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CHARACTERISTIC DETERMINANT OF A BOUNDARY VALUE PROBLEM, WHICH DOES NOT HAVE THE BASIS PROPERTY

M.A. Sadybekov, N.S. Imanbaev

Communicated by Massimo Lanza de Cristoforis

Dedicated to the 70^{th} birthday of Professor Ryskul Oinarov

Key words: ordinary differential operator, boundary value problem, eigenvalues, eigenfunctions, basis property, characteristic determinant.

AMS Mathematics Subject Classification: 34B05, 34B09, 34B10, 34L05, 34L10.

Abstract. In this paper we consider a spectral problem for a two-fold differentiation operator with an integral perturbation of boundary conditions of one type which are regular, but not strongly regular. The unperturbed problem has an asymptotically simple spectrum, and its system of eigenfunctions does not form a basis in L_2. We construct the characteristic determinant of the spectral problem with an integral perturbation of boundary conditions. We show that the set of kernels of the integral perturbation, under which absence of basis properties of the system of root functions persists, is dense in L_2.

1 Introduction and statement of the problem

A well-known fact is that the system of eigenfunctions of an operator given by formally adjoint differential expressions, with arbitrary self-adjoint boundary conditions providing a discrete spectrum, forms an orthonormal basis in L_2. The question of persisting the basis properties under some (weak in definite sense) perturbation of an initial operator has been investigated in many works. For example, the analogous question for the case of a self-adjoint initial operator has been investigated in [1, 2, 3], and for a non-selfadjoint operator in [4, 5, 6].

In the present paper we consider the spectral problem:

$$l(u) \equiv -u''(x) = \lambda u(x), \qquad 0 < x < 1, \tag{1.1}$$

$$U_1(u) \equiv u'(0) - u'(1) - \alpha u(1) = 0, \tag{1.2}$$

$$U_2(u) \equiv u(0) = 0, \tag{1.3}$$

which is close to those considered in [1, 4, 7]. Here $\alpha > 0$ is an arbitrary positive number.

Let $\mathcal{L}_1$ be an operator in $L_2(0,1)$ given by expression (1.1) and by "perturbed" boundary conditions:

$$U_1(u) = \int_0^1 \overline{p(x)} u(x)\,dx, \quad U_2(u) = 0, \quad \text{where } p(x) \in L_2(0,1). \tag{1.4}$$

By $\mathcal{L}_0$ we denote the unperturbed operator (the case $p(x) \equiv 0$).

In our previous papers [6, 7, 8, 9, 10, 11, 12] we considered different variants of the integral perturbation of boundary conditions. In these papers, we constructed the characteristic determinant of the spectral problem for the operator $\mathcal{L}_1$ under the assumption that the unperturbed operator $\mathcal{L}_0$ has a system of eigen- and associated functions (EAF) which form a Riesz basis in $L_2(0,1)$. On the basis of the obtained formula we established the stability or the instability of the Riesz basis properties of EAF of the problem under the integral perturbation of the boundary condition. In [8] the problems of stability of the basis properties of root vectors of the spectral problem, where $\alpha= 0$, and with the integral perturbation of the second boundary condition, have been investigated.

As follows from [4], the system of root vectors of spectral problem (1.1), (1.4) forms a Riesz basis with brackets in $L_2(0,1)$ for any choice of $p \in L_2(0,1)$. However even for $p(x)\equiv 0$ (i.e., in case of the unperturbed problem) the system of the root vectors of the problem does not form a basis in $L_2(0,1)$ (see [13]). Therefore we cannot apply directly the methods of our previous papers. We use a special auxiliary system for constructing the characteristic determinant.

2 Constructing a basis from the eigenfunctions of the operator $\mathcal{L}_0$

Boundary conditions in (1.1) - (1.3) are regular but not strongly regular. The system of root functions of the operator $\mathcal{L}_0$ is a complete system but does not form even an ordinary basis in $L_2(0,1)$ [13]. However, as shown in [14], on the basis of these eigenfunctions one can construct a basis by applying the method of separation of variables for solving initial-boundary value problems with boundary condition (1.2).

