Eurasian Mathematical Journal

2017,   Volume 8,   Number 1

EURASIAN MATHEMATICAL JOURNAL

Editorial Board

Editors–in–Chief

V.I. Burenkov, M. Otelbaev, V.A. Sadovnichy

Editors

Managing Editor

A.M. Temirkhanova

Aims and Scope

The Eurasian Mathematical Journal (EMJ) publishes carefully selected original research papers in all areas of mathematics written by mathematicians, principally from Europe and Asia. However papers by mathematicians from other continents are also welcome.

From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

The contents of EMJ are indexed in Scopus, Web of Science (ESCI), Mathematical Reviews, MathSciNet, Zentralblatt Math (ZMATH), Referativnyi Zhurnal – Matematika, Math-Net.Ru.

The EMJ is included in the list of journals recommended by the Committee for Control of Education and Science (Ministry of Education and Science of the Republic of Kazakhstan) and in the list of journals recommended by the Higher Attestation Commission (Ministry of Education and Science of the Russian Federation).

Information for the Authors

Submission. Manuscripts should be written in LaTeX and should be submitted electronically in DVI, PostScript or PDF format to the EMJ Editorial Office via e-mail (eurasianmj@yandex.kz).

When the paper is accepted, the authors will be asked to send the tex-file of the paper to the Editorial Office.

The author who submitted an article for publication will be considered as a corresponding author. Authors may nominate a member of the Editorial Board whom they consider appropriate for the article. However, assignment to that particular editor is not guaranteed.

Copyright. When the paper is accepted, the copyright is automatically transferred to the EMJ. Manuscripts are accepted for review on the understanding that the same work has not been already published (except in the form of an abstract), that it is not under consideration for publication elsewhere, and that it has been approved by all authors.

Title page. The title page should start with the title of the paper and authors' names (no degrees). It should contain the Keywords (no more than 10), the Subject Classification (AMS Mathematics Subject Classification (2010) with primary (and secondary) subject classification codes), and the Abstract (no more than 150 words with minimal use of mathematical symbols).

Figures. Figures should be prepared in a digital form which is suitable for direct reproduction.

References. Bibliographical references should be listed alphabetically at the end of the article. The authors should consult the Mathematical Reviews for the standard abbreviations of journals' names.

Authors' data. The authors' affiliations, addresses and e-mail addresses should be placed after the References.

Proofs. The authors will receive proofs only once. The late return of proofs may result in the paper being published in a later issue.

Offprints. The authors will receive offprints in electronic form.

Publication Ethics and Publication Malpractice

For information on Ethics in publishing and Ethical guidelines for journal publication see http://www.elsevier.com/publishingethics and http://www.elsevier.com/journal-authors/ethics.

Submission of an article to the EMJ implies that the work described has not been published previously (except in the form of an abstract or as part of a published lecture or academic thesis or as an electronic preprint, see http://www.elsevier.com/postingpolicy), that it is not under consideration for publication elsewhere, that its publication is approved by all authors and tacitly or explicitly by the responsible authorities where the work was carried out, and that, if accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright-holder. In particular, translations into English of papers already published in another language are not accepted.

No other forms of scientific misconduct are allowed, such as plagiarism, falsification, fraudulent data, incorrect interpretation of other works, incorrect citations, etc. The EMJ follows the Code of Conduct of the Committee on Publication Ethics (COPE), and follows the COPE Flowcharts for Resolving Cases of Suspected Misconduct (http : //publicationethics.org/files/u2/New$_C$ode.pdf). To verify originality, your article may be checked by the originality detection service CrossCheck http://www.elsevier.com/editors/plagdetect.

The authors are obliged to participate in peer review process and be ready to provide corrections, clarifications, retractions and apologies when needed. All authors of a paper should have significantly contributed to the research.

The reviewers should provide objective judgments and should point out relevant published works which are not yet cited. Reviewed articles should be treated confidentially. The reviewers will be chosen in such a way that there is no conflict of interests with respect to the research, the authors and/or the research funders.

The editors have complete responsibility and authority to reject or accept a paper, and they will only accept a paper when reasonably certain. They will preserve anonymity of reviewers and promote publication of corrections, clarifications, retractions and apologies when needed. The acceptance of a paper automatically implies the copyright transfer to the EMJ.

The Editorial Board of the EMJ will monitor and safeguard publishing ethics.

The procedure of reviewing a manuscript, established by the Editorial Board of the Eurasian Mathematical Journal

1. Reviewing procedure

1.1. All research papers received by the Eurasian Mathematical Journal (EMJ) are subject to mandatory reviewing.

1.2. The Managing Editor of the journal determines whether a paper fits to the scope of the EMJ and satisfies the rules of writing papers for the EMJ, and directs it for a preliminary review to one of the Editors-in-chief who checks the scientific content of the manuscript and assigns a specialist for reviewing the manuscript.

1.3. Reviewers of manuscripts are selected from highly qualified scientists and specialists of the L.N. Gumilyov Eurasian National University (doctors of sciences, professors), other universities of the Republic of Kazakhstan and foreign countries. An author of a paper cannot be its reviewer.

1.4. Duration of reviewing in each case is determined by the Managing Editor aiming at creating conditions for the most rapid publication of the paper.

1.5. Reviewing is confidential. Information about a reviewer is anonymous to the authors and is available only for the Editorial Board and the Control Committee in the Field of Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan (CCFES). The author has the right to read the text of the review.

1.6. If required, the review is sent to the author by e-mail.

1.7. A positive review is not a sufficient basis for publication of the paper.

1.8. If a reviewer overall approves the paper, but has observations, the review is confidentially sent to the author. A revised version of the paper in which the comments of the reviewer are taken into account is sent to the same reviewer for additional reviewing.

1.9. In the case of a negative review the text of the review is confidentially sent to the author.

1.10. If the author sends a well reasoned response to the comments of the reviewer, the paper should be considered by a commission, consisting of three members of the Editorial Board.

1.11. The final decision on publication of the paper is made by the Editorial Board and is recorded in the minutes of the meeting of the Editorial Board.

1.12. After the paper is accepted for publication by the Editorial Board the Managing Editor informs the author about this and about the date of publication.

1.13. Originals reviews are stored in the Editorial Office for three years from the date of publication and are provided on request of the CCFES.

1.14. No fee for reviewing papers will be charged.

2. Requirements for the content of a review

2.1. In the title of a review there should be indicated the author(s) and the title of a paper.

2.2. A review should include a qualified analysis of the material of a paper, objective assessment and reasoned recommendations.

2.3. A review should cover the following topics:

- compliance of the paper with the scope of the EMJ;
- compliance of the title of the paper to its content;
- compliance of the paper to the rules of writing papers for the EMJ (abstract, key words and phrases, bibliography etc.);
- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);

- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

2.4. The final part of the review should contain an overall opinion of a reviewer on the paper and a clear recommendation on whether the paper can be published in the Eurasian Mathematical Journal, should be sent back to the author for revision or cannot be published.

Web-page

The web-page of EMJ is www.emj.enu.kz. One can enter the web-page by typing Eurasian Mathematical Journal in any search engine (Google, Yandex, etc.). The archive of the web-page contains all papers published in EMJ (free access).

Subscription

For Institutions

- US$ 200 (or equivalent) for one volume (4 issues)
- US$ 60 (or equivalent) for one issue

For Individuals

- US$ 160 (or equivalent) for one volume (4 issues)
- US$ 50 (or equivalent) for one issue.

The price includes handling and postage.

The Subscription Form for subscribers can be obtained by e-mail:

eurasianmj@yandex.kz

This issue contains the first part of the collection of papers sent to the Eurasian Mathematical Journal dedicated to the 70th birthday of Professor R. Oinarov.

The second part of the collection will be published in Volume 8, Number 2.

RYSKUL OINAROV

(to the 70th birthday)

On February 26, 2017 was the 70th birthday of Ryskul Oinarov, member of the Editorial Board of the Eurasian Mathematical Journal, professor of the Department Fundamental Mathematics of the L.N. Gumilyov Eurasian National University, doctor of physical and mathematical sciences (1994), professor (1997), honoured worker of education of the Republic of Kazakhstan (2007), corresponding member of the National Academy of Sciences of the Republic of Kazakhstan (2012). In 2005 he was awarded the breastplate "For the merits in the development of science in the Republic of Kazakhstan", in 2007 and 2014 the state grant "The best university teacher", in 2016 the Order "Kurmet" (= "Honour").

R. Oinarov was born in the village Kul'Aryk, Kazalinsk district, Kyzylorda region. In 1969 he graduated from the S.M. Kirov Kazakh State University (Almaty). Starting with 1972 he worked at the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR (senior engineer, junior researcher, senior researcher, head of a laboratory). In 1981 he defended of the candidate of sciences thesis "Continuity and Lipschitzness of nonlinear integral operators of Uryson's type" at the Tashkent State University of the Uzbek SSR and in 1994 the doctor of sciences thesis "Weighted estimates of integral and differential operators" at the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR.

Starting from 2000 he has been working as a professor at the L.N. Gumilyov Eurasian National University

Scientific works of R. Oinarov are devoted to investigation of linear and non-linear integral and discrete operators in weighted spaces; to studying problems of the well-posedness of differential equations; to weighted inequalities; to embedding theorems for the weighted function spaces of Sobolev type and their applications to the qualitative theory of linear and quasilinear differential equations. A certain class of integral operators is named after him - integral operators with *Oinarov's kernels* or *Oinarov condition*. On the whole, the results obtained by R. Oinarov have laid the groundwork for new perspective directions in the theory of function spaces and its applications to the theory of differential equations.

