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- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

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This issue contains the first part of the collection of papers sent to the Eurasian Mathematical Journal dedicated to the 70th birthday of Professor R. Oinarov.

The first part of the collection was published in Volume 8, Number 1.

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THE WAY TO THE TOP OF SCIENCE OF PROFESSOR R. OINAROV

L.-E. Persson

Dedicated to the 70th birthday of Professor Ryskul Oinarov

1 Introduction

Professor Ryskul Oinarov was born on February 26, 1947 in the village of Kul-Aryk, Kazalinsk district, Kyzylorda region. His parents Baiarystanov Oinar (1905-1975) and Ergaliyeva Sanim (1921-2005) were simple and open people who devoted their entire life to hard farming and brought up their children in this environment. The example of parents, the early recognition of the need for hard work, the high spirituality of simple Kazakh people formed the basis of Ryskul's strong Personality and formed his value orientations. In 1953 he went to the 1st grade of a rural school in the village of Tursinbike. But he studied only three classes under the parental care.

Photo 1. R. Oinarov 5 years old.

From 1956 to 1959 he studied in the city of Aralsk. From 1959 to 1964 he continued his studies in the village of Soltrest, which is located 15 km from the Aralsk city, where he graduated

from the eleven-year secondary school. In the same year Ryskul entered the Mechanics and Mathematics Faculty of the S.M. Kirov Kazakh State University.

Beginning with the second year of study at the University, Ryskul was regularly awarded the Kirov scholarship, which was awarded to students with especially high academic performance. In 1969 Ryskul graduated from the KazSU with honors and as one of the best graduate students he was sent to post-graduate studies at the M.V. Lomonosov Moscow State University.

However, the M.V. Lomonosov MSU post-graduate programme did not take place. Immediately after graduation from the University he was drafted into the Soviet Army and between 1969 and 1971 he served as an officer in the most southern and hottest point of the Soviet Union, namely Kushka city. It was a difficult but at the same time an interesting period of his life, of which he has a lot of memories. Service in the Army enriched his experience with the skills of working with people, strengthened the spirit and accustomed him with accountability.

Photo 2. R. Oinarov as a school teacher.

After the Army service he worked for some time as a school teacher in his home city Aralsk. In the autumn of 1972 Ryskul moved to Alma-Ata. At first he worked at his "alma mater"[1] at the Mechanics and Mathematics Faculty of the S.M. Kirov Kazakh State University in the position of an assistant at the Department of Mathematical Analysis. After that he transferred to the position of a senior engineer at the Laboratory of Mechanics of Rocks of the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR. Here he independently investigated the inverse problem for the Newtonian potential and reported some of his results at the scientific seminar "Functional Analysis and its Applications" led by M. Otelbaev and T.Sh. Kalmenov. M. Otelbaev was impressed and suggested the young mathematician to work at his laboratory. Thus, from November 1979 Ryskul Oinarov became an employee of the Laboratory of Applied Analysis of the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR. He worked at this laboratory for more than 20 years, having passed the way from a senior engineer to the Head of the Laboratory, giving a lot of power and

[1]Alma mater (Latin: "nourishing/kind mother") is an allegorical Latin phrase for a university or college. In modern usage, it is a school or university which an individual has attended, or a song or hymn associated with that school.

creative energy to the development of the institute and the formation of Kazakhstan mathematics. In 1981 he defended his thesis on the topic "Continuity and Lipschitzness of nonlinear integral operators of Urysohn type" (under the guidance of M. Otelbaev). And in 1994 he defended his doctoral thesis on the topic "Weighted estimates for integral and differential operators". Within the walls of Laboratory of Applied Analysis of the IMM of the Academy of Sciences of the Kazakh SSR his scientific potential was fully realized, and manager's abilities were revealed.

Photo 3. In 1994 R. Oinarov during defence of Doctor of Science thesis and Charman of the Commission Academician U.M. Sultangazin.

In the period from 1997 to 1999 R. Oinarov worked as the Head of the Department of Higher Mathematics in Shymkent city by the invitation of Academician of the National Academy of Sciences T.Sh. Kalmenov (at that time the rector of the South Kazakhstan State University).

