infty} \sum_{k=-K}^{K} \sum_{n=1}^{n_k^-} \left(\int_{m_{n-1}^k}^{m_n^k} W^{p'} v^{p'} \right)^{r/p'} \left(\int_{b^{-1}(m_{n-1}^k)}^{\eta_{k+1}} w^q \right)^{-r/q'} =: \sum_{k \in \mathbb{Z}} \sum_{n=1}^{n_k^-} \mu_{k,n}, \tag{2.36}
$$

which is true for $q > 1$ as well.

In order to establish the estimate $\|\mathcal{H}\| \gg \mathbb{B}_{q<1}^-$ we write, using (2.32), taking into account $-r/q' > 0$, and denoting $m_{-1}^k = m_0^{k-1}$,

$$
\begin{aligned}
[\mathbb{B}_{q<1}^-]^r &\le \sum_{k \in \mathbb{Z}} \sum_{n=1}^{n_k^-} \int_{m_{n-1}^k}^{m_n^k} \left[\int_{\delta_\rho^-(t)} W^{p'} v^{p'} \right]^{r/p'} W(t)^{-r/p'} w^q(t)\,\mathrm{d}t \\
&\ll \sum_{k \in \mathbb{Z}} \sum_{n=1}^{n_k^-} \left(\int_{a(m_{n-1}^k)}^{m_n^k} W^{p'} v^{p'} \right)^{r/p'} \left(\int_{b^{-1}(m_n^k)}^{\eta_{k+1}} w^q \right)^{-r/q'} \\
&\le \sum_{k \in \mathbb{Z}} \sum_{n=1}^{n_k^-} \left(\int_{m_0^{k-1}}^{m_n^k} W^{p'} v^{p'} \right)^{r/p'} \left(\int_{b^{-1}(m_n^k)}^{\eta_{k+1}} w^q \right)^{-r/q'}
\end{aligned}
$$

$$
\begin{aligned}
&\le \sum_{k \in \mathbb{Z}} \sum_{n=0}^{n_k^-} \left(\int_{m_0^{k-1}}^{m_n^k} W^{p'} v^{p'} \right)^{r/p'} \left(\int_{b^{-1}(m_n^k)}^{\eta_{k+1}} w^q \right)^{-r/q'} \\
&\le \sum_{k \in \mathbb{Z}} \left[\sum_{n=0}^{n_k^-} \left(\sum_{l=0}^{n} \int_{m_{l-1}^k}^{m_l^k} W^{p'} v^{p'} \right)^{r/p'} \left(\int_{b^{-1}(m_n^k)}^{\eta_{k+1}} w^q \right)^{-r/q'} \right].
\end{aligned}
\tag{2.37}
$$

Since $r/p' < 1$ and for each $k \in \mathbb{Z}$ the γ_n^k form strongly decreasing sequence, we obtain by [5, Proposition 2.1(b)]:

$$\sum_{n=0}^{n_k^-}\Big(\sum_{l=0}^{n}\int_{m_{l-1}^k}^{m_l^k} W^{p'}v^{p'}\Big)^{r/p'}\Big(\int_{b^{-1}(m_n^k)}^{\eta_{k+1}} w^q\Big)^{-r/q'}$$
$$\ll \sum_{n=0}^{n_k^-}\Big(\int_{m_{n-1}^k}^{m_n^k} W^{p'}v^{p'}\Big)^{r/p'}\Big(\int_{b^{-1}(m_n^k)}^{\eta_{k+1}} w^q\Big)^{-r/q'}$$
$$\le 2\sum_{n=1}^{n_{k-1}^-}\Big(\int_{m_{n-1}^{k-1}}^{m_n^{k-1}} W^{p'}v^{p'}\Big)^{r/p'}\Big(\int_{b^{-1}(m_n^{k-1})}^{\eta_k} w^q\Big)^{-r/q'}$$
$$+\sum_{n=1}^{n_k^-}\Big(\int_{m_{n-1}^k}^{m_n^k} W^{p'}v^{p'}\Big)^{r/p'}\Big(\int_{b^{-1}(m_n^k)}^{\eta_{k+1}} w^q\Big)^{-r/q'}.$$