R. Oinarov has published more than 100 scientific papers. The list of his most important publications may be seen on the web-page

$$https://scholar.google.com/citations?user = NzXYMS4AAAAJhl = ruoi = ao$$

Under his supervision 26 theses have been defended: 1 doctor of sciences thesis, 15 candidate of sciences theses and 10 PhD theses. The Editorial Board of the Eurasian Mathematical Journal congratulates Ryskul Oinarov on the occasion of his 70th birthday and wishes him good health and new achievements in mathematics and mathematical education.

EMBEDDING RELATIONS BETWEEN WEIGHTED COMPLEMENTARY LOCAL MORREY-TYPE SPACES AND WEIGHTED LOCAL MORREY-TYPE SPACES

A. Gogatishvili, R.Ch. Mustafayev, T. Ünver

Communicated by V.D. Stepanov

Dedicated to the 70th birthday of Professor Ryskul Oinarov

Key words: local Morrey-type spaces, embeddings, iterated Hardy inequalities

AMS Mathematics Subject Classification: 46E30, 26D10.

Abstract. In this paper embedding relations between weighted complementary local Morrey-type spaces $^{\mathrm{c}}LM_{p\theta,\omega}(\mathbb{R}^n, v)$ and weighted local Morrey-type spaces $LM_{p\theta,\omega}(\mathbb{R}^n, v)$ are characterized. In particular, two-sided estimates of the optimal constant c in the inequality

$$\left(\int_0^\infty \left(\int_{B(0,t)} f(x)^{p_2} v_2(x)\, dx \right)^{\frac{q_2}{p_2}} u_2(t)\, dt \right)^{\frac{1}{q_2}}$$
$$\leq c \left(\int_0^\infty \left(\int_{{}^{\mathrm{c}}B(0,t)} f(x)^{p_1} v_1(x)\, dx \right)^{\frac{q_1}{p_1}} u_1(t)\, dt \right)^{\frac{1}{q_1}}, \quad f \geq 0$$

are obtained, where p_1, p_2, q_1, $q_2 \in (0,\infty)$, $p_2 \leq q_2$ and u_1, u_2 and v_1, v_2 are weights on $(0,\infty)$ and $\mathbb{R}^n$, respectively. The proof is based on the combination of the duality techniques with estimates of optimal constants of the embedding relations between weighted local Morrey-type and complementary local Morrey-type spaces and weighted Lebesgue spaces, which allows to reduce the problem to using of the known Hardy-type inequalities.

1 Introduction

The classical Morrey spaces $\mathcal{M}_{p,\lambda} \equiv \mathcal{M}_{p,\lambda}(\mathbb{R}^n)$, were introduced by C. Morrey in [26] in order to study regularity questions which appear in the Calculus of Variations, and defined as follows: for $0 \leq \lambda \leq n$ and $1 \leq p \leq \infty$,

$$\mathcal{M}_{p,\lambda} := \left\{ f \in L_p^{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{\mathcal{M}_{p,\lambda}} := \sup_{x \in \mathbb{R}^n,\, r>0} r^{\frac{\lambda-n}{p}} \|f\|_{L_p(B(x,r))} < \infty \right\},$$

where $B(x,r)$ is the open ball centered at x of radius r.

Note that $\mathcal{M}_{p,0}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$ and $\mathcal{M}_{p,n}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$.

These spaces describe local regularity more precisely than Lebesgue spaces and appeared to be quite useful in the study of the local behavior of solutions to partial differential equations, a priori estimates and other topics in the theory of partial differential equations (cf. [16]).

The classical Morrey spaces were widely investigated during the last decades, including the study of classical operators of Harmonic and Real Analysis - maximal, singular and potential operators - in generalizations of these spaces (the so-called Morrey-type spaces). The local Morrey-type spaces and the complementary local Morrey-type spaces introduced by Guliyev in his doctoral thesis [23].

The research on local Morrey-type spaces mainly includes the study of the boundedness of classical operators in these spaces (see, for instance, [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 24]), and the investigation of the functional-analytic properties of them and relation of these spaces with other known function spaces (see, for instance, [12, 1, 27, 17, 18]). We refer the reader to the surveys [2] and [3] for a comprehensive discussion of the history of $LM_{p\theta,\omega}$ and $^cLM_{p\theta,\omega}$.

Let A be any measurable subset of $\mathbb{R}^n$, $n \geq 1$. By $\mathfrak{M}(A)$ we denote the set of all measurable functions on A. The symbol $\mathfrak{M}^+(A)$ stands for the collection of all $f \in \mathfrak{M}(A)$ which are non-negative on A. The family of all weight functions (also called just weights) on A, that is, measurable, positive and finite a.e. on A, is given by $\mathcal{W}(A)$.

For $p \in (0, \infty]$, we define the functional $\| \cdot \|_{p,A}$ on $\mathfrak{M}(A)$ by

$$\|f\|_{p,A} := \begin{cases} \left(\int_A |f(x)|^p \, dx \right)^{1/p} & \text{if} \quad p < \infty \\ \operatorname{ess\,sup}_A |f(x)| & \text{if} \quad p = \infty \end{cases}.$$

If $w \in \mathcal{W}(A)$, then the weighted Lebesgue space $L_p(w, A)$ is given by

$$L_p(w, A) \equiv L_{p,w}(A) := \{ f \in \mathfrak{M}(A) : \ \|f\|_{p,w,A} := \|fw\|_{p,A} < \infty \}.$$

When $A = \mathbb{R}^n$, we often write simply $L_{p,w}$ and $L_p(w)$ instead of $L_{p,w}(A)$ and $L_p(w, A)$, respectively.

Throughout the paper, we always denote by c and C positive constants, which are independent of main parameters but it may vary from line to line. However a constant with subscript such as c_1 does not change in different occurrences. By $a \lesssim b$, $(b \gtrsim a)$ we mean that $a \leq \lambda b$, where $\lambda > 0$ depends on inessential parameters. If $a \lesssim b$ and $b \lesssim a$, we write $a \approx b$ and say that a and b are equivalent. We will denote by $\mathbf{1}$ the function $\mathbf{1}(x) = 1$, $x \in \mathbb{R}$.

Given two quasi-normed vector spaces X and Y, we write $X = Y$ if X and Y are equal in the algebraic and the topological sense (their quasi-norms are equivalent). The symbol $X \hookrightarrow Y$ $(Y \hookleftarrow X)$ means that $X \subset Y$ and the natural embedding I of X in Y is continuous, that is, there exist a constant $c > 0$ such that $\|z\|_Y \leq c\|z\|_X$ for all $z \in X$. The best constant of the embedding $X \hookrightarrow Y$ is $\| \operatorname{I} \|_{X \to Y}$.

The weighted local Morrey-type spaces $LM_{p\theta,\omega}(\mathbb{R}^n, v)$ and weighted complementary local Morrey-type spaces $LM_{p\theta,\omega}(\mathbb{R}^n, v)$ are defined as follows: Let $0 < p, \theta \leq \infty$. Assume that $\omega \in \mathfrak{M}^+(0, \infty)$ and $v \in \mathcal{W}(\mathbb{R}^n)$.

$$LM_{p\theta,\omega}(\mathbb{R}^n, v) := \left\{ f \in L_{p,v}^{\text{loc}}(\mathbb{R}^n) : \ \|f\|_{LM_{p\theta,\omega}(\mathbb{R}^n, v)} < \infty \right\},$$

where

$$\|f\|_{LM_{p\theta,\omega}(\mathbb{R}^n, v)} := \left\| \|f\|_{p,v,B(0,r)} \right\|_{\theta,\omega,(0,\infty)},$$

and

$$^cLM_{p\theta,\omega}(\mathbb{R}^n, v) := \left\{ f \in \bigcap_{t>0} L_{p,v}(^cB(0,t)) : \ \|f\|^c_{LM_{p\theta,\omega}(\mathbb{R}^n, v)} < \infty \right\},$$

where

$$\|f\|^c_{LM_{p\theta,\omega}(\mathbb{R}^n, v)} := \left\| \|f\|_{p,v,^cB(0,r)} \right\|_{\theta,\omega,(0,\infty)}.$$

Remark 1. In [5] and [7] it was proved that the spaces $LM_{p\theta,\omega}(\mathbb{R}^n) := LM_{p\theta,\omega}(\mathbb{R}^n, \mathbf{1})$ and ${}^{c}LM_{p\theta,\omega}(\mathbb{R}^n) := {}^{c}LM_{p\theta,\omega}(\mathbb{R}^n, \mathbf{1})$ are non-trivial, i.e. consists not only of functions equivalent to 0 on $\mathbb{R}^n$, if and only if

$$\|\omega\|_{\theta,(t,\infty)} < \infty, \qquad \text{for some} \qquad t > 0, \tag{1.1}$$

and

$$\|\omega\|_{\theta,(0,t)} < \infty, \qquad \text{for some} \qquad t > 0, \tag{1.2}$$

respectively. The same conclusion is true for $LM_{p\theta,\omega}(\mathbb{R}^n, v)$ and ${}^{c}LM_{p\theta,\omega}(\mathbb{R}^n, v)$ for any $v \in \mathcal{W}(\mathbb{R}^n)$.

The proof of the following statement is straightforward.