In the autumn of 2000 Professor R. Oinarov moved with his family to Astana by the invitation of the Rector of the L.N. Gumilyov Eurasian National University Professor M.Zh. Zholdasbekov, and after that he has been working all time as a professor of the Department of Fundamental Mathematics at this university.

2 R. Oinarov's Family

After graduating from the University in July 1969 R. Oinarov got married. His spouse Kerimbaeva Maiya Saparbekovna, Candidate of Pedagogical Sciences, worked as a director of the Institute for Improvement of Teachers of Astana City. At present she is a pensioner. R. Oinarov has two sons: Azamat (born 1970) and Galymzhan (born 1976), two granddaughters: Asem, Saiya and a grandson Chingiz.

Photo 4. In 1969 R. Oinarov got married.

Photo 5. R. Oinarov's family.

3 Some information from the impressive CV of R. Oinarov

R. Oinarov is a member of the editorial board of three international journals: Journal of Mathematical Inequalities (JMI), Journal of Inequalities in Pure and Applied Mathematics and Eurasian Mathematical Journal.

He is a scientific supervisor of 28 Students: Doctor of Philology Sh. Blyal (1997), Doctor of Physical and Mathematical Sciences B.L. Baideldinov (1998), Candidates of Physics and Mathematics: A.E. Zhumagalieva (1985), G.Zh. Pshayeva (1985), A. Ibatov (1986), I.N. Parasidi (1989), Sh. Blyalov (1991), A.P. Stikharnyi (1991), S.S. Sagintaeva (1995), S.Kh. Shalginbaeva (1999), M.Zh. Omirbek (2000), B.O. Sagindikov (2000), A.A. Kalybay (2002 Candidate of Physics and Mathematics, 2006 PhD doctor), A.M. Abylayeva (2007 Candidate of Physics and Mathematics, 2016 PhD doctor), K.R. Myrzataeva (2008), A. Alimaganbetova (2009), M. Aldai (2010), PhD doctors: Z. Abdikalikova (2009), S. Kudabayeva (2011), Zh. Taspaganbetova (2013), L. Arendarenko (2013), A. Temirkhanova (2015), S. Shaimardan (2016), H.S. Ramazanova (2016), and A. Yeskermessuly (2017).

Since 2005, under the presidential decree of the Republic of Kazakhstan, a Doctorate for PhD doctoral students together with professors from foreign universities has been opened at the L.N. Gumilyov Eurasian National University (ENU). R. Oinarov was one of the first among ENU professors in 2005 to sign the first agreement with Professor Lars-Erik Persson for joint PhD doctoral studies between Luleå University of Technology in Sweden and ENU. Morever, in 2008 a similar agreement was signed with Professor Massimo Lanza de Cristoforis on the joint preparation of PhD doctoral students between the Padova University in Italy and ENU. Thanks to these agreements, 12 ENU post-graduate students have defended their theses, namely A. Kalybay (2006) (2 diplomas of PhD in Kazakhstan and Sweden), Z.T. Abdikalikova (2009), L. Arendarenko (2013), A.M. Temirkhanova (2015) (2 diplomas of PhD in Kazakhstan and Sweden), A.M. Abylayeva (2016) (2 diplomas of PhD in Kazakhstan and Sweden), under the supervision of R. Oinarov and L.-E. Persson, L. Sarybekova (2009) under the supervision of N.T. Tleukhanova and L.-E. Persson (2 PhD degrees from Kazakhstan and Sweden); S.E. Kud-

abaeva (2011), Zh. Taspaganbetova (2013) under the supervision of R. Oinarov and M. Lanza de Cristophoris, L. Zhapsarbaeva under the supervision of M. Otelbaev and M. Lanza de Cristoforis, A. Myrzagalieva under the supervision of L. Kusainova and M. Lanza de Cristoforis (2 diplomas of PhD in Kazakhstan and Italy).

Photo 6. 2006, after defence of a PhD degree, A. Kalybay with Professors R. Oinarov and L.-E. Persson in Sweden.

Photo 7. 2009, after defence of a Licentiate degree [2], Z. Abdikalikova, L. Sarybekova and A. Temirkhanova with Professors, R. Oinarov, L. Maligranda and L.-E. Persson in Sweden.

