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YESMUKHANBET SAIDAKHMETOVICH SMAILOV

(to the 70th birthday)

On October 18, 2016 was the 70th birthday of Yesmukhabet Saidakhmetovich Smailov, member of the Editorial Board of the Eurasian Mathematical Journal, director of the Institute of Applied Mathematics (Karaganda), doctor of physical and mathematical sciences (1997), professor (1993), honoured worker of the E.A. Buketov Karaganda State University, honorary professor of the Sh. Valikanov Kokshetau State University, honorary citizen of the Tarbagatai district of the East-Kazakhstan region. In 2011 he was awarded the Order "Kurmet" (= "Honour").

Y.S. Smailov was born in the Kyzyl-Kesek village (the Aksuat district of the Semipalatinsk region of the Kazakh SSR). He graduated from the S.M. Kirov Kazakh State University (Almaty) in 1968 and in 1971 he completed his postgraduate studies at the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR (Almaty). Starting with 1972 he worked at the E.A. Buketov Karaganda State University (senior lecturer, associate professor, professor, head of the Department of Mathematical Analysis, dean of the Mathematical Faculty; from 2004 director of the Institute of Applied Mathematics).

In 1999 the American Biographical Institute declared professor Smailov "Man of the Year" and published his biography in the "Biographical encyclopedia of professional leaders of the Millennium".

Professor Smailov is one of the leading experts in the theory of functions and functional analysis and a major organizer of science in the Republic of Kazakhstan. He had a great influence on the formation of the Mathematical Faculty of the E.A. Buketov Karaganda State University and he made a significant contribution to the development of mathematics in Central Kazakhstan. Due to the efforts of Y.S. Smailov, in Karaganda an actively operating Mathematical School on the function theory was established, which is well known in Kazakhstan and abroad.

He has published more than 140 scientific papers, two textbooks for students and one monograph. 10 candidate of sciences and 4 doctor of sciences dissertations have been defended under his supervision.

Research interests of Professor Smailov are quite broad: the embedding theory of function spaces; approximation of functions of real variables; interpolation of function spaces and linear operators; Fourier series for general orthogonal systems; Fourier multipliers; difference embedding theorems.

The Editorial Board of the Eurasian Mathematical Journal congratulates Yesmukhanbet Saidakhmetovich Smailov on the occasion of his 70th birthday and wishes him good health and new achievements in mathematics and mathematical education.

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ON THE SOLVABILITY OF PARABOLIC FUNCTIONAL DIFFERENTIAL EQUATIONS IN BANACH SPACES

A.M. Selitskii

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Key words: functional differential equations, Lipschitz domain, Banach spaces.

AMS Mathematics Subject Classification: 39A14.

Abstract. In this paper, a parabolic functional differential equation is considered in the spaces $C(0,T;H_p^1(Q))$ for p close to 2. The transformations of the space argument are supposed to be multiplicators of the Sobolev spaces with a small smoothness exponent. The machinery of the investigation is based on the semigroup theory. In particular, it is proved that the elliptic part of the operator is a generator of a strongly continuous semigroup.

1 Introduction

We consider the second boundary-value problem for the following parabolic functional differential equation

$$u_t - \sum_{i,j=1}^{n} \left(A_{ij} u_{x_j}\right)_{x_i} + \sum_{i=1}^{n} B_i u_{x_i} + Cu = f(x,t) \qquad ((x,t) \in Q_T) \tag{1.1}$$

in a bounded domain $Q \subset \mathbb{R}^n$ with a Lipschitz boundary ∂Q, where the operators A_{ij}, B_i, and C are bounded in $L_2(Q)$, with the boundary condition

$$\sum_{i,j=1}^{n} A_{ij} u_{x_j} \cos(\nu,\, x_i) = 0 \qquad ((x,t) \in \Gamma_T)\,, \tag{1.2}$$

and the initial condition

$$u|_{t=0} = \varphi(x) \qquad (x \in Q)\,, \tag{1.3}$$

where $Q_T = Q \times (0,\, T)$, $0 < T < \infty$, $\Gamma_T = \partial Q \times (0,\, T)$, ν is the external unit normal to Γ_T (it exists at almost every point of Γ_T), $f \in L_2(Q_T)$, and $\varphi \in L_2(Q)$.

For differential-difference operators this problem was solved in $L_p(0,T;H_2^1(Q))$ $(1 < p < \infty)$ (see [8]). In this article we suppose only that operators A_{ij} are bounded in $H^s(Q)$ for small $|s|$ and all the coefficients belong to $L_p(Q)$ for p close to 2.