Lemma 1.1. (i) *If* $\|\omega\|_{\theta,(t_1,\infty)} = \infty$ *for some* $t_1 > 0$, *then*

$$f \in LM_{p\theta,\omega}(\mathbb{R}^n, v) \Rightarrow f = 0 \quad a.e. \ on \quad B(0, t_1).$$

(ii) *If* $\|\omega\|_{\theta,(0,t_2)} = \infty$ *for some* $t_2 > 0$, *then*

$$f \in {}^{c}LM_{p\theta,\omega}(\mathbb{R}^n, v) \Rightarrow f = 0 \quad a.e. \ on \quad {}^{c}B(0, t_2).$$

Let $0 < \theta \le \infty$. We denote by

$$\Omega_\theta := \left\{ \omega \in \mathfrak{M}^+(0, \infty) : 0 < \|\omega\|_{\theta,(t,\infty)} < \infty, \ t > 0 \right\},$$
$${}^{c}\Omega_\theta := \left\{ \omega \in \mathfrak{M}^+(0, \infty) : 0 < \|\omega\|_{\theta,(0,t)} < \infty, \ t > 0 \right\}.$$

Let $v \in \mathcal{W}(\mathbb{R}^n)$. It is easy to see that $LM_{p\theta,\omega}(\mathbb{R}^n, v)$ and ${}^{c}LM_{p\theta,\omega}(\mathbb{R}^n, v)$ are quasi-normed vector spaces when $\omega \in \Omega_\theta$ and $\omega \in {}^{c}\Omega_\theta$, respectively.

The following statements are immediate consequences of Fubini's Theorem and were observed in [5] and [7], for $v = 1$, respectively.

Lemma 1.2. *Let* $0 < p \le \infty$ *and* $v \in \mathcal{W}(\mathbb{R}^n)$. *Then*
(i) $LM_{pp,\omega}(\mathbb{R}^n, v) = L_p(w)$, *where* $w(x) := v(x)\|\omega\|_{p,(|x|,\infty)}$, $x \in \mathbb{R}^n$.
(ii) ${}^{c}LM_{pp,\omega}(\mathbb{R}^n, v) = L_p(w)$, *where* $w(x) := v(x)\|\omega\|_{p,(0,|x|)}$, $x \in \mathbb{R}^n$.

Recall that the embedding relations between weighted local Morrey-type spaces and weighted Lebesgue spaces, that is, the embeddings

$$L_{p_1}(v_1) \hookrightarrow LM_{p_2\theta,\omega}(\mathbb{R}^n, v_2), \tag{1.3}$$
$$L_{p_1}(v_1) \hookrightarrow {}^{c}LM_{p_2\theta,\omega}(\mathbb{R}^n, v_2), \tag{1.4}$$
$$L_{p_1}(v_1) \hookleftarrow LM_{p_2\theta,\omega}(\mathbb{R}^n, v_2), \tag{1.5}$$
$$L_{p_1}(v_1) \hookleftarrow {}^{c}LM_{p_2\theta,\omega}(\mathbb{R}^n, v_2) \tag{1.6}$$

are completely characterized in [27].

Our principal goal in this paper is to investigate the embedding relations between weighted complementary local Morrey–type spaces and weighted local Morrey type spaces and vice versa, that is, the embeddings

$$
{}^{c}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n, v_1) \hookrightarrow LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n, v_2), \tag{1.7}
$$
$$
LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n, v_1) \hookrightarrow {}^{c}LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n, v_2). \tag{1.8}
$$

The approach used in this paper consists of a the duality argument combined with estimates of optimal constants of embeddings (1.3) - (1.6), which allows us to reduce the problem to using the following Hardy-type inequalities

$$\left\| \left\| H^* f \right\|_{p,u,(0,\cdot)} \right\|_{q,w,(0,\infty)} \leq c \left\| f \right\|_{\theta,v,(0,\infty)}, \ f \in \mathfrak{M}^+(0,\infty), \tag{1.9}$$

with

$$(H^* f)(t) := \int_t^\infty f(\tau) \, d\tau, \quad t > 0,$$

where u, v, w are weights on $(0,\infty)$ and $0 < p, q \leq \infty$, $1 < \theta < \infty$. There exists different criteria for the validity of these inequalities (for more detailed information see, for instance, [19] and [20]). We will use characterizations from [21] and [22].

Note that in view of Lemma 1.2, embeddings (1.7) - (1.8) contain embeddings (1.3) - (1.6) as a special case. Moreover, by the change of variables $x = y/|y|^2$ and $t = 1/\tau$, it is easy to see that (1.8) is equivalent to the embedding

$$^{\mathrm{c}} LM_{p_1\theta_1,\tilde{\omega}_1}(\mathbb{R}^n, \tilde{v}_1) \hookrightarrow LM_{p_2\theta_2,\tilde{\omega}_2}(\mathbb{R}^n, \tilde{v}_2),$$

where $\tilde{v}_i(y) = v_i(y/|y|^2)|y|^{-2n/p_i}$ and $\tilde{\omega}_i(\tau) = \tau^{-2/\theta_i}\omega_i(1/\tau)$, $i = 1, 2$. This observation allows us to concentrate our attention on characterization of (1.7). On the negative side of things we have to admit that the duality approach works only in the case when, in (1.7) - (1.8), one has $p_2 \leq \theta_2$. Unfortunately, in the case when $p_2 > \theta_2$ the problem of characterization of these embeddings remains open.

In particular, we obtain two-sided estimates of the optimal constant c in the inequality

$$\left(\int_0^\infty \left(\int_{B(0,t)} f(x)^{p_2} v_2(x) \, dx \right)^{\frac{q_2}{p_2}} u_2(t) \, dt \right)^{\frac{1}{q_2}}$$

$$\leq c \left(\int_0^\infty \left(\int_{^{\mathrm{c}}B(0,t)} f(x)^{p_1} v_1(x) \, dx \right)^{\frac{q_1}{p_1}} u_1(t) \, dt \right)^{\frac{1}{q_1}},$$

where p_1, p_2, q_1, $q_2 \in (0,\infty)$, $p_2 \leq q_2$ and u_1, u_2 and v_1, v_2 are weights on $(0,\infty)$ and $\mathbb{R}^n$, respectively.

The paper is organized as follows. We start with formulations of our main results in Section 2. The proofs of the main results are presented in Section 3.

2   Statement of the main results

We adopt the following usual conventions.

Convention 1. (i) *Throughout the paper we put* $0/0 = 0$, $0 \cdot (\pm\infty) = 0$ *and* $1/(\pm\infty) = 0$.
   (ii) *We put*

$$p' := \begin{cases} \frac{p}{1-p} & if \quad 0 < p < 1, \\ \infty & if \quad p = 1, \\ \frac{p}{p-1} & if \quad 1 < p < \infty, \\ 1 & if \quad p = \infty. \end{cases}$$

   (iii) *To state our results we use the notation* $p \to q$ *for* $0 < p, q \leq \infty$ *defined by*

$$\frac{1}{p \to q} = \frac{1}{q} - \frac{1}{p} \qquad if \qquad q < p,$$

and $p \to q = \infty$ *if* $q \geq p$.

(iv) *If* $I = (a,b) \subseteq \mathbb{R}$ *and* g *is a monotone function on* I, *then by* $g(a)$ *and* $g(b)$ *we mean the limits* $\lim_{t \to a+} g(t)$ *and* $\lim_{t \to b-} g(t)$, *respectively.*

Our main results are the following theorems. Throughout the paper we will denote

$$\widetilde{V}(x) := \|v_1^{-1} v_2\|_{p_1 \to p_2, B(0,x)}, \quad \text{and} \quad \mathcal{V}(t,x) := \frac{\widetilde{V}(t)}{\widetilde{V}(t) + \widetilde{V}(x)} \quad (t > 0, \ x > 0).$$

Theorem 2.1. *Let* $0 < \theta_2 = p_2 \leq p_1 = \theta_1 < \infty$. *Assume that* $v_1, v_2 \in \mathcal{W}(\mathbb{R}^n)$, $\omega_1 \in {}^{c}\Omega_{\theta_1}$ *and* $\omega_2 \in \Omega_{\theta_2}$. *Then*

$$\| \mathrm{I} \|_{{}^{c}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)} \approx \left\| \|\omega_1\|_{p_1,(0,|\cdot|)}^{-1} \ \|\omega_2\|_{p_2,(|\cdot|,\infty)} \right\|_{p_1 \to p_2, v_1^{-1} v_2, \mathbb{R}^n}.$$

Theorem 2.2. *Let* $0 < p_1, p_2, \theta_1, \theta_2 < \infty$ *and* $\theta_2 \neq p_2 \leq p_1 = \theta_1$. *Assume that* $v_1, v_2 \in \mathcal{W}(\mathbb{R}^n)$, $\omega_1 \in {}^{c}\Omega_{\theta_1}$ *and* $\omega_2 \in \Omega_{\theta_2}$.
(i) *If* $p_1 \leq \theta_2$, *then*

$$\| \mathrm{I} \|_{{}^{c}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)} \approx \sup_{t \in (0,\infty)} \left\| \|\omega_1\|_{p_1,(0,|\cdot|)}^{-1} \right\|_{p_1 \to p_2, v_1^{-1} v_2, B(0,t)} \|\omega_2\|_{\theta_2,(t,\infty)};$$

(ii) *If* $\theta_2 < p_1$, *then*

$$\| \mathrm{I} \|_{{}^{c}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)}$$
$$\approx \left(\int_0^\infty \left\| \|\omega_1\|_{p_1,(0,|\cdot|)}^{-1} \right\|_{p_1 \to p_2, v_1^{-1} v_2, B(0,t)}^{p_1 \to \theta_2} d\left(-\|\omega_2\|_{\theta_2,(t,\infty)}^{p_1 \to \theta_2} \right) \right)^{\frac{1}{p_1 \to \theta_2}}.$$

Theorem 2.3. *Let* $0 < p_1, p_2, \theta_1, \theta_2 < \infty$ *and* $\theta_2 = p_2 \leq p_1 \neq \theta_1$. *Assume that* $v_1, v_2 \in \mathcal{W}(\mathbb{R}^n)$, $\omega_1 \in {}^{c}\Omega_{\theta_1}$ *and* $\omega_2 \in \Omega_{\theta_2}$.
(i) *If* $\theta_1 \leq p_2$, *then*

$$\| \mathrm{I} \|_{{}^{c}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)} \approx \sup_{t \in (0,\infty)} \|\omega_1\|_{\theta_1,(0,t)}^{-1} \left\| \|\omega_2\|_{p_2,(|\cdot|,\infty)} \right\|_{p_1 \to p_2, v_1^{-1} v_2, B(0,t)};$$

(ii) *If* $p_2 < \theta_1$, *then*

$$\| \mathrm{I} \|_{{}^{c}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)}$$
$$\approx \left(\int_0^\infty \left\| \|\omega_2\|_{p_2,(|\cdot|,\infty)} \right\|_{p_1 \to p_2, v_1^{-1} v_2, B(0,t)}^{\theta_1 \to p_2} d\left(-\|\omega_1\|_{\theta_1,(0,t)}^{-\theta_1 \to p_2} \right) \right)^{\frac{1}{\theta_1 \to p_2}}$$
$$+ \|\omega_1\|_{\theta_1,(0,\infty)}^{-1} \left\| \|\omega_2\|_{p_2,(|\cdot|,\infty)} \right\|_{p_1 \to p_2, v_1^{-1} v_2, \mathbb{R}^n}.$$

In view of Lemma 1.2, Theorems 2.1 - 2.3 are straightforward corrolaries of [27, Theorem 3.1] and [27, Theorem 4.2].