This yields

$$[\mathbb{B}_{q<1}^-]^r \ll \sum_{k\in\mathbb{Z}}\Big[\sum_{n=1}^{n_{k-1}^-}\Big(\int_{m_{n-1}^{k-1}}^{m_n^{k-1}} W^{p'}v^{p'}\Big)^{r/p'}\Big(\int_{b^{-1}(m_{n-1}^{k-1})}^{\eta_k} w^q\Big)^{-r/q'}$$
$$+\sum_{n=1}^{n_k^-}\Big(\int_{m_{n-1}^k}^{m_n^k} W^{p'}v^{p'}\Big)^{r/p'}\Big(\int_{b^{-1}(m_{n-1}^k)}^{\eta_{k+1}} w^q\Big)^{-r/q'}\Big]$$
$$\le 2\sum_{k\in\mathbb{Z}}\sum_{n=1}^{n_k^-}\Big(\int_{m_{n-1}^k}^{m_n^k} W^{p'}v^{p'}\Big)^{r/p'}\Big(\int_{b^{-1}(m_{n-1}^k)}^{\eta_{k+1}} w^q\Big)^{-r/q'} = \sum_{k\in\mathbb{Z}}\sum_{n=1}^{n_k^-}\mu_{k,n}. \quad (2.38)$$

In combination with (2.36) this implies that$\|\mathcal{H}\| \gg \mathbb{B}_{q<1}^-$.

To establish that $\|\mathcal{H}\| \gg (\mathbb{B}_{q>1}^-)^*$ and $\|\mathcal{H}\| \gg (\mathbb{B}_{q<1}^-)^*$ notice that by the construction (see e.g. (2.34)) and in view of $r/p'-1=r/q'<0$,

$$\frac{r}{p'}\cdot\mu_{k,n} = \int_{m_{n-1}^k}^{m_n^k}\Big(\int_t^{m_n^k} W(y)^{p'}\,v^{p'}(y)\,\mathrm{d}y\Big)^{r/q'} W(t)^{p'}\,v^{p'}(t)\,\mathrm{d}t\Big(\int_{b^{-1}(m_{n-1}^k)}^{\eta_{k+1}} w^q\Big)^{-r/q'}$$
$$\ge \int_{m_{n-1}^k}^{m_n^k}\Big(\int_t^{a^{-1}(t)} W(y)^{p'}\,v^{p'}(y)\,\mathrm{d}y\Big)^{r/q'}\Big(\int_{b^{-1}(m_{n-1}^k)}^{\eta_{k+1}} w^q\Big)^{-r/q'} W(t)^{p'}\,v^{p'}(t)\,\mathrm{d}t$$
$$\approx \int_{m_{n-1}^k}^{m_n^k}\Big(\int_t^{a^{-1}(t)} W(y)^{p'}\,v^{p'}(y)\,\mathrm{d}y\Big)^{r/q'} W(t)^{p'-r/q'}\,v^{p'}(t)\,\mathrm{d}t.$$

For the same reasons,

$$\frac{r}{p'}\cdot\mu_{k,n} = \int_{m_{n-1}^k}^{m_n^k}\Big(\int_{m_{n-1}^k}^{t} W(y)^{p'}\,v^{p'}(y)\,\mathrm{d}y\Big)^{r/q'} W(t)^{p'}\,v^{p'}(t)\,\mathrm{d}t\Big(\int_{b^{-1}(m_{n-1}^k)}^{\eta_{k+1}} w^q\Big)^{-r/q'}$$
$$\ge \int_{m_{n-1}^k}^{m_n^k}\Big(\int_{a(t)}^{t} W(y)^{p'}\,v^{p'}(y)\,\mathrm{d}y\Big)^{r/q'}\Big(\int_{b^{-1}(m_{n-1}^k)}^{\eta_{k+1}} w^q\Big)^{-r/q'} W(t)^{p'}\,v^{p'}(t)\,\mathrm{d}t$$
$$\approx \int_{m_{n-1}^k}^{m_n^k}\Big(\int_{a(t)}^{t} W(y)^{p'}\,v^{p'}(y)\,\mathrm{d}y\Big)^{r/q'} W(t)^{p'-r/q'}\,v^{p'}(t)\,\mathrm{d}t.$$

Combination of these relations with (2.36) gives the required inequalities $\|\mathcal{H}\| \gg (\mathbb{B}^-_{q>1})^*$ and $\|\mathcal{H}\| \gg (\mathbb{B}^-_{q<1})^*$.