In this section we introduce results from [14] and make additional calculations which are necessary for our further work. Spectral problem (1.1) - (1.3) is easily reduced to the characteristic determinant of the problem

$$\Delta_0(\lambda) = \sqrt{\lambda}\left(1 - \cos\sqrt{\lambda}\right) - \alpha \sin\sqrt{\lambda} = 0. \tag{2.1}$$

Therefore the problem has two sequences of eigenvalues

$$\lambda_k^{(1)} = (2\pi k)^2,\ k = 1, 2, \ldots, \quad \lambda_k^{(2)} = (2\beta_k)^2,\ k = 0, 1, 2, \ldots.$$

Here β_k are roots of the equation

$$\mathrm{tg}\beta = \alpha/2\beta, \quad \beta > 0\ . \tag{2.2}$$

They are positive and satisfy the inequalities

$$\pi k < \beta_k < \pi k + \pi/2,\ k = 0, 1, 2, \ldots.$$

Two-sided estimates

$$\frac{\alpha}{2\pi k}\left(1 - \frac{1}{2\pi k}\right) < \delta_k < \frac{\alpha}{2\pi k}\left(1 + \frac{1}{2\pi k}\right) \tag{2.3}$$

hold for the difference $\delta_k = \beta_k - \pi k$ for large enough k .

The eigenfunctions of (1.1) - (1.3) have the form

$$y_k^{(1)}(x) = \sin(2\pi k x),\ k = 1, 2, \ldots, \quad y_k^{(2)}(x) = \sin(2\beta_k x),\ k = 0, 1, 2, \ldots.$$

This system is almost normalized but does not form even an ordinary basis in $L_2(0,1)$. However, as shown in [14], the auxiliary system

$$y_0(x) = y_0^{(2)}(x)(2\beta_0)^{-1},\ y_{2k}(x) = y_k^{(1)}(x),$$

$$y_{2k-1}(x) = \left(y_k^{(2)}(x) - y_k^{(1)}(x)\right)(2\delta_k)^{-1},\ k=1,2,\ldots$$

constructed from this system, already forms a Riesz basis in $L_2(0,1)$. The system

$$v_0(x) = 2\beta_0 v_0^{(2)}(x),$$

$$v_{2k}(x) = v_k^{(2)}(x) + v_k^{(1)}(x),\quad v_{2k-1}(x) = 2\delta_k v_k^{(2)}(x),\ k = 1,2,\ldots$$

is biorthogonal to the auxiliary system. This system is constructed from the eigenfunctions of the problem

$$v_k^{(1)}(x) = C_k^{(1)}\left(\cos(2\pi kx) - \frac{\alpha}{2\pi k}\sin(2\pi kx)\right),\ k = 1,2,\ldots,$$

$$v_k^{(2)}(x) = C_k^{(2)}\left(\cos(2\beta_k x) + \frac{\alpha}{2\beta_k}\sin(2\beta_k x)\right),\ k = 0,1,2,\ldots$$

adjoint to (1.1) - (1.3). The constants $C_k^{(j)}$ are chosen from the orthogonality relation $\left(y_k^{(j)}, v_k^{(j)}\right) = 1,\quad j = 1,2$. It is evident that the system $\{v_k(x)\}$ forms a Riesz basis in $L_2(0,1)$.

By direct calculation it is easy to check that

$$C_k^{(1)} = -\frac{4\pi k}{\alpha},\quad C_k^{(2)} = \frac{4\pi k}{\alpha} + O\left(\frac{1}{k}\right). \tag{2.4}$$

It is easy to see that $\|y_k^{(1)}\|\|v_k^{(1)}\| = 1 + \frac{2\pi k}{\alpha}$. Therefore $\lim_{k\to\infty}\|y_k^{(1)}\|\|v_k^{(1)}\| = \infty$. That is, the necessary condition of the basis property does not hold. Hence, the systems $\left\{y_k^{(1)},\ y_k^{(2)}\right\}$ and $\left\{v_k^{(1)},\ v_k^{(2)}\right\}$ do not form an unconditional basis in $L_2(0,1)$.