Note that the modeling of the optical system with 2D feedback leads to a functional differential parabolic equation with transformation of argument in the unknown function, see, e.g., [11] and literature therein.

2 Weak and strong solvability

Let $H^1(Q)$ be the Sobolev space of complex-valued functions belonging to $L_2(Q)$ having all generalized derivatives of the first order belonging to $L_2(Q)$.

Introduce the sesquilinear form $\Phi(v,\, w)$ in $L_2(Q)$ with the domain $H^1(Q)$ by the formula

$$\Phi(v,\, w) = \sum_{i,j=1}^{n} \big(A_{ij} v_{x_j},\, w_{x_i}\big)_{L_2(Q)} + \sum_{i=1}^{n} (B_i v_{x_i},\, w)_{L_2(Q)} + (Cv,\, w)_{L_2(Q)}\,. \tag{2.1}$$

By assumption, the operators A_{ij}, B_i, $C\colon L_2(Q) \to L_2(Q)$ are bounded. Therefore, it follows that there exists a constant $c_0 > 0$ such that

$$|\Phi(v,\, w)| \le c_0 \|v\|_{H^1(Q)} \|w\|_{H^1(Q)} \qquad (v, w \in H^1(Q)). \tag{2.2}$$

Since the sesquilinear form $\Phi(v,\, w)$ is continuous with respect to w in $H^1(Q)$, there exists a linear bounded operator $A\colon H^1(Q) \to [H^1(Q)]' = \widetilde{H}^{-1}(Q)$, such that

$$\langle Av,\, \overline{w}\rangle = \Phi(v,\, w) \qquad (v, w \in H^1(Q)), \tag{2.3}$$

where $\langle\cdot,\cdot\rangle$ denotes the dual pairing with respect to the scalar product in $L_2(Q)$.

We suppose that the form $\Phi(v,\, w)$ is coercive, i.e., there exist numbers $c_1 > 0$ and $c_2 \ge 0$ such that

$$\mathrm{Re}\ \Phi(v,\, v) \ge c_1 \|v\|^2_{H^1(Q)} - c_2 \|v\|^2_{L_2(Q)} \qquad (v \in H^1(Q)). \tag{2.4}$$

We can assume that $c_2 = 0$ in inequality (2.4). Otherwise, we set $u = z\, e^{c_2 t}$ that transforms operator A to $A + c_2 I$.

Denote by $\mathcal{A}$ the operator A with the domain $D(\mathcal{A}) = \{u \in H^1(Q) : Au \in L_2(Q)\}$ with the graph norm.

To formulate the definition of a weak solution of problem (1.1)–(1.3) we introduce the following space

$$W(A) = \Big\{u \in L_p\big(0,\, T;\, H^1(Q)\big) : u' \in L_p\big(0,\, T;\, \widetilde{H}^{-1}(Q)\big)\Big\}. \tag{2.5}$$

Definition 1. A function $u \in W(A)$ is said to be a weak solution of problem (1.1)–(1.3) if it satisfies the equation

$$\frac{du}{dt} + Au = f \qquad \text{for almost all } t \in (0,\ T) \tag{2.6}$$

and the initial condition

$$u|_{t=0} = \varphi. \tag{2.7}$$

Note that from [4, Ch. XVIII, §1, Sec. 4, Proposition 9] it follows that for a function $u \in W(A)$ the value $u(0) \in \widetilde{H}^{-1}(Q)$ is defined.

We need the notation of Besov spaces. Consider the modulus of smoothness of the second order for a function $f \in L_p(Q)$:

$$\omega_p^2(f, t) = \sup_{|h| \le t} \big\| f(x + 2h) - 2f(x+h) + f(x)\big\|_{L_p(Q)}.$$

Let $1 \le p, \theta < \infty$ and $s > 0$ and $s = \tilde{s} + \alpha$, where $\tilde{s}$ is an integer and $0 < \alpha \le 1$.