Photo 8. 2016, after defence of PhD degree, A. Abylayeva in Sweden.

[2] Licentiate degree in Sweden means three years of studies after Master Degree. The PhD programme in Sweden is five years.

Professor R. Oinarov was awarded with a badge of the Ministry of Education and Science of the Republic of Kazakhstan "For merits in the development of science in the Republic of Kazakhstan" in 2005, "Honored Worker of the Education Area in the Republic of Kazakhstan" in 2007, the diploma "Kurmet" ("Honour") of the Republic of Kazakhstan in 2008 and the order "Kurmet" ("Honour") of the Republic of Kazakhstan in 2016.

In 2007 and 2014 Professor R. Oinarov, got a prestigious award "The best university teacher".

Professor R. Oinarov has been an author of more than 100 scientific papers and 1 book. He has influented the mathematics all over the world in a substansial way. For example when I wrote my book in 2003 with A.Kufner his results were very crucial and he has even influented the terminology we used in the book.

4 Some selected scientific results of Oinarov

It is absolutely impossible to fully and fairly describe Professor R. Oinarov's scientific results so just judge what I write below as examples of such results I have chosen. Pers(s)onally with my bounded knowledge.

• Let $F \subseteq R_n$, $G \subseteq R_m$, $n,\ m \geq 1$, be measurable sets. It is known that a linear integral operator K acts from $L_1(F)$ to $L_q(G)$, $1 \leq q \leq \infty$, if and only if its kernel $K(\cdot,\cdot)$ satisfies the condition

$$C \equiv \operatorname*{ess\,sup}_{s \in F} \|K(\cdot, s)\|_{L_q(G)} < \infty,$$

and $\|K\|_{1\to q} = C$.

In the paper R. Oinarov and M. Otelbaev, *Criteria of Lipschitzness and compressibility of nonlinear integral operators*,. Sib. Mat. Journal, Vol. 25, No. 6, 116-127, this assertion was extended to the nonlinear integral operator

$$Uf(x) = \int_F K(x, s, f(s))ds$$

in the following way: the operator U acts from $L_1(F)$ to $L_q(G)$, $1 \leq q \leq \infty$, and it is a Lipschitz mapping if and only if $\int_F K(x,\ s,\ 0)ds \in L_q(G)$ and

$$C \equiv \operatorname*{ess\,sup}_{x \in R, h>0} \operatorname*{ess\,sup}_{s \in F} \left\| \frac{K(\cdot, s, x+h) - K(\cdot, s, x)}{h} \right\|_{L_q(G)} < \infty,$$

where $C \approx L$ and L is a Lipschitz constant of the operator U.

• Let Ω be a bounded open set in R^n, $n \geq 2$, with the boundary $\partial\Omega \in C^1$. We denote by $\overset{0}{W}{}_p^1(\Omega, \rho, \vartheta)$ the completion of the set $C_0^\infty(\Omega)$ with respect to the norm

$$\|f\|_{W_p^1} = \|\rho\,|\nabla f|\|_{p,\Omega} + \|\vartheta f\|_{p,\Omega},$$

where

$$|\nabla f| = \left(\sum_{i=1}^{n} \left| \frac{\partial f}{\partial x_i} \right|^2 \right)^{\frac{1}{2}}.$$

Moreover, we denote by $W_p^1(\Omega,\rho,\vartheta)$ the completion of the set $\left\{f : f \in C^\infty(\overline{\Omega}), \|f\|_{W_p^1} < \infty\right\}$ with respect to the norm $\|\cdot\|_{W_p^1}$.

Here and below $\rho > 0$ in Ω, $\rho \in L_p^{loc}(\Omega)$ and $\vartheta \in L_p^{loc}(\Omega)$.