To state further results we need the following definitions.

Definition 1. Let U be a continuous, strictly increasing function on $[0,\infty)$ such that $U(0) = 0$ and $\lim_{t \to \infty} U(t) = \infty$. Then we say that U is admissible.

Let U be an admissible function. We say that a function φ is U-quasiconcave if φ is equivalent to an increasing function on $(0, \infty)$ and φ/U is equivalent to a decreasing function on $(0, \infty)$. We say that a U-quasiconcave function φ is non-degenerate if

$$\lim_{t \to 0+} \varphi(t) = \lim_{t \to \infty} \frac{1}{\varphi(t)} = \lim_{t \to \infty} \frac{\varphi(t)}{U(t)} = \lim_{t \to 0+} \frac{U(t)}{\varphi(t)} = 0.$$

The family of non-degenerate U-quasiconcave functions is denoted by Q_U.

Definition 2. Let U be an admissible function, and let w be a non-negative measurable function on $(0, \infty)$. We say that the function φ, defined by

$$\varphi(t) = U(t) \int_0^\infty \frac{w(\tau)\, d\tau}{U(\tau) + U(t)}, \qquad t \in (0, \infty),$$

is a fundamental function of w with respect to U. One will also say that $w(\tau)\, d\tau$ is a representation measure of φ with respect to U.

Remark 2. Let φ be the fundamental function of w with respect to U. Assume that

$$\int_0^\infty \frac{w(\tau)\, d\tau}{U(\tau) + U(t)} < \infty, \ t > 0, \qquad \int_0^1 \frac{w(\tau)\, d\tau}{U(\tau)} = \int_1^\infty w(\tau)\, d\tau = \infty.$$

Then $\varphi \in Q_U$.

Remark 3. Suppose that $\varphi(x) < \infty$ for all $x \in (0, \infty)$, where φ is defined by

$$\varphi(x) = \operatorname{ess\,sup}_{t \in (0,x)} U(t) \operatorname{ess\,sup}_{\tau \in (t,\infty)} \frac{w(\tau)}{U(\tau)}, \quad t \in (0, \infty).$$

If

$$\limsup_{t \to 0+} w(t) = \limsup_{t \to +\infty} \frac{1}{w(t)} = \limsup_{t \to 0+} \frac{U(t)}{w(t)} = \limsup_{t \to +\infty} \frac{w(t)}{U(t)} = 0,$$

then $\varphi \in Q_U$.

Theorem 2.4. *Let $0 < p_1, p_2, \theta_1, \theta_2 < \infty$, $p_2 < p_1$, $\theta_1 \le p_2 < \theta_2$. Assume that $v_1, v_2 \in \mathcal{W}(\mathbb{R}^n)$, $\omega_1 \in {}^c\Omega_{\theta_1}$ and $\omega_2 \in \Omega_{\theta_2}$. Suppose that $\widetilde{V}$ is admissible and*

$$\varphi_1(x) := \sup_{t \in (0,\infty)} \widetilde{V}(t)\, \mathcal{V}(x,t)\, \|\omega_1\|_{\theta_1,(0,t)}^{-1} \in Q_{\widetilde{V}^{\frac{1}{p_1 \to p_2}}}.$$

(i) *If $p_1 \le \theta_2$, then*

$$\| \operatorname{I} \|_{{}^c LM_{p_1 \theta_1, \omega_1}(\mathbb{R}^n, v_1) \to LM_{p_2 \theta_2, \omega_2}(\mathbb{R}^n, v_2)} \approx \sup_{x \in (0,\infty)} \varphi_1(x) \sup_{t \in (0,\infty)} \mathcal{V}(t,x)\, \|\omega_2\|_{\theta_2,(t,\infty)}.$$

(ii) *If $\theta_2 < p_1$, then*

$$\| \operatorname{I} \|_{{}^c LM_{p_1 \theta_1, \omega_1}(\mathbb{R}^n, v_1) \to LM_{p_2 \theta_2, \omega_2}(\mathbb{R}^n, v_2)}$$

$$\approx \sup_{x \in (0,\infty)} \varphi_1(x) \left(\int_0^\infty \mathcal{V}(t,x)^{p_1 \to \theta_2}\, d\left(-\|\omega_2\|_{\theta_2,(t,\infty)}^{p_1 \to \theta_2} \right) \right)^{\frac{1}{p_1 \to \theta_2}}.$$

Theorem 2.5. *Let $0 < p_1, p_2, \theta_1, \theta_2 < \infty$, $p_2 < p_1$ and $p_2 < \min\{\theta_1, \theta_2\}$. Assume that $v_1, v_2 \in \mathcal{W}(\mathbb{R}^n)$, $\omega_1 \in {}^c\Omega_{\theta_1}$ and $\omega_2 \in \Omega_{\theta_2}$. Suppose that $\widetilde{V}$ is admissible and*

$$\varphi_2(x) := \left(\int_0^\infty [\widetilde{V}(t)\mathcal{V}(x,t)]^{\theta_1 \to p_2} \, d\left(-\|\omega_1\|_{\theta_1,(0,t)}^{-\theta_1 \to p_2} \right) \right)^{\frac{1}{\theta_1 \to p_2}} \in Q_{\widetilde{V}^{\frac{1}{p_1 \to p_2}}}.$$

(i) *If $\max\{p_1, \theta_1\} \leq \theta_2$, then*

$$\| \mathrm{I} \|_{{}^cLM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)} \approx \sup_{x \in (0,\infty)} \varphi_2(x) \sup_{t \in (0,\infty)} \mathcal{V}(t,x)\|\omega_2\|_{\theta_2,(t,\infty)}$$
$$+ \|\omega_1\|_{\theta_1,(0,\infty)}^{-1} \sup_{t \in (0,\infty)} \widetilde{V}(t)\|\omega_2\|_{\theta_2,(t,\infty)};$$

(ii) *If $p_1 \leq \theta_2 < \theta_1$, then*

$$\| \mathrm{I} \|_{{}^cLM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)}$$
$$\approx \left(\int_0^\infty \varphi_2(x)^{\frac{\theta_1 \to \theta_2 \cdot \theta_1 \to p_2}{\theta_2 \to p_2}} \widetilde{V}(x)^{\theta_1 \to p_2} \left(\sup_{t \in (0,\infty)} \mathcal{V}(t,x)\|\omega_2\|_{\theta_2,(t,\infty)} \right)^{\theta_1 \to \theta_2} \right.$$
$$\left. \times d\left(-\|\omega_1\|_{\theta_1,(0,x)}^{-\theta_1 \to p_2} \right) \right)^{\frac{1}{\theta_1 \to \theta_2}}$$
$$+ \|\omega_1\|_{\theta_1,(0,\infty)}^{-1} \sup_{t \in (0,\infty)} \widetilde{V}(t)\|\omega_2\|_{\theta_2,(t,\infty)};$$

(iii) *If $\theta_1 \leq \theta_2 < p_1$, then*

$$\| \mathrm{I} \|_{{}^cLM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)}$$
$$\approx \sup_{x \in (0,\infty)} \varphi_2(x) \left(\int_0^\infty \mathcal{V}(t,x)^{p_1 \to \theta_2} d\left(-\|\omega_2\|_{\theta_2,(t,\infty)}^{p_1 \to \theta_2} \right) \right)^{\frac{1}{p_1 \to \theta_2}}$$
$$+ \|\omega_1\|_{\theta_1,(0,\infty)}^{-1} \left(\int_0^\infty \widetilde{V}(t)^{p_1 \to \theta_2} d\left(-\|\omega_2\|_{\theta_2,(t,\infty)}^{p_1 \to \theta_2} \right) \right)^{\frac{1}{p_1 \to \theta_2}};$$

(iv) *If $\theta_2 < \min\{p_1, \theta_1\}$, then*

$$\| \mathrm{I} \|_{{}^cLM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)}$$
$$\approx \left(\int_0^\infty \varphi_2(x)^{\frac{\theta_1 \to \theta_2 \cdot \theta_1 \to p_2}{\theta_2 \to p_2}} \widetilde{V}(x)^{\theta_1 \to p_2} \left(\int_0^\infty \mathcal{V}(t,x)^{p_1 \to \theta_2} d\left(-\|\omega_2\|_{\theta_2,(t,\infty)}^{p_1 \to \theta_2} \right) \right)^{\frac{\theta_1 \to \theta_2}{p_1 \to \theta_2}} \right.$$
$$\left. \times d\left(-\|\omega_1\|_{\theta_1,(0,x)}^{-\theta_1 \to p_2} \right) \right)^{\frac{1}{\theta_1 \to \theta_2}}$$
$$+ \|\omega_1\|_{\theta_1,(0,\infty)}^{-1} \left(\int_0^\infty \widetilde{V}(t)^{p_1 \to \theta_2} d\left(-\|\omega_2\|_{\theta_2,(t,\infty)}^{p_1 \to \theta_2} \right) \right)^{\frac{1}{p_1 \to \theta_2}}.$$