It is said that a function $f \in L_p(\mathbb{R}^n)$ belongs to the function space $B^s_{p,\theta}(\mathbb{R}^n)$ if it has all partial derivatives up to the $\tilde{s}$th order and

$$\|f\|_{B^s_{p,\theta}(\mathbb{R}^n)} = \|f\|_{W^{\tilde{s}}_p(\mathbb{R}^n)} + \left(\int_0^\infty \left|\frac{\omega_p^2(f^{(\tilde{s})}, t)}{t^\alpha}\right|^\theta \frac{dt}{t}\right)^{1/\theta} < \infty. \tag{2.8}$$

Here $W^{\tilde{s}}_p(\mathbb{R}^n)$ is the Sobolev space. The space $B^s_{p,\theta}(Q)$ is defined as restriction of the space $B^s_{p,\theta}(\mathbb{R}^n)$ on Q with inf-norm (see [12, Sec. 4.2.1]). Set $B^0_{p,p}(Q) = L_p(Q)$ and let $\widetilde{B}^{-s}_{p',\theta'}(Q)$ denote the dual of $B^s_{p,\theta}(Q)$ (here $1/p + 1/p' = 1/\theta + 1/\theta' = 1$).

Remark 1. *We can use interpolation theorems for functions defined on $\mathbb{R}^n$ because Theorem 1 from Sec. 4.3.1 in [12] is true for Lipschitz domain as well by virtue of existence of an extension operator independent of p, θ, and s (see [7]). See the definitions of real and complex interpolations in [12].*

Theorem 2.1. *Let the form $\Phi(v, w)$ be coercive with $c_2 = 0$, and $f \in L_p(0, T; \widetilde{H}^{-1}(Q))$ $(1 < p < \infty)$. Then problem (1.1)–(1.3) has a unique weak solution $u \in W(A)$ if and only if $\varphi \in B^{1-\frac{2}{p}}_{2,p}(Q)$ for $p \ge 2$ and $\varphi \in \widetilde{B}^{1-\frac{2}{p}}_{2,p}(Q)$ for $p < 2$. Moreover, this solution is given by the formula*

$$u(x,\, t) = T_t\varphi(x) + \int\limits_0^t T_{t-s} f(x,\, s)\, ds, \tag{2.9}$$

where $\{T_t\}$ $(t \ge 0)$ is the analytic semigroup generated by the operator $-A$.

Proof. By virtue of [5, Theorem 1.55], the operator $-A$ is a generator of the strongly continuous analytic semigroup T_t. Then the statement of the theorem follows by Theorems 3.6 and 3.7 in [3] for $\varphi \in \big(\widetilde{H}^{-1}(Q),\, H^1(Q)\big)_{1-\frac{1}{p},p}$ (see. [3, Ch. 1, Sec. 3]).

By [9, Corollary 3.3] it follows that $D(\mathcal{A}^{1/2}) = H^1(Q)$ that is equal to the identity $D(A^\alpha) = [\widetilde{H}^{-1}(Q),\, H^1(Q)]_\alpha$ for $0 \le \alpha \le 1$ (see [1, Theorem 3.5]).

For $p \ge 2$, using formula (2) of Sec. 1.15.4 in [12], we obtain

$$\begin{aligned}(\widetilde{H}^{-1}(Q),\, H^1(Q))_{1-\frac{1}{p},p} &= (D(A^0), D(A))_{1-\frac{1}{p},p} = \\ &= (D(A^{1/2}), D(A))_{1-\frac{2}{p},p} = (L_2(Q), H^1(Q))_{1-\frac{2}{p},p} = B^{1-\frac{2}{p}}_{2,p}(Q).\end{aligned} \tag{2.10}$$

For $p < 2$, using Theorem 2 of Sec. 1.10.3 and the Theorem on duality of Sec. 1.11.2 in [12], we obtain

$$\begin{aligned}(\widetilde{H}^{-1}(Q),\, H^1(Q))_{1-\frac{1}{p},p} &= (H^1(Q), \widetilde{H}^{-1}(Q))_{\frac{1}{p},p} = (\widetilde{H}^{-1}(Q),\, H^1(Q))'_{\frac{1}{p},p'} = \\ &= \Big([\widetilde{H}^{-1}(Q),\, H^1(Q)]_{1/2}, [\widetilde{H}^{-1}(Q),\, H^1(Q)]_1\Big)'_{\frac{2}{p}-1,p'} = \\ &= (L_2(Q), H^1(Q))'_{\frac{2}{p}-1,p'} = \Big(B^{\frac{2}{p}-1}_{2,p'}(Q)\Big)' = \widetilde{B}^{1-\frac{2}{p}}_{2,p}(Q).\end{aligned} \tag{2.11}$$

$\square$

Remark 2. By virtue of [10, Theorem 3.2], if $p = 2$ and $f \in L_2(Q_T)$, then Definition 1 is equivalent to the definition of a weak solution via integral equality.