Let the real functions $a_{i,j}(\cdot)$, $i,j = 1,...,n$, be measurable in Ω and satisfy the symmetry condition $a_{i,j}(\cdot) \equiv a_{j,i}(\cdot)$ and ρ - ellipticity that is $\exists c_1,\ c_2\ (0 < c_1 < c_2)$:

$$c_1\rho^2(x)|\xi|^2 \le \sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\overline{\xi}_j \le c_2\rho^2(x)|\xi|^2,$$

$$\forall x \in \overline{\Omega},\ \ \xi \in C^n.$$

Let $\vartheta \ge 1$. Then the form

$$\int_\Omega \left(\sum_{i,j=1}^{n} a_{i,j}(x)\frac{\partial f}{\partial x_i}\frac{\partial \overline{g}}{\partial x_j} + \vartheta^2(x) f\overline{g}\right)dx$$

generates a positive definite selfadjoint Dirichlet operator L_D and Neumann operator L_N on $\overset{0}{W}{}_2^1$, W_2^1 respectively.

In the paper R. Oinarov, *Necessary and sufficient condition for the coincidence of the extension of an elliptic operator corresponding to the Dirichlet and Neumann problems*, Izvestiya AN Kaz SSR, Ser. fiz.-mat. 1986, No. 5, 31-35, he proved that the equality $L_D = L_N$ holds only if and only if the following condition

$$\left\|\rho^{-2} + \vartheta^2\right\|_{L_1(\Omega \bigcap B_x)} = \infty,\ \ \forall x \in \partial\Omega,$$

holds for any ball B_x with the center $x \in \partial\Omega$.

This result is a corollary of a more general result: the equality $\overset{0}{W}{}_p^1 = W_p^1$, $1 < p < \infty$, holds if and only if $\left\|\rho^{-p'} + \vartheta^p\right\|_{L_1(\Omega \bigcap B_x)} = \infty$, $\forall x \in \partial\Omega$.

- We consider the Schrödinger operator

$$Au = -\Delta u + \vartheta u,\quad u \in D(A),$$

$$D(A) = \left\{u : u \in L_p(R^n),\ \ \vartheta u \in L_p^{loc}(R^n),\quad -\Delta u + \vartheta u \in L_p(R^n)\right\},$$

with a locally summable potential $\vartheta \ge 1$ in the space $L_p(R^n)$, $n \ge 3$, where the expression $-\Delta u + \vartheta u$ is understood as a distribution on $C_0^\infty(R^n)$. The operator A is called separable in the space $L_p(R^n)$, if from $u \in D(A)$ it follows that $\Delta u \in L_p(R^n)$.

The separability of the one-dimensional and multidimensional Schrödinger operators has been extensively studied by many authors. It is shown that the operator A, in general, is unseparable in $L_p(R^n)$, $n \ge 1$, $1 < p < \infty$, without any restrictions on the potential $\vartheta \ge 1$.

In the work R. Oinarov, *On the separability of the Schrödinger operator in the space of summable functions*, Doklady of the Academy of Sciences of the USSR, V. 285, No. 5, 1062-1064, it was proved that the operator A is separable in the space $L_1(R^n)$ and the following estimates hold:

$$\|\vartheta u\|_1 \le \|Au\|_1,\ \ u \in D(A),$$

$$\|\Delta u\|_1 \le 2\,\|Au\|_1\,, \quad u \in D(A).$$

• In 1991 in the paper *Weighted inequalities for a class of integral operators*, Doklady of the Academy of Sciences of the USSR, 1991, V. 319, No. 5 1076-1078, R. Oinarov obtained necessary and sufficient conditions for the weighted inequality

$$\|uKf\|_q \le C(\|\rho f\|_p + \|\vartheta Kf\|_p), \quad f \ge 0,$$

to hold in the case $1 < p \le q < \infty$ for the integral operator

$$Kf(x) = \int_0^x K(x,s)f(s)ds, \quad x > 0,$$

with a kernel $K(\cdot,\cdot)$ satisfying the following conditions:

(1) $K(x,s) \ge 0, \quad x \ge s > 0$;

(2) $\exists d \ge 1$ и $\forall (x,t,s): \quad x \ge t \ge s > 0$

$$\frac{1}{d}(K(x,t) + K(t,s)) \le K(x,s) \le d(K(x,t) + K(t,s)).$$

As a corollary of this statement a criterion follows for the boundedness of the integral operator K from $L_p(0,\infty)$ to $L_q(0,\infty)$ for $1 < p \le q < \infty$ (if $\vartheta = 0$).