Theorem 2.6. *Let $0 < \theta_1 < p < \theta_2 < \infty$. Assume that $v_1, v_2 \in \mathcal{W}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, $\omega_1 \in {}^c\Omega_{\theta_1}$ and $\omega_2 \in \Omega_{\theta_2}$.*

$$\| \mathrm{I} \|_{{}^cLM_{p\theta_1,\omega_1}(\mathbb{R}^n,v_1) \to LM_{p\theta_2,\omega_2}(\mathbb{R}^n,v_2)} \approx \sup_{t \in (0,\infty)} \left\| \|\omega_1\|_{\theta_1,(0,|\cdot|)}^{-1} \right\|_{\infty,v_1^{-1}v_2,B(0,t)} \|\omega_2\|_{\theta_2,(t,\infty)}.$$

Theorem 2.7. *Let $0 < \theta_1, \theta_2 < \infty$ and $0 < p < \min\{\theta_1, \theta_2\}$. Assume that $v_1, v_2 \in \mathcal{W}(\mathbb{R}^n)$ such that $v_1^{-1} v_2 \in C(\mathbb{R}^n)$. Suppose that $\omega_1 \in {}^c\Omega_{\theta_1}$, $\omega_2 \in \Omega_{\theta_2}$ and*

$$0 < \|\omega_2^{-1}\|_{\theta_2 \to p, (x,\infty)} < \infty$$

holds for all $x > 0$.
  (i) *If $\theta_1 \leq \theta_2$, then*

$$\| \mathrm{I} \|_{{}^c LM_{p\theta_1, \omega_1}(\mathbb{R}^n, v_1) \to LM_{p\theta_2, \omega_2}(\mathbb{R}^n, v_2)}$$

$$\approx \sup_{x \in (0,\infty)} \left(\widetilde{V}(x)^{\theta_1 \to p} \int_x^\infty d\left(- \|\omega_1\|_{\theta_1, (0,t)}^{-\theta_1 \to p} \right) \right.$$

$$+ \int_0^x \widetilde{V}(t)^{\theta_1 \to p} d\left(- \|\omega_1\|_{\theta_1, (0,t)}^{-\theta_1 \to p} \right) \bigg)^{\frac{1}{\theta_1 \to p}} \|\omega_2\|_{\theta_2, (x,\infty)}$$

$$+ \|\omega_1\|_{\theta_1, (0,\infty)}^{-1} \sup_{t \in (0,\infty)} \widetilde{V}(t) \|\omega_2\|_{\theta_2, (t,\infty)};$$

  (ii) *If $\theta_2 < \theta_1$, then*

$$\| \mathrm{I} \|_{{}^c LM_{p\theta_1, \omega_1}(\mathbb{R}^n, v_1) \to LM_{p\theta_2, \omega_2}(\mathbb{R}^n, v_2)}$$

$$\approx \left(\int_0^\infty \left(\int_x^\infty d\left(- \|\omega_1\|_{\theta_1, (0,t)}^{-\theta_1 \to p} \right) \right)^{\frac{\theta_1 \to \theta_2}{\theta_2 \to p}} \left(\sup_{0 < \tau \leq x} \widetilde{V}(\tau) \|\omega_2\|_{\theta_2, (\tau,\infty)} \right)^{\theta_1 \to \theta_2} \right.$$

$$\times d\left(- \|\omega_1\|_{\theta_1, (0,x)}^{-\theta_1 \to p} \right) \bigg)^{\frac{1}{\theta_1 \to \theta_2}}$$

$$+ \left(\int_0^\infty \left(\int_0^x \widetilde{V}(t)^{\theta_1 \to p} d\left(- \|\omega_1\|_{\theta_1, (0,t)}^{-\theta_1 \to p} \right) \right)^{\frac{\theta_1 \to \theta_2}{\theta_2 \to p}} \widetilde{V}(x)^{\theta_1 \to p} \|\omega_2\|_{\theta_2, (t,\infty)}^{\theta_1 \to \theta_2} \right.$$

$$\times d\left(- \|\omega_1\|_{\theta_1, (0,x)}^{-\theta_1 \to p} \right) \bigg)^{\frac{1}{\theta_1 \to \theta_2}}$$

$$+ \|\omega_1\|_{\theta_1, (0,\infty)}^{-1} \sup_{t \in (0,\infty)} \widetilde{V}(t) \|\omega_2\|_{\theta_2, (t,\infty)}.$$

3   Proofs of main results

Before proceeding to the proof of our main results we recall the following integration in polar coordinates formula.

  We denote the unit sphere $\{x \in \mathbb{R}^n : |x| = 1\}$ in $\mathbb{R}^n$ by S^{n-1}. If $x \in \mathbb{R}^n \backslash \{0\}$, the polar coordinates of x are

$$r = |x| \in (0, \infty), \qquad x' = \frac{x}{|x|} \in S^{n-1}.$$

There is a unique Borel measure $\sigma = \sigma_{n-1}$ on S^{n-1} such that if f is Borel measurable on $\mathbb{R}^n$ and $f \geq 0$ or $f \in L^1(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} f(x)\, dx = \int_0^\infty \int_{S^{n-1}} f(rx') r^{n-1} d\sigma(x') dr$$

(see, for instance, [15, p. 78]).

It should be noted that ${}^{c}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n, v_1) \not\hookrightarrow LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n, v_2)$ when $0 < p_1, p_2, \theta_1, \theta_2 \leq \infty$ and $p_1 < p_2$, where $v_1, v_2 \in \mathcal{W}(\mathbb{R}^n)$, $\omega_1 \in {}^{c}\Omega_{\theta_1}$ and $\omega_2 \in \Omega_{\theta_2}$. To see this, assume that ${}^{c}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n, v_1) \hookrightarrow LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n, v_2)$ holds. Then there exist $c > 0$ such that

$$\|f\|_{LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)} \leq c\, \|f\|_{{}^{c}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)}$$

holds for all $f \in \mathfrak{M}^{+}(\mathbb{R}^n)$. Let $\tau \in (0,\infty)$ and $f \in \mathfrak{M}(\mathbb{R}^n)$: $\operatorname{supp} f \subset B(0,\tau)$. It is easy to see that

$$\begin{aligned}
\|f\|_{LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)} &= \big\|\|f\|_{p_2,v_2,B(0,t)}\big\|_{\theta_2,\omega_2,(0,\infty)} \\
&\geq \big\|\|f\|_{p_2,v_2,B(0,t)}\big\|_{\theta_2,\omega_2,(\tau,\infty)} \\
&\geq \|\omega_2\|_{\theta_2,(\tau,\infty)} \|f\|_{p_2,v_2,B(0,\tau)}
\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}
\|f\|_{{}^{c}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)} &= \big\|\|f\|_{p_1,v_1,{}^{c}B(0,t)}\big\|_{\theta_1,\omega_1,(0,\infty)} \\
&= \big\|\|f\|_{p_1,v_1,{}^{c}B(0,t)}\big\|_{\theta_1,\omega_1,(0,\tau)} \\
&\leq \|\omega_1\|_{\theta_1,(0,\tau)} \|f\|_{p_1,v_1,B(0,\tau)}.
\end{aligned} \tag{3.2}$$

Combining (3.1) with (3.2), we can assert that

$$\|\omega_2\|_{\theta_2,(\tau,\infty)} \|f\|_{p_2,v_2,B(0,\tau)} \leq c\, \|\omega_1\|_{\theta_1,(0,\tau)} \|f\|_{p_1,v_1,B(0,\tau)}.$$

Since $\omega_1 \in {}^{c}\Omega_{\theta_1}$ and $\omega_2 \in \Omega_{\theta_2}$, we conclude that $L_{p_1}(B(0,\tau), v_1) \hookrightarrow L_{p_2}(B(0,\tau), v_2)$, which is a contradiction.

The following lemma is true.

Lemma 3.1. *Let $0 < p_1, p_2, \theta_1, \theta_2 < \infty$, $p_2 \leq p_1$ and $p_2 < \theta_2$. Assume that $v_1, v_2 \in \mathcal{W}(\mathbb{R}^n)$, $\omega_1 \in {}^{c}\Omega_{\theta_1}$ and $\omega_2 \in \Omega_{\theta_2}$. Then*

$$\| \mathrm{I} \|_{{}^{c}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)}$$

$$= \left\{ \sup_{g \in \mathfrak{M}^{+}(0,\infty)} \frac{\| \mathrm{I} \|^{p_2}_{{}^{c}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \to L_{p_2}\left(v_2(\cdot)H^{*}g(|\cdot|)^{\frac{1}{p_2}}\right)}}{\|g\|_{\frac{\theta_2}{\theta_2-p_2},\omega_2^{-p_2},(0,\infty)}} \right\}^{\frac{1}{p_2}}.$$