3 Characteristic determinant of spectral problem (1.1), (1.4)

Representing the general solution to equation (1.1) by the formula

$$u(x,\lambda) = C_1\cos\sqrt{\lambda}x + C_2\sin\sqrt{\lambda}x,$$

and watching it to the boundary conditions (1.4), we get that $C_1 = 0$ and

$$C_2\left[\sqrt{\lambda}\left(1 - \cos\sqrt{\lambda}\right) - \alpha\sin\sqrt{\lambda} - \int_0^1 \overline{p(x)}\sin\sqrt{\lambda}x\,dx\right] = 0.$$

Therefore the characteristic determinant of (1.1), (1.4) has a form

$$\Delta_1(\lambda) = \sqrt{\lambda}\left(1 - \cos\sqrt{\lambda}\right) - \lambda\sin\sqrt{\lambda} - \int_0^1 \overline{p(x)}\sin\sqrt{\lambda}x\,dx. \tag{3.1}$$

It is easy to see that the characteristic determinant of unperturbed problem (1.1) - (1.3) is obtained here for $p(x)=0$. As in (2.1), we denote it by

$$\Delta_0(\lambda)=\sqrt{\lambda}\left(1-\cos\sqrt{\lambda}\right)-\alpha\sin\sqrt{\lambda}\,.$$

By virtue of Section 2, we represent the function p in the form of the Fourier series with respect to the auxiliary system $\{v_k(x)\}$:

$$p(x)=a_0v_0(x)+\sum_{k=1}^{\infty}[a_kv_{2k}(x)+b_kv_{2k-1}(x)]. \tag{3.2}$$

Using (3.2), we find a more convenient representation of the determinant $\Delta_1(\lambda)$. To do so, we first compute the integral which appears in (3.1).

By a simple calculation we show that the following inequalities take place.

$$\begin{aligned}\int_0^1\overline{v_0(x)}\sin\sqrt{\lambda}x\,dx&=2\beta_0C_0^{(2)}\int_0^1\left(\cos(2\beta_0x)+\frac{\alpha}{2\beta_0}\sin(2\beta_0x)\right)\sin\sqrt{\lambda}x\,dx\\&=\frac{2\beta_0C_0^{(2)}}{\lambda-(2\beta_0)^2}\left\{\sqrt{\lambda}\left(1-\cos\sqrt{\lambda}\cos(2\beta_0)-\frac{\alpha}{2\beta_0}\sin(2\beta_0)\cos\sqrt{\lambda}\right)\right\}\\&\quad+\frac{2\beta_0C_0^{(2)}}{\lambda-(2\beta_0)^2}\left\{\sin\sqrt{\lambda}\,[\alpha\cos(2\beta_0)-(2\beta_0)\sin(2\beta_0)]\right\}.\end{aligned} \tag{3.3}$$

From (2.1) we obtain that $2\beta_0(1-\cos(2\beta_0))=\alpha\sin(2\beta_0)$. Therefore in the first summand from (3.3) inside the round brackets we have:

$$\left(1-\cos\sqrt{\lambda}\cos(2\beta_0)-\frac{\alpha}{2\beta_0}\sin(2\beta_0)\cos\sqrt{\lambda}\right)=1-\cos\sqrt{\lambda}\,.$$

Using (2.2), we find that

$$\sin(2\beta_0)=\frac{2\mathrm{tg}(\beta_0)}{1+\mathrm{tg}^2(\beta_0)}=\frac{4\alpha\beta_0}{(2\beta_0)^2+\alpha^2},\ \cos(2\beta_0)=\frac{1-\mathrm{tg}^2(\beta_0)}{1+\mathrm{tg}^2(\beta_0)}=\frac{(2\beta_0)^2-\alpha^2}{(2\beta_0)^2+\alpha^2}.$$