In the same year, under condition (1)-(2) the boundedness of the operator K from $L_p(0,\infty)$ to $L_q(0,\infty)$ was obtained in the work by Bloom S. and Kerman R. by some other method.

At present, there are a lot of works devoted to the investigation of various properties of the integral operator K with the conditions (1)-(2). Moreover, this condition in the mathematical literature is called the "Oinarov's condition" or "Oinarov's kernel".

• In the paper *The boundedness and complexity of Volterra type integral operators*, Sib. Mat. Zhurnal, 2007, V. 48, No. 5, 1100-1115, R. Oinarov introduced the expanding classes of kernels $\mathcal{O}_n$, $\mathcal{O}_{n+1} \supset \mathcal{O}_n$, $n \ge 0$, where the kernel satisfying condition (1)-(2) belongs to the class $\mathcal{O}_1$, and found for the operator K with the kernel in the class $\mathcal{O}_n$, $n \ge 0$, a criteria on of boundedness and compactness from $L_p(0,\infty)$ to $L_q(0,\infty)$ for $1 < p \le q < \infty$.

• In the works R. Oinarov, M. Otelbaev, *Criterion for the discreteness of the spectrum of the general Sturm-Liouville operator and embedding theorems associated with it*, Differential equations, 1988, V. 24, No. 4, 584-591, and R. Oinarov, *On weighted norm inequalities with three weights*, J. London Math. Soc. 1993. V. 48, No. 2, 103-116, a criterion was derived for the validity of the inequality:

$$\|uf\|_q \le C(\|\rho f^{'}\|_p + \|\vartheta f\|_p)$$

for $1 \le p, q \le \infty$. From this inequality with $p = q = 2$ in particular follows the discreteness of the spectrum of the Sturm-Liouville operator

$$Ly = -(\rho y^{'})^{'} + \vartheta y.$$

This result widely generalizes the well-known Molchanov's criterion.

Based on the above works in the works R. Oinarov, *Boundedness of integral operators from a weighted Sobolev space to a weighted Lebesgue space*, Complex Variables and Elliptic Equations, 2011, V. 56, No. 10-11, 1021-1038, and R. Oinarov, *Boundedness of integral operators in weighted Sobolev spaces*, Izvestiya of the Academy of Sciences of Russia. Ser. Math., 2014,

V. 78, No. 4, 207-223, a criterion was obtained for the boundedness from the Sobolev space $W_p^1(R_+,\rho,\vartheta)$ to the Sobolev space $W_p^1(R_+,\omega,u)$ for the integral operator K with the kernel from the class $\mathcal{O}_n$, $n\geq 0$.

The idea of the investigation method in these works was developed in the series of works by V.D. Stepanov, D.V. Prokhorov and E.P. Ushakova for the description of the associated space to the space $W_p^1(R_+,\rho,\vartheta)$.

• In the work V.I. Burenkov, R. Oinarov, *Necessary and sufficient conditions for boundedness of the Hardy-type operator from a weighted Lebesgue space to a Morrey-type space*, Math. Ineq. Appl., 2013, V. 16, No. 1, 1-19, necessary and sufficient conditions were obtained for the boundedness of the multidimensional Hardy operator

$$(H_{n,\varphi}\,f)(x)=\varphi(|x|)\int_{B_{|x|}} f(y)dy,\quad x\in R^n$$

from the weighted Lebesgue spaces $L_{p,u}(R^n)$ to the local Morrey-type spaces $LM_{q,\theta\,,\omega}\,(R_n)$, where

$$LM_{q,\theta\,,\omega}\equiv\{f\in L_q^{loc}:\|\omega(r)\|f\|_{L_q(B_r)}\|_{L_\theta(0,\infty)}<\infty\}.$$

A part of the proof of this result it was the estimate of the form

$$\left(\int_0^\infty \omega^\theta(r)\left(\int_0^r |H_{1,\varphi}\,g|^q dt\right)^{\frac{\theta}{q}} dr\right)^{\frac{1}{\theta}}\leq C\left(\int_0^\infty |ug|^p dt\right)^{\frac{1}{p}}.$$

Studies of such estimates were further developed in the works by A. Gogatashvili, V.D. Stepanov, D.V. Prokhorov, R. Mustafayev and others.