Remark 3. We have to define boundary condition (1.2) for a function $u(\cdot, t) \in H^1(Q)$. Rewrite it in the form

$$T^+u = 0 \qquad \text{on } \Gamma_T. \tag{2.12}$$

The value of operator $T^+ \colon H^1(Q) \to \widetilde{H}^{-1/2}(\partial Q))$ (where $\widetilde{H}^{-1/2}(\partial Q)$ denotes the dual to $H^{1/2}(\partial Q) = [L_2(\partial Q), H^1(\partial Q)]_{1/2}$) on a function $u \in W_p(A)$ is defined by the Green formula

$$\left\langle f(\cdot,\, t),\, \overline{w}\right\rangle = \left\langle u_t,\, \overline{w}\right\rangle - \left\langle T^+u,\, \overline{w}|_{\partial Q}\right\rangle_\Gamma + \left\langle Au(\cdot, t),\, \overline{w}\right\rangle, \qquad w \in H^1(Q). \tag{2.13}$$

If the function u is sufficiently smooth (that in case of differeintial-difference operators, as shown in [10], takes place near a smooth part of the boundary), then

$$T^+u = \sum_{ij=1}^{n} A_{ij} u_{x_j} \cos(\nu, x_i). \tag{2.14}$$

Introduce the Hilbert space

$$W_p(\mathcal{A}) = \left\{ w \in L_p(0,\, T;\, D(\mathcal{A})) : w_t \in L_p(0,\, T;\, L_2(Q)) \right\}.$$

Definition 2. A function $u \in W_p(\mathcal{A})$ is called a strong solution of problem (1.1)–(1.3) if it satisfies the equation

$$\frac{du}{dt} + \mathcal{A}u(\cdot, t) = f \qquad \text{for almost all } t \in (0,\ T) \tag{2.15}$$

and the initial condition

$$u|_{t=0} = \varphi. \tag{2.16}$$

Theorem 2.2. *Let the form $\Phi[v, w]$ be coercive with $c_2 = 0$ and $1 < p \le 2$. Then for any $f \in L_p(0,\, T;\, L_2(Q))$ and $\varphi \in B_{2,p}^{2-\frac{2}{p}}(Q)$ problem (1.1)–(1.3) has a unique strong solution defined by formula (2.9).*

Proof. By virtue of [10, Theorem 4.1], the operator $-\mathcal{A}$ is a generator of the strongly continuous analytic semigroup $\mathcal{T}_t$, and $\mathcal{T}_t u = T_t u$ for $u \in L_2(Q)$ (see [5, Theorem 1.55]). Then the statement of the theorem follows by Theorems 3.6 and 3.7 in [3] for $\varphi \in \big(L_2(Q),\, D(\mathcal{A})\big)_{1-\frac{1}{p},p}$ (see. [3, Ch. 1, Sec. 3]).

By virtue of Theorem 2 of Sec. 1.10.3 in [12],

$$\begin{aligned}(L_2(Q),\, D(\mathcal{A}))_{1-\frac{1}{p},p} &= \Big(L_2(Q),\, [L_2(Q),\, D(\mathcal{A})]_{1/2}\Big)_{2-\frac{2}{p},p} = \\ &= \big(L_2(Q),\, H^1(Q)\big)_{2-\frac{2}{p},p} = B_{2,p}^{2-\frac{2}{p}}(Q).\end{aligned} \tag{2.17}$$

□

Remark 4. As it follows from the proof, the restriction $p \le 2$ can be omitted, but we should consider then the space $\big(L_2(Q),\, D(\mathcal{A})\big)_{1-\frac{1}{p},p}$ instead of $\big(L_2(Q),\, H^1(Q)\big)_{2-\frac{2}{p},p}$. Moreover, in general (without an assumption on smoothness of a solution), the description of the domain $D(\mathcal{A})$ is unknown. So we use the fact that $D(\mathcal{A}^{1/2}) = H^1(Q)$.

3 Generalization to Banach spaces

We need to consider the space $H_p^1(\mathbb{R}^n)$. It can be defined for $1 < p < \infty$ as the space of all distributions in S' with finite norm

$$\|u\|_{H_p^1(\mathbb{R}^n)} = \|\Lambda u\|_{L_p(\mathbb{R}^n)}, \tag{3.1}$$

where $\Lambda = F^{-1}(1+|\xi|^2)^{1/2}F$, here F is the Fourier transform in the sense of distributions. The space $H_p^1(Q)$ is defined as a restriction of the space $H_p^1(\mathbb{R}^n)$ on Q with the inf-norm. For details see book [1].