It remains to confirm that $\mathbb{B}_{q<1} \ll \mathbb{B}_{q>1}$ for $q < 1$ and $\mathbb{B}_{q>1} \ll \mathbb{B}_{q<1}$ if $q > 1$. Since the proofs of these inequalities are similar to each other, we shall establish the one for $q < 1$ only.

Observe that if $n_k^- > 1$ then, by (2.31) and in view of (1.5),

$$\int_{b^{-1}(m^k_{n-1})}^{\eta_{k+1}} w^q = 2\int_{b^{-1}(m^k_{n-1})}^{b^{-1}(m^k_n)} w^q = \int_{b^{-1}(m^k_n)}^{\eta_{k+1}} w^q, \qquad 1 \le n \le n_k^- - 1,$$

where $m_1^k = b(m_0^k)$. Thus, we obtain if $n_k^- > 3$, taking into account (2.34),

$$\begin{aligned}\sum_{n=1}^{n_k^-} \mu_{k,n} &\ll \sum_{n=1}^{n_k^- -2} \left(\int_{m^k_{n-1}}^{\eta_{k+1}} W^{p'} v^{p'}\right)^{r/p'} \int_{b^{-1}(m^k_n)}^{b^{-1}(m^k_{n+1})} w^q \left(\int_{b^{-1}(m^k_{n-1})}^{\eta_{k+1}} w^q\right)^{-r/p'}\\
&\le \sum_{n=1}^{n_k^- -2} \int_{b^{-1}(m^k_n)}^{b^{-1}(m^k_{n+1})} \left(\int_{b^{-1}(t)}^{a^{-1}(t)} W^{p'} v^{p'}\right)^{r/p'} w^q(t) \left(\int_{b^{-1}(m^k_{n-1})}^{\eta_{k+1}} w^q\right)^{-r/p'} \mathrm{d}t\\
&\approx \sum_{n=1}^{n_k^- -2} \int_{b^{-1}(m^k_n)}^{b^{-1}(m^k_{n+1})} \left(\int_{b^{-1}(t)}^{a^{-1}(t)} W^{p'} v^{p'}\right)^{r/p'} W(t)^{-r/p'} w^q(t)\,\mathrm{d}t\\
&\le \int_{\eta_k}^{\eta_{k+1}} \left(\int_{b^{-1}(t)}^{a^{-1}(t)} W^{p'} v^{p'}\right)^{r/p'} W(t)^{-r/p'} w^q(t)\,\mathrm{d}t.\end{aligned}$$

If $n_k^- \le 3$ we have

$$\begin{aligned}\sum_{n=1}^{n_k^-} \mu_{k,n} &\ll \left(\int_{\eta_k}^{\eta_{k+1}} W^{p'} v^{p'}\right)^{r/p'} \int_{\eta_k}^{\min\{b(\eta_k),\eta_{k+1}\}} W(t)^{-r/p'} w^q(t)\,\mathrm{d}t\\
&\le \int_{\eta_k}^{\min\{b(\eta_k),\eta_{k+1}\}} \left(\int_{b^{-1}(t)}^{a^{-1}(t)} W^{p'} v^{p'}\right)^{r/p'} W(t)^{-r/p'} w^q(t)\,\mathrm{d}t\\
&\le \int_{\eta_k}^{\eta_{k+1}} \left(\int_{b^{-1}(t)}^{a^{-1}(t)} W^{p'} v^{p'}\right)^{r/p'} W(t)^{-r/p'} w^q(t)\,\mathrm{d}t.\end{aligned}$$

Together with (2.38) these lead to

$$\mathbb{B}^-_{q<1} \ll \mathbb{B}_{q>1}. \tag{2.39}$$

Analogously, but by starting with establishing the estimate for $\mathbb{B}^+_{q<1}$ similar to that in (2.38), we can prove that $\mathbb{B}^+_{q<1} \ll \mathbb{B}_{q>1}$. Together with (2.39), this gives $\mathbb{B}_{q<1} \ll \mathbb{B}_{q>1}$. □

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