• In the paper Kufner A., Kuliev K., Oinarov R. *Some criteria for boundedness and compactness of the Hardy operator with some special kernels*, J. Inequal. Appl. 2013, 2013:310, 1-15, the boundedness and compactness from $L_{p,\vartheta}$ to $L_{q,u}$ was studied of the integral operator K with the degenerate kernel

$$K(x,s)=\sum_{i=1}^m a_i(x)b_i(x),\quad m>1.$$

• In the paper Maligranda L., Oinarov R., Persson L.-E., *On Hardy q-Inequalities*, Czech. Math. J. 2014, V. 64. No. 64, 659-682, a q - analogue of the weighted Hardy inequality

$$\int_0^b\left(x^{\alpha-1}\int_0^x t^{-\alpha}f(t)d_qt\right)^p d_qx\leq C\int_0^b f^p(t)d_qt,\quad 0<b\leq\infty,$$

was derived with the best constant $C=\frac{1}{\left[\frac{(p-1)}{p}-\alpha\right]_q^p}$ for $1\leq p<\infty$ or $p<0\quad(f>0)$, $\alpha<(p-1)/p$. If $0<p<1$, then this inequality holds in the opposite direction still with the same sharp constant.

• In the paper Oinarov R., Ramazanova K., Tiryaki A., *An extension of the weighted Hardy inequalities and its application to half - linear equations*, Taiwanese J. Math. V. 19, No. 6, 1693-1711, the three-weighted inequality

$$\|uf\|_p^p\leq C(\|\rho f^{'}\|_p^p+\|\vartheta f\|_p^p)$$

was investigated with a specific estimate for the best constant C. On the basis of this result, a criterion was found for the oscillation and non-oscillation of the following second-order differential equation:

$$(\rho(t)|y'(t)|^{p-2}y'(t))' + \vartheta(t)|y(t)|^{p-2}y(t) = 0, \quad t \geq 0, \quad 1 < p < \infty, \quad \rho(t) > 0, \quad t > 0.$$

5 On my scientific cooperation with Oinarov

I have known Ryskul Oinarov since 1996, when he first time came to Luleå, Sweden, as an invited guest researcher. But our main scientific cooperation started during the conference in Chech Republic in June 2004 dedicated to the 70th birthday of Professor Alois Kufner. There Ryskul Oinarov came with his student Aigerim Kalybay. In particular, we discussed a possible program for the preparation of her PhD thesis and its defence at the Luleå University of Technology. We carried out this plan and as a result she defended her PhD in 2006.

In 2005, the President of the Republic of Kazakhstan awarded the L.N. Gumilyov Eurasian National University (ENU) by a financial support for the preparation of PhD students jointly with foreign professors. R. Oinarov invited me to further jointly supervise PhD students at ENU and I accepted his invitation. We had prepared a training program and agreement between the L.N. Gumilyov Eurasian National University and Luleå University of Technology (LTU), which were signed in 2005 during my visit to Astana by the invitation of the ENU. The ENU team awarded me the title of "Honorary Professor of the ENU" for my scientific achievements and support of their new international Program for PhD studies.

According to the signed agreement on preparation of PhD students at ENU, I am a responsible supervisor at the Luleå University of Technology and R. Oinarov has the same responsibility at the ENU. From 2005 up to the present days an essential number of PhD students from ENU worked part time at the LTU. Among them are the following:

A) Defended theses for the degree of Licentiate in Sweden:

1. Zamira Abdikalikova 2007. Current position: Associate Professor at the International Information Technology University, Almaty, Kazakhstan. Title of Licentiate thesis: Embedding Theorems for Spaces with Multiweighted Derivatives.

2. Lyazzat Sarybekova 2009. Current position: Associate Professor at the Kazakh-British Technical University, Almaty, Kazakhstan. Title of Licentiate thesis: Some new Lizorkin Multiplier Theorems for Fourier Series and Transforms.

3. Ainur Temirkhanova 2009. Current position: Associate Professor at the L.N. Gumilyov Eurasian National University, Astana, Kazakhstan. Title of Licentiate thesis: Some New Boundedness and Compactness Results for Discrete Hardy Type Operators with Kernels.