Therefore in the second summand from (3.3) inside the square brackets we will have:

$$[\alpha\cos(2\beta_0)-(2\beta_0)\sin(2\beta_0)]=\left[\alpha\frac{(2\beta_0)^2-\alpha^2}{(2\beta_0)^2+\alpha^2}-(2\beta_0)\frac{4\alpha\beta_0}{(2\beta_0)^2+\alpha^2}\right]=-\alpha.$$

Finally we obtain:

$$\begin{aligned}&\int_0^1\overline{v_0(x)}\sin\sqrt{\lambda}x\,dx\\&=\frac{2\beta_0C_0^{(2)}}{\lambda-(2\beta_0)^2}\left\{\sqrt{\lambda}\left(1-\cos\sqrt{\lambda}\right)-\alpha\sin\sqrt{\lambda}\right\}=\frac{2\beta_0C_0^{(2)}}{\lambda-(2\beta_0)^2}\Delta_0(\lambda).\end{aligned} \tag{3.4}$$

Analogously, we calculate the integral

$$\int_0^1\overline{v_{2k-1}(x)}\sin\sqrt{\lambda}x\,dx=2\delta_kC_k^{(2)}\frac{1}{\lambda-(2\beta_k)^2}\Delta_0(\lambda). \tag{3.5}$$

Further we have

$$\int_0^1 \overline{v_{2k}(x)} \sin \sqrt{\lambda} x \, dx = \int_0^1 \left(v_k^{(2)}(x) + v_k^{(1)}(x) \right) \sin \sqrt{\lambda} x \, dx$$

$$= C_k^{(2)} \frac{1}{\lambda - (2\beta_k)^2} \Delta_0(\lambda) + C_k^{(1)} \frac{1}{\lambda - (2\pi k)^2} \Delta_0(\lambda).$$

And so, we finally obtain

$$\int_0^1 \overline{p(x)} \sin \sqrt{\lambda} x \, dx = \Delta_0(\lambda) A(\lambda),$$

$$A(\lambda) = \frac{2a_0 \beta_0 C_0^{(2)}}{\lambda - (2\beta_0)^2} + \sum_{k=1}^{\infty} \left[a_k \left(\frac{C_k^{(2)}}{\lambda - (2\beta_k)^2} + \frac{C_k^{(1)}}{\lambda - (2\pi k)^2} \right) + \frac{2b_k \delta_k C_k^{(2)}}{\lambda - (2\beta_k)^2} \right]. \tag{3.6}$$

The convergence of the obtained numerical series for $\lambda \neq (2\beta_k)^2$ and $\lambda \neq (2\pi k)^2$ is provided by the asymptotic behaviors in (2.3) and (2.4). From these formulas it follows that the parentheses inside the sign of sum cannot be removed, otherwise the corresponding series may diverge.

In representation (3.6) the function $A(\lambda)$ has poles at $\lambda = (2\beta_k)^2$ and $\lambda = (2\pi k)^2$. But at the same points the function $\Delta_0(\lambda)$ has zeros. So the function $\Delta_0(\lambda) A(\lambda)$ is an entire analytic function of the variable λ.

Now we substitute all the calculations into (3.1). Let us formulate the obtained result in the form of a theorem.

Theorem 3.1. *The characteristic determinant of problem* (1.1), (1.4) *with the perturbed boundary conditions can be represented in form*

$$\Delta_1(\lambda) = \Delta_0(\lambda)(1 - A(\lambda)), \tag{3.7}$$

where $\Delta_0(\lambda)$ *is the characteristic determinant of unperturbed problem* (1.1)- (1.3), $A(\lambda)$ *is given by* (3.6), *in which* a_k *and* b_k *are the Fourier coefficients of the biorthogonal expansion* (3.2) *of the function* p *with respect to the auxiliary system* $\{v_k\}$.