Proof. By duality, interchanging suprema, we have that

$$\| \mathrm{I} \|_{{}^{c}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)}$$

$$= \sup_{f \in \mathfrak{M}^{+}(\mathbb{R}^n)} \frac{\|f\|_{LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)}}{\|f\|_{{}^{c}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)}}$$

$$= \sup_{f \in \mathfrak{M}^{+}(\mathbb{R}^n)} \frac{1}{\|f\|_{{}^{c}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)}} \sup_{g \in \mathfrak{M}^{+}(0,\infty)} \frac{\left(\int_{0}^{\infty} \left(\int_{B(0,\tau)} f(x)^{p_2} v_2(x)^{p_2}\, dx\right) g(\tau)\, d\tau\right)^{\frac{1}{p_2}}}{\|g\|_{\frac{\theta_2}{\theta_2-p_2},\omega_2^{-p_2},(0,\infty)}^{\frac{1}{p_2}}}$$

$$= \sup_{g \in \mathfrak{M}^{+}(0,\infty)} \frac{1}{\|g\|_{\frac{\theta_2}{\theta_2-p_2},\omega_2^{-p_2},(0,\infty)}^{\frac{1}{p_2}}} \sup_{f \in \mathfrak{M}^{+}(\mathbb{R}^n)} \frac{\left(\int_{0}^{\infty} \left(\int_{B(0,\tau)} f(x)^{p_2} v_2(x)^{p_2}\, dx\right) g(\tau)\, d\tau\right)^{\frac{1}{p_2}}}{\|f\|_{{}^{c}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)}}.$$

Applying Fubini's Theorem, we get that

$$\| \mathrm{I} \|_{^cLM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)}$$

$$= \sup_{g \in \mathfrak{M}^+(0,\infty)} \frac{1}{\|g\|_{\frac{\theta_2}{\theta_2-p_2},\omega_2^{-p_2},(0,\infty)}^{\frac{1}{p_2}}} \sup_{f \in \mathfrak{M}^+(\mathbb{R}^n)} \frac{\left(\int_{\mathbb{R}^n} f(x)^{p_2} v_2(x)^{p_2} \left(\int_{|x|}^{\infty} g(\tau)\,d\tau \right) dx \right)^{\frac{1}{p_2}}}{\|f\|_{^cLM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)}}$$

$$= \sup_{g \in \mathfrak{M}^+(0,\infty)} \frac{1}{\|g\|_{\frac{\theta_2}{\theta_2-p_2},\omega_2^{-p_2},(0,\infty)}^{\frac{1}{p_2}}} \| \mathrm{I} \|_{^cLM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \to L_{p_2}\left(v_2(\cdot) H^* g(|\cdot|)^{\frac{1}{p_2}}\right)}. \qquad (3.3)$$

$\square$

Proof of Theorem 2.4. By Lemma 3.1, we have that

$$\| \mathrm{I} \|_{^cLM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)}$$

$$= \sup_{g \in \mathfrak{M}^+(0,\infty)} \frac{1}{\|g\|_{\frac{\theta_2}{\theta_2-p_2},\omega_2^{-p_2},(0,\infty)}^{\frac{1}{p_2}}} \| \mathrm{I} \|_{^cLM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \to L_{p_2}\left(v_2(\cdot) H^* g(|\cdot|)^{\frac{1}{p_2}}\right)}.$$

Since $\theta_1 \leq p_2$, applying [27, Theorem 4.2, (a)], we obtain that

$$\| \mathrm{I} \|_{^cLM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)}$$

$$\approx \left\{ \sup_{g \in \mathfrak{M}^+(0,\infty)} \frac{\sup\limits_{t \in (0,\infty)} \|\omega_1\|_{\theta_1,(0,t)}^{-p_2} \|H^* g(|\cdot|)\|_{\frac{p_1}{p_1-p_2},(v_1^{-1}v_2)^{p_2},B(0,t)}}{\|g\|_{\frac{\theta_2}{\theta_2-p_2},\omega_2^{-p_2},(0,\infty)}} \right\}^{\frac{1}{p_2}}.$$

By using polar coordinates, we have that

$$\|H^* g(|\cdot|)\|_{\frac{p_1}{p_1-p_2},(v_1^{-1}v_2)^{p_2},B(0,t)} = \|H^* g\|_{\frac{p_1}{p_1-p_2},\tilde{v}^{\frac{p_1-p_2}{p_1}},(0,t)}, \qquad t > 0,$$

where

$$\tilde{v}(r) := \int_{S^{n-1}} (v_1^{-1}v_2)(rx')^{\frac{p_1 p_2}{p_1-p_2}} r^{n-1} d\sigma(x'), \quad r > 0.$$

Thus, we obtain that

$$\| \mathrm{I} \|_{^cLM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)}$$

$$\approx \left\{ \sup_{g \in \mathfrak{M}^+(0,\infty)} \frac{\sup\limits_{t \in (0,\infty)} \|\omega_1\|_{\theta_1,(0,t)}^{-p_2} \|H^* g\|_{\frac{p_1}{p_1-p_2},\tilde{v}^{\frac{p_1-p_2}{p_1}},(0,t)}}{\|g\|_{\frac{\theta_2}{\theta_2-p_2},\omega_2^{-p_2},(0,\infty)}} \right\}^{\frac{1}{p_2}}.$$

Taking into account that

$$\int_0^t \tilde{v}(r)\,dr = \int_0^t \int_{S^{n-1}} (v_1^{-1}v_2)(rx')^{\frac{p_1 p_2}{p_1-p_2}} \, d\sigma(x') r^{n-1} dr$$

$$= \int_{B(0,t)} (v_1^{-1}v_2)^{\frac{p_1 p_2}{p_1-p_2}}(x)\,dx = \widetilde{V}(t)^{\frac{p_1 p_2}{p_1-p_2}}, \qquad (3.4)$$

(i) if $p_1 \leq \theta_2$, then applying [21, Theorem 3.2, (i)], we arrive at

$$\|\,\mathrm{I}\,\|_{{}^cLM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)\to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)} \approx \sup_{x\in(0,\infty)} \varphi_1(x) \sup_{t\in(0,\infty)} \mathcal{V}(t,x)\|\omega_2\|_{\theta_2,(t,\infty)};$$

(ii) if $\theta_2 < p_1$, then applying [21, Theorem 3.2, (ii)], we arrive at

$$\|\,\mathrm{I}\,\|_{{}^cLM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)\to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)}$$

$$\approx \sup_{x\in(0,\infty)} \varphi_1(x) \left(\int_0^\infty \mathcal{V}(t,x)^{p_1\to\theta_2} d\Big(-\|\omega_2\|_{\theta_2,(t,\infty)}^{p_1\to\theta_2} \Big) \right)^{\frac{1}{p_1\to\theta_2}}.$$

$\square$

Remark 4. In view of Remark 3, if

$$\limsup_{t\to 0+} \widetilde{V}(t)\|\omega_1\|_{\theta_1,(0,t)}^{-1} = \limsup_{t\to+\infty} \widetilde{V}(t)\|\omega_1\|_{\theta_1,(0,t)}$$

$$= \limsup_{t\to 0+} \|\omega_1\|_{\theta_1,(0,t)} = \limsup_{t\to+\infty} \|\omega_1\|_{\theta_1,(0,t)}^{-1} = 0,$$

then $\varphi_1 \in Q_{\widetilde{V}^{\frac{1}{p_1\to p_2}}}$.

Proof of Theorem 2.5. By Lemma 3.1, applying [27, Theorem 4.2, (c)], we have that

$$\|\,\mathrm{I}\,\|_{{}^cLM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)\to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)}$$

$$\approx \|\omega_1\|_{\theta_1,(0,\infty)}^{-1} \left\{ \sup_{g\in\mathfrak{M}^+(0,\infty)} \frac{\|H^*g(|\cdot|)\|_{\frac{p_1}{p_1-p_2},(v_1^{-1}v_2)^{p_2},\mathbb{R}^n}}{\|g\|_{\frac{\theta_2}{\theta_2-p_2},\omega_2^{-p_2},(0,\infty)}} \right\}^{\frac{1}{p_2}}$$

$$+ \left\{ \sup_{g\in\mathfrak{M}^+(0,\infty)} \frac{\left(\int_0^\infty \|H^*g(|\cdot|)\|_{\frac{p_1}{p_1-p_2},(v_1^{-1}v_2)^{p_2},B(0,t)}^{\frac{\theta_1}{\theta_1-p_2}} d\Big(-\|\omega_1\|_{\theta_1,(0,t)}^{-\frac{\theta_1 p_2}{\theta_1-p_2}} \Big) \right)^{\frac{\theta_1-p_2}{\theta_1}}}{\|g\|_{\frac{\theta_2}{\theta_2-p_2},\omega_2^{-p_2},(0,\infty)}} \right\}^{\frac{1}{p_2}}.$$

By using polar coordinates, we have that

$$\|\,\mathrm{I}\,\|_{{}^cLM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)\to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)}$$

$$\approx \|\omega_1\|_{\theta_1,(0,\infty)}^{-1} \left\{ \sup_{g\in\mathfrak{M}^+(0,\infty)} \frac{\|H^*g\|_{\frac{p_1}{p_1-p_2},\tilde{v}^{\frac{p_1-p_2}{p_1}},(0,\infty)}}{\|g\|_{\frac{\theta_2}{\theta_2-p_2},\omega_2^{-p_2},(0,\infty)}} \right\}^{\frac{1}{p_2}}$$

$$+ \left\{ \sup_{g\in\mathfrak{M}^+(0,\infty)} \frac{\left(\int_0^\infty \|H^*g\|_{\frac{p_1}{p_1-p_2},\tilde{v}^{\frac{p_1-p_2}{p_1}},(0,t)}^{\frac{\theta_1}{\theta_1-p_2}} d\Big(-\|\omega_1\|_{\theta_1,(0,t)}^{-\frac{\theta_1 p_2}{\theta_1-p_2}} \Big) \right)^{\frac{\theta_1-p_2}{\theta_1}}}{\|g\|_{\frac{\theta_2}{\theta_2-p_2},\omega_2^{-p_2},(0,\infty)}} \right\}^{\frac{1}{p_2}}$$

$$:= C_1 + C_2.$$

Assume first that $p_1 \leq \theta_2$. By using the characterization of the boundedness of the operator H^* in weighted Lebesgue spaces (see, for instance, [28, 25]), we arrive at

$$C_1 \approx \|\omega_1\|_{\theta_1,(0,\infty)}^{-1} \sup_{t\in(0,\infty)} \widetilde{V}(t)\,\|\omega_2\|_{\theta_2,(t,\infty)}.$$

(i) Let $\theta_1 \leq \theta_2$. By applying [21, Theorem 3.1, (i)], we obtain that

$$C_2 \approx \sup_{x\in(0,\infty)} \varphi_2(x) \sup_{t\in(0,\infty)} \mathcal{V}(t,x)\,\|\omega_2\|_{\theta_2,(t,\infty)}.$$

Consequently, the proof is completed in this case.