4. Aigerim Kopezhanova 2012. Current position: Associate Professor at the L.N. Gumilyov Eurasian National University, Astana, Kazakhstan. Title of Licentiate thesis: Relations Between Functions from some Lorentz Type Spaces and Summability of Their Fourier Coefficients.

5. Larissa Arendarenko 2012. Title of Licentiate thesis: Hardy-Type Inequalities for Integral Operators with Kernels.

6. Raya Akhmetkaliyeva 2013. Current position: Associate Professor at the L.N. Gumilyov Eurasian National University, Astana, Kazakhstan. Title of Licentiate thesis: Coercive Estimates for the Solutions of some Singular Differential Equations and their Applications.

7. Serikbol Shaimardan 2015. Current position: Teacher at Eurasian National University, Astana, Kazakhstan. Title of Licentiate thesis: Some New Hardy-type Inequalities in q-Analysis.

B) Defended theses for PhD degree in Sweden:

1. Aigerim Kalybay 2006. Current position: Associate Professor in mathematics at the KIMEP University, Almaty, Kazakhstan. Title of PhD thesis: A New Development of Nikolskii-Lizorkin and Hardy Type Inequalities with Applications.

2. Zamira Abdikalikova 2009. Current position: Associate Professor at the International Information Technology University, Almaty, Kazakhstan Title of PhD thesis: Some New Results Concerning Boundedness and Compactness for Embeddings between Spaces with Multiweighted Derivatives.

3. Lyazzat Sarybekova 2011. Current position: Associate Professor at the Kazakh-British Technical University, Almaty, Kazakhstan. Title of PhD thesis: Some New Fourier Multiplier Results of Lizorkin and Hörmander Types.

4. Larissa Arendarenko 2013. Title of PhD thesis: Estimates for Hardy-Type Integral Operators in Weighted Lebesgue Spaces.

5. Ainur Temirkhanova 2015. Current position: Associate Professor at the L.N. Gumilyov Eurasian National University, Astana, Kazakhstan. Title of PhD thesis: Estimates for Discrete Hardy-type Operators in Weighted Sequence Spaces.

6. Akbota Abylayeva 2016. Current position: Associate Professor at the L.N. Gumilyov Eurasian National University, Astana, Kazakhstan. Title of PhD thesis: Inequalities for Some Classes of Hardy Type Operators and Compactness in Weighted Lebesgue Spaces.

C) The following PhD students are preparing to defend their PhD theses soon at Luleå University of Technology: Aigerim Kopezhanova (8 June 2017), Raya Akhmetkaliyeva, Serikbol Shaimardan.

During the time of our scientific cooperation with scientists and students from ENU the following papers were published:

1. Three weights higher order Hardy type inequalities (together with Aigerim Kalybay, Almaty), J. Funct. Spaces Appl. 4 (2006) No. 2, 163-191.

2. Weighted inequalities of Hardy type for matrix operators: The case q<p (together with Christopher Okpoti, Winneba, and Ryskul Oinarov, Astana), Math. Inequal. Appl. 10 (2007), No. 4, 841-859.

3. Spectral properties of a class of singular differential operators (together with Aigerim Kalybay, Almaty, and Ryskul Oinarov, Astana), Math. Inequal. Appl. 1 (2007), No. 3, 355-376.

4. Multidimensional generalization of the Lizorkin theorem on Fourier multipliers (together with Lyazzat Sarybekova, Astana, and Nazerke Tleukhanova, Astana), J. Proc. A. Razmadze Inst. 151 (2009), 83-101.

5. Weighted inequalities for a class of matrix operators: the case $p \leq q$ (together with Ryskul Oinarov, Astana, and Ainur Temirkhanova, Astana), Math. Inequal. Appl. 12 (2009), No. 4, 891-903.

6. Relations between summability of the Fourier coefficients in regular systems and functions from some Lorentz type spaces (together with Aigerim Kopezhanova, Astana, and Erlan Nursultanov, Astana), J. Proc. A. Razmadze Inst. 152 (2010), 73-88.

7. On summability of the Fourier coefficients in bounded orthonormal systems for functions from some Lorentz type spaces (together with Aigerim Kopezhanova, Astana), Eurasian

Mathematical Journal 1 (2010), No. 2, 76-85.