We note that in all previous works the basis properties of the system of root functions of the unperturbed problem have been necessarily required for constructing the characteristic determinant. The principal distinction of the present paper is that characteristic determinant (3.7) is constructed without such a requirement.

4 The case of a simple form of characteristic determinant (3.7)

The case of a simple form of characteristic determinant (3.7) takes place when p is represented in form (3.2) with a finite second sum. That is, when there is a number N such that $a_k = 0$ and $b_k = 0$ for all $k > N$. In this case, formula (3.7) takes the form

$$\Delta_1(\lambda) = \Delta_0(\lambda) \left(1 - a_0 \frac{2\beta_0 C_0^{(2)}}{\lambda - (2\beta_0)^2} \right.$$

$$\left. - \sum_{k=1}^{N} \left[a_k \left(C_k^{(2)} \frac{1}{\lambda - (2\beta_k)^2} + C_k^{(1)} \frac{1}{\lambda - (2\pi k)^2} \right) + b_k \frac{2\beta_k C_k^{(2)}}{\lambda - (2\beta_k)^2} \right] \right). \tag{4.1}$$

On the basis of this particular case of formula (3.7), one can readily prove the following theorem.

Theorem 4.1. *For any prescribed numbers, a complex number $\widehat{\lambda}$ and a natural one $\widehat{m}$, there is always a function p such that $\widehat{\lambda}$ is an eigenvalue of problem* (1.1), (1.4) *of the multiplicity $\widehat{m}$.*

From the analysis of formula (4.1) it is also easy to see that $\Delta_1(\lambda_k^{(1)}) = \Delta_1(\lambda_k^{(2)}) = 0$ for all $k > N$. Hence all the eigenvalues $\lambda_k^{(1)}, \lambda_k^{(2)},\ k > N$ of unperturbed problem (1.1)-(1.3) are eigenvalues of perturbed problem (1.1), (1.4). Also it is not difficult to see that the multiplicity of the eigenvalues $\lambda_k^{(1)}, \lambda_k^{(2)},\ k > N$ is also preserved.

Moreover from the biorthogonality condition of the system of eigenfunctions $\left\{y_k^{(1)}(x), y_k^{(2)}(x)\right\}$ and $\left\{v_k^{(1)}(x), v_k^{(2)}(x)\right\}$ of the adjoint problems it follows that in this case

$$\int_0^1 \overline{p(x)} y_k^{(j)}(x)\,dx = 0,\ \ j = 1, 2,\ \ k > N.$$

So the eigenfunctions $\left\{y_k^{(1)}(x), y_k^{(2)}(x)\right\}$ of problem (1.1)-(1.3) for $k > N$ satisfy boundary conditions (1.4) and hence, are eigenfunctions of problem (1.1), (1.4). Thus in this case the system of eigenfunctions of problem (1.1), (1.4) and the system of eigenfunctions of problem (1.1)-(1.3) (not forming a basis) coincide except for a finite number of the first terms. Consequently, also the system of eigenfunctions of problem (1.1), (1.4) is not a basis in $L_2(0,1)$.

By the Riesz basis property in $L_2(0,1)$ of the system $\{v_k\}$, the set of all functions p, represented by finite sums in (3.2) is dense in $L_2(0,1)$. Hence the following statement follows.

Theorem 4.2. *The set of all functions $p \in L_2(0,1)$, for which the system of eigenfunctions of problem* (1.1), (1.4) *is a not basis in $L_2(0,1)$, is dense in $L_2(0,1)$.*

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Makhmud Abdysametovich Sadybekov
Institute of Mathematics and Mathematical Modeling
125 Pushkin street, 050010 Almaty, Kazakhstan
E-mail: sadybekov@math.kz

Nurlan Sayramovich Imanbaev
Institute of Mathematics and Mathematical Modeling
125 Pushkin street, 050010 Almaty, Kazakhstan, and
South Kazakhstan State Pedagogical Institute
16 G. Ilyaev street, 160012, Shymkent, Kazahstan
E-mail: imanbaevnur@mail.ru

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