(ii) Let $\theta_2 < \theta_1$. By using [21, Theorem 3.1, (ii)], we have that

$$C_2 \approx \left(\int_0^\infty \varphi_2(x)^{\frac{\theta_1\to\theta_2\cdot\theta_1\to p_2}{\theta_2\to p_2}} \widetilde{V}(x)^{\theta_1\to p_2} \left(\sup_{t\in(0,\infty)} \mathcal{V}(t,x)\|\omega_2\|_{\theta_2,(t,\infty)} \right)^{\theta_1\to\theta_2} \right.$$
$$\left. \times d\left(-\|\omega_1\|_{\theta_1,(0,x)}^{-\theta_1\to p_2} \right) \right)^{\frac{1}{\theta_1\to\theta_2}},$$

and the statement follows in this case.

Let us now assume that $\theta_2 < p_1$. Then, using the characterization of the boundedness of the operator H^* in weighted Lebesgue spaces, we have that

$$C_1 \approx \|\omega_1\|_{\theta_1,(0,\infty)}^{-1} \left(\int_0^\infty \widetilde{V}(t)^{p_1\to\theta_2}\, d\left(-\|\omega_2\|_{\theta_2,(t,\infty)}^{p_1\to\theta_2} \right) \right)^{\frac{1}{p_1\to\theta_2}}.$$

(iii) Let $\theta_1 \leq \theta_2$, then [21, Theorem 3.1, (iii)] yields that

$$C_2 \approx \sup_{x\in(0,\infty)} \varphi_2(x) \left(\int_0^\infty \mathcal{V}(t,x)^{p_1\to\theta_2} d\left(-\|\omega_2\|_{\theta_2,(t,\infty)}^{p_1\to\theta_2} \right) \right)^{\frac{1}{p_1\to\theta_2}},$$

and these completes the proof in this case.

(iv) If $\theta_2 < \theta_1$, then on using [21, Theorem 3.1, (iv)], we arrive at

$$C_2 \approx \left(\int_0^\infty \varphi_2(x)^{\frac{\theta_1\to\theta_2\cdot\theta_1\to p_2}{\theta_2\to p_2}} \widetilde{V}(x)^{\theta_1\to p_2} \left(\int_0^\infty \mathcal{V}(t,x)^{p_1\to\theta_2} d\left(-\|\omega_2\|_{\theta_2,(t,\infty)}^{p_1\to\theta_2} \right) \right)^{\frac{\theta_1\to\theta_2}{p_1\to\theta_2}} \right.$$
$$\left. \times d\left(-\|\omega_1\|_{\theta_1,(0,x)}^{-\theta_1\to p_2} \right) \right)^{\frac{1}{\theta_1\to\theta_2}},$$

and in this case the proof is completed. □

Remark 5. Assume that $\varphi_2(x) < \infty,\ x > 0$. In view of Remark 2, if

$$\int_0^1 \left(\int_0^t \omega_1^{\theta_1} \right)^{-\frac{\theta_1}{\theta_1-p_2}} \omega_1^{\theta_1}(t)\,dt = \int_1^\infty \widetilde{V}(t)^{\frac{\theta_1 p_2}{\theta_1-p_2}} \left(\int_0^t \omega_1^{\theta_1} \right)^{-\frac{\theta_1}{\theta_1-p_2}} \omega_1^{\theta_1}(t)\,dt = \infty,$$

then $\varphi_2 \in Q_{\widetilde{V}^{\frac{1}{p_1\to p_2}}}$.

Proof of Theorem 2.6. By Lemma 3.1, applying [27, Theorem 4.2, (b)], we get that

$$\| \mathrm{I} \|_{{}^{c}LM_{p\theta_1,\omega_1}(\mathbb{R}^n,v_1)\to LM_{p\theta_2,\omega_2}(\mathbb{R}^n,v_2)}$$

$$= \left\{ \sup_{g\in\mathfrak{M}^+(0,\infty)} \frac{\displaystyle\sup_{t\in(0,\infty)} \|\omega_1\|_{\theta_1,(0,t)}^{-p} \|H^*g(|\cdot|)\|_{\infty,(v_1^{-1}v_2)^p,B(0,t)}}{\|g\|_{\frac{\theta_2}{\theta_2-p},\omega_2^{-p},(0,\infty)}} \right\}^{\frac{1}{p}}.$$

Recall that, whenever F,G are non-negative measurable functions on $(0,\infty)$ and F is non-increasing, then

$$\operatorname*{ess\,sup}_{t\in(0,\infty)} F(t)G(t) = \operatorname*{ess\,sup}_{t\in(0,\infty)} F(t) \operatorname*{ess\,sup}_{\tau\in(0,t)} G(\tau). \tag{3.5}$$

Observe that

$$\|H^*g(|\cdot|)\|_{\infty,(v_1^{-1}v_2)^p,B(0,t)} = \sup_{\tau\in(0,t)} \sup_{|y|=\tau} \left(v_1^{-1}(y)v_2(y)\right)^p H^*g(|y|) = \|H^*g\|_{\infty,\tilde{\tilde{v}},(0,t)} \tag{3.6}$$

holds for all $t > 0$, where $\tilde{\tilde{v}}(\tau) := \left(\sup_{|y|=\tau} v_1^{-1}(y)v_2(y)\right)^p$, $\tau > 0$.
By using (3.5), we get that

$$\| \mathrm{I} \|_{{}^{c}LM_{p\theta_1,\omega_1}(\mathbb{R}^n,v_1)\to LM_{p\theta_2,\omega_2}(\mathbb{R}^n,v_2)} = \left\{ \sup_{g\in\mathfrak{M}^+(0,\infty)} \frac{\displaystyle\sup_{t\in(0,\infty)} \|\omega_1\|_{\theta_1,(0,t)}^{-p} \|H^*g\|_{\infty,\tilde{\tilde{v}},(0,t)}}{\|g\|_{\frac{\theta_2}{\theta_2-p},\omega_2^{-p},(0,\infty)}} \right\}^{\frac{1}{p}}$$

$$= \left\{ \sup_{g\in\mathfrak{M}^+(0,\infty)} \frac{\|H^*g\|_{\infty,\|\omega_1\|_{\theta_1,(0,\cdot)}^{-p}\tilde{\tilde{v}}(\cdot),(0,\infty)}}{\|g\|_{\frac{\theta_2}{\theta_2-p},\omega_2^{-p},(0,\infty)}} \right\}^{\frac{1}{p}}.$$

By using the characterization of the boundedness of H^* in weighted Lebesgue spaces, we obtain that

$$\| \mathrm{I} \|_{{}^{c}LM_{p\theta_1,\omega_1}(\mathbb{R}^n,v_1)\to LM_{p\theta_2,\omega_2}(\mathbb{R}^n,v_2)}$$

$$\approx \sup_{t\in(0,\infty)} \|\omega_2\|_{\theta_2,(t,\infty)} \left(\sup_{s\in(0,t)} \|\omega_1\|_{\theta_1,(0,s)}^{-1} \tilde{\tilde{v}}(s)^{\frac{1}{p}} \right)$$

$$= \sup_{t\in(0,\infty)} \|\omega_2\|_{\theta_2,(t,\infty)} \left(\sup_{s\in(0,t)} \sup_{|y|=s} \|\omega_1\|_{\theta_1,(0,|y|)}^{-1} v_1^{-1}(y)v_2(y) \right)$$

$$= \sup_{t\in(0,\infty)} \|\omega_2\|_{\theta_2,(t,\infty)} \left(\sup_{x\in B(0,t)} \|\omega_1\|_{\theta_1,(0,|x|)}^{-1} v_1^{-1}(x)v_2(x) \right)$$

$$= \sup_{t\in(0,\infty)} \|\omega_2\|_{\theta_2,(t,\infty)} \left\| \|\omega_1\|_{\theta_1,(0,|\cdot|)}^{-1} \right\|_{\infty,v_1^{-1}v_2,B(0,t)}.$$

$\square$

Proof of Theorem 2.7. By Lemma 3.1, applying [27, Theorem 4.2, (d)], and using (3.6), we get

that

$$\| \, \mathrm{I} \, \|_{{}^cLM_{p\theta_1,\omega_1}(\mathbb{R}^n,v_1)\to LM_{p\theta_2,\omega_2}(\mathbb{R}^n,v_2)}$$

$$\approx \|\omega_1\|_{\theta_1,(0,\infty)}^{-1} \left\{ \sup_{g\in\mathfrak{M}^+(0,\infty)} \frac{\|H^*g\|_{\infty,\tilde{\tilde{v}},(0,\infty)}}{\|g\|_{\frac{\theta_2}{\theta_2-p},\omega_2^{-p},(0,\infty)}} \right\}^{\frac{1}{p}}$$

$$+ \left\{ \sup_{g\in\mathfrak{M}^+(0,\infty)} \frac{\left(\int_0^\infty \|H^*g\|_{\infty,\tilde{\tilde{v}},(0,t)}^{\frac{\theta_1}{\theta_1-p}} d\left(-\|\omega_1\|_{\theta_1,(0,t)}^{-\frac{\theta_1 p}{\theta_1-p}}\right)\right)^{\frac{\theta_1-p}{\theta_1}}}{\|g\|_{\frac{\theta_2}{\theta_2-p},\omega_2^{-p},(0,\infty)}} \right\}^{\frac{1}{p}}$$

$$:= \ C_3 + C_4.$$

Again, by using the characterization of the boundedness of H^* in weighted Lebesgue spaces, we obtain that

$$C_3 \approx \|\omega_1\|_{\theta_1,(0,\infty)}^{-1} \sup_{t\in(0,\infty)} \widetilde{V}(t)\|\omega_2\|_{\theta_2,(t,\infty)}.$$

(i) Let $\theta_1 \le \theta_2$, then by [22, Theorem 4.1], we have that

$$C_4 \approx \sup_{x\in(0,\infty)} \left(\widetilde{V}(x)^{\theta_1\to p} \int_x^\infty d\left(-\|\omega_1\|_{\theta_1,(0,t)}^{-\theta_1\to p}\right)\right.$$

$$\left. + \int_0^x \widetilde{V}(t)^{\theta_1\to p} d\left(-\|\omega_1\|_{\theta_1,(0,t)}^{-\theta_1\to p}\right)\right)^{\frac{1}{\theta_1\to p}} \|\omega_2\|_{\theta_2,(x,\infty)},$$

and the statement follows in this case.