8. Boundedness and Compactness of the Embedding between Spaces with Multiweighted Derivatives when $1 < q < p$ (together with Ryskul Oinarov, Astana, and Zamira Abdikalikova, Astana), Czechoslovak Math. J. 61 (2011), No. 1, 7-26.

9. On inequalities for the Fourier transform of functions from Lorentz spaces (together with Aigerim Kopezhanova, Astana, and Erlan Nursultanov, Astana), Russian Academy of Sciences, Mathematical Notes 90 (2011), No. 5-6, 767-770.

10. On the boundedness of some classes of integral operators in weighted Lebesgue spaces (together with Larissa Arendarenko, Astana, and Ryskul Oinarov, Astana), Eurasian Mathematical Journal 1 (2012), No. 3, 5-17.

11. A Lizorkin theorem on Fourier series multipliers for strong regular systems (together with Lyazzat Sarybekova, Astana, and Nazerke Tleukhanova, Astana), In: Analysis for science, engineering and beyond: the tribute workshop in honour of Gunnar Sparr, Lund, 2008, Eds: K. Åström, et. al. Springer Proceedings in Mathematics, 2012, Ch. 11, 305-317.

12. Some New Hardy-type Integral Inequalities on Cones of Monotone Fuctions (together with Larissa Arendarenko, Astana, and Ryskul Oinarov, Astana), Advances in Harmonic Analysis and Operator Theory 229 (2013), 77-89.

13. Some new results concerning a class of third-order differential equations (together with Raya Akhmetkaliyeva, Astana, and Kordan Ospanov, Astana), Applicable Analysis, 94 (2015), No. 2, 419–434.

14. On Hardy q-inequalities (together with Lech Maligranda, Ryskul Oinarov, Astana), Czechoslovak Mathematical Journal 64 (2014), No.3, 659-682.

15. A new discrete Hardy-type inequality with kernels and monotone functions (together with Aigerim Kalybay, Ainur Temirkhanova, Astana), Journal of Inequalities and Applications. 2015, 2015:321, 10 pp.

16. Some new Hardy-type inequalities for Riemann-Liouville fractional q-integral operator (together with Serikbol Shaimardan), Journal of Inequalities and Applications, 2015, 2015:296, 17 pp.

17. Some New Hardy type inequalities in q-analysis (together with Askar Baiarystanov, Serikbol Shaimardan, and Ainur Temirkhanova, Astana), Journal of Mathematical Inequalities. 10 (2016), No.3, 761-781.

18. Boundedness and compactness of a class of Hardy type operators (together with Akbota Abylayeva, Ryskul Oinarov), Journal of Inequalities and Applications, 2016, 2016:324, 18 pp.

19. Hardy type inequalities with logarithmic singularities (together with Akbota Abylayeva), Research report 2016-05, ISSN: 1400-4003, Department of Engineering Sciences and Mathematics, Luleå University of Technology, Sweden, 14 pp.

20. Additive weighted L_p estimates of some classes of integral operators involving generalized Oinarov kernels (together with Akbota Abylayeva, Astana, Askar Baiarystanov, Astana, and Peter Wall, Luleå), Journal of Mathematical Inequalities (JMI-2460), to appear 2017.

21. Some new two-sided inequalities concerning the Fourier transform (together with Aigerim Kopezhanova, Astana and Erlan Nursultanov, Astana) Math. Inequal. Appl., to appear 2017. (10 pp.).

22. A new generalization of Boas theorem for some Lorentz spaces (together with Aigerim Kopezhanova, Astana and Erlan Nursultanov, Astana), Journal of Mathematical Inequalities, submitted in 2016. (16 pp.).

23. Hardy type inequalities and compactness of a class of integral operators with logarithmic singularities (together with Akbota Abylayeva, Astana), submitted in 2016. (17 pp.).

6 Thank you R. Oinarov. Congratulations

- **Thank you for all wonderful new knowledge you have given to mathematics.**
- **Thank you for providing us, including all students, with such positive and supportative.**
- **Thank you for always looking forward new adventures in the future.**
- **Thank you for beeing my Pers(s)onal friend and main collaborator.**

And cordial congratulations to your +25 birthday

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