Suppose that $u \in H_p^1(Q)$ and $v \in H_q^1(Q)$ for $1 < p < \infty$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$. If operators A_{ij}, B_i, and C are bounded in $L_p(Q)$, then there exists $c_3 > 0$ such that

$$|\Phi(u, v)| \le c_3 \|u\|_{H_p^1(Q)} \|v\|_{H_q^1(Q)} \quad (u \in H_p^1(Q),\ v \in H_q^1(Q)).$$

Denote the corresponding operator by $A_p\colon H_p^1(Q) \to \widetilde{H}_p^{-1}(Q) = \big[H_q^1(Q)\big]'$. It can be described in the following way. For each $u \in H_p^1(Q)$ the form $\Phi(u,\, v)$ defines an antilinear bounded functional on $H_q^1(Q)$, i.e., there exists $f_u \in \widetilde{H}_p^{-1}(Q)$ such that $\Phi(u,\, v) = \langle f_u,\, \overline{v}\rangle = (\mathcal{E}_0 f_u,\, \mathcal{E} v)_{\mathbb{R}^n}$, where $\mathcal{E}_0$ is an extension operator by zero and $\mathcal{E}$ is an extension operator [7], the form $(\cdot,\cdot)_{\mathbb{R}^n}$ is the extension of the form $(\cdot,\cdot)_{L_2(\mathbb{R}^n)}$ on $\widetilde{H}_p^{-1}(\mathbb{R}^n) \times H_q^1(\mathbb{R}^n)$. We used the fact that elements from $\widetilde{H}_p^{-1}(Q)$ are supported in $\overline{Q}$. Then we set $A_p u = f_u$. The operator A_p is bounded:

$$\|A_p u\|_{H_p^{-1}(Q)} \le \sup_{v \ne 0} \frac{|\Phi(u,\, v)|}{\|v\|_{H_q^1(Q)}} \le c_3 \|u\|_{H_p^1(Q)}.$$

Obviously, $A_p u = Au$ if $u \in H_p^1(Q)$ $(p \le 2)$. We consider the following problem

$$u_t + A_p = f(x,\, t), \tag{3.2}$$

$$u|_{t=0} = \varphi(x). \tag{3.3}$$

Definition 3. A function $u \in C([0,T]; H_p^{-1}(Q)) \cap C^1((0,T); H_p^{-1}(Q))$ is called a classical solution of (3.2)-(3.3) if $u(t,\cdot) \in H_p^1(Q)$ for $0 < t < T$ and equalities (3.2) and (3.3) are satisfied on $[0,\, T)$.

Theorem 3.1. *Let $f \in L_1(0,\, T; H_p^{-1}(Q))$ be Lipschitz continuous on $[0,\, T]$ and $\varphi \in H_p^1(Q)$. Then there exists $\delta > 0$ such that for $\left|\dfrac{1}{2} - \dfrac{1}{p}\right| < \delta$ problem (3.2)-(3.3) has a unique classical solution.*

Proof. Note that the spaces $H_p^1(Q)$ form an interpolation scale: $\big[H_{p_1}^1(Q),\, H_{p_2}^1(Q)\big]_\theta = H_p^1(Q)$, where $\dfrac{1}{p} = \dfrac{1-\theta}{p_1} + \dfrac{\theta}{p_2}$. Operator A_p is bounded, for example, for $\dfrac{1}{2} \le \dfrac{1}{p} \le \dfrac{3}{2}$, and $A_2 = A$ is invertible (see Sec. 2). Applying the Shneiberg theorem on extrapolation of invertibility (see, e.g., Theorem 13.7.3 in [1]), we obtain the existence of such $0 < \delta < \dfrac{1}{2}$, that operator A_p remains invertible for $\left|\dfrac{1}{p} - \dfrac{1}{2}\right| < \delta$.

The same theorem was used in [2] to prove the following estimate

$$\|u\|_{H^1_p(Q)} + |\lambda|\|u\|_{H^{-1}_p(Q)} \le c_4\|f\|_{H^{-1}_p(Q)} \tag{3.4}$$

for $(A_p - \lambda I)u = f$, where $c_4 > 0$ does not depend of f and $\lambda \in \left\{\mu \in \mathbb{C} : |\arg\mu| > \dfrac{\pi}{2} - \delta_1\right\} \cup B_{\varepsilon_1}(0)$ for some $\varepsilon_1 > 0$ and $\delta_1 > 0$. Another proof was given in [1, Sec. 17.4].

From estimate (3.4) it follows that operator $-A_p$ is a generator of an analytic semigroup (see, e.g., Theorem 5.2 in [6]). Therefore, we can apply Corollary 2.11 in [6]. $\square$

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