(ii) Let $\theta_2 < \theta_1$, then [22, Theorem 4.4] yields that

$$C_4 \approx \left(\int_0^\infty \left(\int_x^\infty d\left(-\|\omega_1\|_{\theta_1,(0,t)}^{-\theta_1\to p}\right)\right)^{\frac{\theta_1\to\theta_2}{\theta_2\to p}} \left(\sup_{0<\tau\le x} \widetilde{V}(\tau)\|\omega_2\|_{\theta_2,(\tau,\infty)}\right)^{\theta_1\to\theta_2}\right.$$

$$\times d\left(-\|\omega_1\|_{\theta_1,(0,x)}^{-\theta_1\to p}\right)\bigg)^{\frac{1}{\theta_1\to\theta_2}}$$

$$+ \left(\int_0^\infty \left(\int_0^x \widetilde{V}(t)^{\theta_1\to p} d\left(-\|\omega_1\|_{\theta_1,(0,t)}^{-\theta_1\to p}\right)\right)^{\frac{\theta_1\to\theta_2}{\theta_2\to p}} \widetilde{V}(x)^{\theta_1\to p}\|\omega_2\|_{\theta_2,(t,\infty)}^{\theta_1\to\theta_2}\right.$$

$$\times d\left(-\|\omega_1\|_{\theta_1,(0,x)}^{-\theta_1\to p}\right)\bigg)^{\frac{1}{\theta_1\to\theta_2}},$$

and the proof is completed in this case. □

Acknowledgments

References

[1] Ts. Batbold, Y. Sawano, *Decompositions for local Morrey spaces,* Eurasian Math. J. 5 (2014), no. 3, 9–45.

[2] V.I. Burenkov, *Recent progress in studying the boundedness of classical operators of real analysis in general Morrey-type spaces. I,* Eurasian Math. J. 3 (2012), no. 3, 11–32.

[3] V.I. Burenkov, *Recent progress in studying the boundedness of classical operators of real analysis in general Morrey-type spaces. II,* Eurasian Math. J. 4 (2013), no. 1, 21–45.

[4] V.I. Burenkov, M.L. Goldman, *Necessary and sufficient conditions for the boundedness of the maximal operator from Lebesgue spaces to Morrey-type spaces,* Math. Inequal. Appl. 17 (2014), no. 2, 401–418.

[5] V.I. Burenkov, H.V. Guliyev, *Necessary and sufficient conditions for boundedness of the maximal operator in local Morrey-type spaces,* Studia Math. 163 (2004), no. 2, 157–176.

[6] V.I. Burenkov, H.V. Guliyev, V.S. Guliyev, *Necessary and sufficient conditions for the boundedness of fractional maximal operators in local Morrey-type spaces,* J. Comput. Appl. Math. 208 (2007), no. 1, 280–301.

[7] V.I. Burenkov, H.V. Guliyev, V.S. Guliyev, *On boundedness of the fractional maximal operator from complementary Morrey-type spaces to Morrey-type spaces,* The interaction of analysis and geometry, Contemp. Math. 424 (2007) Amer. Math. Soc. Providence, RI, 17–32.

[8] V.I. Burenkov, V.S. Guliyev, A. Serbetci, T.V. Tararykova, *Necessary and sufficient conditions for the boundedness of genuine singular integral operators in local Morrey-type spaces,* Eurasian Math. J. 1 (2010), no. 1, 32–53.

[9] V.I. Burenkov, A. Gogatishvili, V.S. Guliyev, R.Ch. Mustafayev, *Boundedness of the fractional maximal operator in local Morrey-type spaces,* Complex Var. Elliptic Equ. 55 (2010), no. 8–10, 739–758.

[10] V.I. Burenkov, A. Gogatishvili, V.S. Guliyev, R.Ch. Mustafayev, *Boundedness of the Riesz potential in local Morrey-type spaces,* Potential Anal. 35 (2011), no. 1, 67–87.

[11] V.I. Burenkov, P. Jain, T.V. Tararykova, *On boundedness of the Hardy operator,* Eurasian Math. J. 2 (2011), no. 1, 52-80.

[12] V.I. Burenkov, E.D. Nursultanov, *Description of interpolation spaces for local Morrey-type spaces,* Trudy Mat. Inst. Steklova RAS 269 (2010), Teoriya Funktsii i Differentsialnye Uravneniya, 52–62 (in Russian). English transl. in Proc. Steklov Inst. Math. 269 (2010), no. 1, 46–56.

[13] V.I. Burenkov, E.D. Nursultanov, D.K. Chiganbaeva, *Description of the interpolation spaces for a pair of local Morrey-type spaces and their generalizations,* Trudy Mat. Inst. Steklova RAS 284 (2014), 105–137 (in Russian). English transl. in Proc. Steklov Inst. Math. 284 (2014), 97–128.

[14] V.I. Burenkov, R. Oinarov, *Necessary and sufficient conditions for boundedness of the Hardy operator from a weghted Lebesque space to a Morrey-type space,* Math. Inequal. Appl. 16 (2013), no. 1, 1-19.

[15] G.B. Folland, *Real analysis,* Pure and Applied Mathematics (New York) 2nd ed., Modern techniques and their applications; A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1999.

[16] D. Gilbarg, N.S. Trudinger, *Elliptic partial differential equations of second order,* 2nd ed., Springer-Verlag, Berlin, 1983.

[17] A. Gogatishvili, R.Ch. Mustafayev, *Dual spaces of local Morrey-type spaces,* Czechoslovak Math. J. 61 (136) (2011), no. 3, 609–622.

[18] A. Gogatishvili, R.Ch. Mustafayev, *New pre-dual space of Morrey space,* J. Math. Anal. Appl. 397 (2013), no. 2, 678–692.

[19] A. Gogatishvili, R.Ch. Mustafayev, *Weighted iterated Hardy-type inequalities,* accepted in Math. Inequal. Appl. (2016).

[20] A. Gogatishvili, R.Ch. Mustafayev, *Iterated Hardy-type inequalities involving suprema,* accepted in Math. Inequal. Appl. (2016).

[21] A. Gogatishvili, R.Ch. Mustafayev, L.-E. Persson, *Some new iterated Hardy-type inequalities,* J. Funct. Spaces Appl. (2012) Art. ID 734194, 30.

[22] A. Gogatishvili, B. Opic, L. Pick, *Weighted inequalities for Hardy-type operators involving suprema,* Collect. Math. 57 (2006), no. 3, 227–255.

[23] V.S. Guliyev, *Integral operators on function spaces on the homogeneous groups and on domains in $\mathbb{R}^n$,* Doctor's degree dissertation. Mat. Inst. Steklov, Moscow, 1994 (in Russian).

[24] V.S. Guliev, R.Ch. Mustafaev, *Fractional integrals in spaces of functions defined on spaces of homogeneous type,* Russian, with English and Russian summaries, Anal. Math. 24 (1998), no. 3, 181–200.

[25] A. Kufner, L.-E. Persson, *Weighted inequalities of Hardy type,* World Scientific Publishing Co. Inc., River Edge, NJ, 2003.

[26] C.B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations,* Trans. Amer. Math. Soc. 43 (1938), no. 1, 126–166.

[27] R.Ch. Mustafayev, T. Ünver, *Embeddings between weighted local Morrey-type spaces and weighted Lebesgue spaces,* J. Math. Inequal. 9 (2015), no. 1, 277–296.

[28] B. Opic, A. Kufner, *Hardy-type inequalities,* Pitman Research Notes in Mathematics Series, 219, Longman Scientific & Technical, Harlow, 1990.

Amiran Gogatishvili
Institute of Mathematics
Academy of Sciences of the Czech Republic
Žitná 25,
115 67 Praha 1, Czech Republic
E-mails: gogatish@math.cas.cz

Rza Mustafayev
Institute of Mathematics and Mechanics
Academy of Sciences of Azerbaijan
B. Vahabzade St. 9,
Baku, AZ 1141, Azerbaijan
Department of Mathematics
Faculty of Science and Arts
Kirikkale University,
71450 Yahsihan, Kirikkale, Turkey
E-mail: rzamustafayev@gmail.com

Tuğçe Ünver
Department of Mathematics
Faculty of Science and Arts
Kirikkale University,
71450 Yahsihan, Kirikkale, Turkey
E-mail: tugceunver@gmail.com