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YESMUKHANBET SAIDAKHMETOVICH SMAILOV

(to the 70th birthday)

On October 18, 2016 was the 70th birthday of Yesmukhabet Saidakhmetovich Smailov, member of the Editorial Board of the Eurasian Mathematical Journal, director of the Institute of Applied Mathematics (Karaganda), doctor of physical and mathematical sciences (1997), professor (1993), honoured worker of the E.A. Buketov Karaganda State University, honorary professor of the Sh. Valikanov Kokshetau State University, honorary citizen of the Tarbagatai district of the East-Kazakhstan region. In 2011 he was awarded the Order "Kurmet" (= "Honour").

Y.S. Smailov was born in the Kyzyl-Kesek village (the Aksuat district of the Semipalatinsk region of the Kazakh SSR). He graduated from the S.M. Kirov Kazakh State University (Almaty) in 1968 and in 1971 he completed his postgraduate studies at the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR (Almaty). Starting with 1972 he worked at the E.A. Buketov Karaganda State University (senior lecturer, associate professor, professor, head of the Department of Mathematical Analysis, dean of the Mathematical Faculty; from 2004 director of the Institute of Applied Mathematics).

In 1999 the American Biographical Institute declared professor Smailov "Man of the Year" and published his biography in the "Biographical encyclopedia of professional leaders of the Millennium".

Professor Smailov is one of the leading experts in the theory of functions and functional analysis and a major organizer of science in the Republic of Kazakhstan. He had a great influence on the formation of the Mathematical Faculty of the E.A. Buketov Karaganda State University and he made a significant contribution to the development of mathematics in Central Kazakhstan. Due to the efforts of Y.S. Smailov, in Karaganda an actively operating Mathematical School on the function theory was established, which is well known in Kazakhstan and abroad.

He has published more than 140 scientific papers, two textbooks for students and one monograph. 10 candidate of sciences and 4 doctor of sciences dissertations have been defended under his supervision.

Research interests of Professor Smailov are quite broad: the embedding theory of function spaces; approximation of functions of real variables; interpolation of function spaces and linear operators; Fourier series for general orthogonal systems; Fourier multipliers; difference embedding theorems.

The Editorial Board of the Eurasian Mathematical Journal congratulates Yesmukhanbet Saidakhmetovich Smailov on the occasion of his 70th birthday and wishes him good health and new achievements in mathematics and mathematical education.

Short communications

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ON STABILITY OF SOLUTIONS TO CERTAIN DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS RIGHT-HAND SIDES

V.I. Bezyaev

Communicated by K.N. Ospanov

Key words: stability and instability of solutions, differential equations and systems with discontinuous right-hand sides, normal matrices.

AMS Mathematics Subject Classification: 34D20.

Abstract. For a class of nonlinear nonautonomous systems of differential equations with discontinuous right-hand sides effective conditions for stability and instability of their solutions are given.

1 Introduction

Differential equations and systems with discontinuous right-hand sides have been actively studied in connection with important applications in engineering [2]. Questions of stability of solutions of quasi-linear ordinary differential equations (ODE) and systems, including quasi-linear ODE and systems with discontinuous right-hand side, are considered, for example, in $[1, 4-6, 8]$. In this paper, for quasi-linear systems with normal $(A^*(x,t)A(x,t) \equiv A(x,t)A^*(x,t))$ defining matrices (discontinuous in general) the method (see e.g. [3], [7]), which gives constructive conditions of stability and instability of the solutions of such systems, is further developed. The applicability of the proposed approach to the analysis of a wider class of quasilinear systems is also proved, an analogue of the superposition principle for quasilinear systems is formulated. The proposed statements are obtained without using the method of Lyapunov functions. The application of that method the case under consideration is very problematic.

A piecewise continuous function $f(x,t)$ (which can be a matrix one) in a bounded domain G in $\mathbb{R}^{n+1}_{x,t}$ is a function continuous up to the boundary on each of the domains $G_i\,(i = \overline{1,k})$, for which

$$G = \left(\bigcup_{i=1}^{k} G_i\right) \bigcup M, \quad G_i \bigcap G_j = \varnothing \quad \text{if} \quad i \neq j, \quad M \subset \bigcup_{i=1}^{k} \partial G_i, \quad meas M = 0$$

($meas$ - the Lebesgue measure). If a domain G is unbounded, then it is additionally required that the each bounded subdomain of G has common points only with a finite family of the domains G_i ([4], § 4). The most common case is the one when the set M of discontinuity points of the function $f(x,t)$ consists of a finite family of hypersurfaces.

Moreover, we assume that for each domain G_i for almost all t, the cross section of the boundary by the plane $t = const$ coincides with the boundary of the cross section of the domain G_i by the same plane.

We consider the system of differential equations

$$\dot{x} = f(x,t) \tag{1.1}$$

with piecewise continuous vector-functions $f(x,t)$ in the domain G. Following A.F. Filippov ([5], §4, part 2a) it is extended to the differential inclusion

$$\dot{x} \in F(x,t), \tag{1.2}$$

where a multi-valued function $F(x,t)$ is defined for almost all t and all x, for which $(x,t) \in G$. In this case $F(x,t)$ is the smallest convex closed set containing all the limit values of the vector-function $f(\tilde{x},t)$, when $(\tilde{x},t) \notin M,\ \tilde{x} \to x,\ t = const$ and the multivalued function $F(x,t)$ is β-continuous (semi-continuous from above with respect to the inclusion) in the domain G. These properties of the function $F(x,t)$ ensure the existence of solution to the problem $\dot{x} \in F(x,t),\ x(t_0) = x^0$ in some neighborhood of any point $(x^0, t_0) \in G$ and the possibility of extending it up to the boundary of a closed bounded domain $D \subset G$ ([4]).

A solution to system (1.1) is called a solution to differential inclusion (1.2).

2 Main results

Theorem 1. *Let for the nonautonomous quasilinear system*

$$\dot{x} = A(x,t)x \tag{2.1}$$

where the matrix $A(x,t)$ is piecewise continuous in the domain $\Omega = \{(x,t) \in \mathbb{R}^{n+1} : |x| < \delta,\ t > 0\}$ and normal in $\Omega \backslash M$ (M is the set of all points of discontinuity of $A(x,t)$), the spectrum $\{\lambda_j(x,t)\}_1^n$ of $A(x,t)$ satisfies in $\Omega \backslash M$ the inequalities

$$Re\lambda_j(x,t) \le \varkappa(x,t) \quad \text{or} \quad Re\lambda_j(x,t) \ge \nu(x,t) \qquad (\, j = \overline{1,n}\,),$$

where $\varkappa$ and ν are continuous functions in Ω.

Then for any solution $x(t)$ of system (2.1) for almost all $t \in I \subseteq [0,\infty)$, where I is the interval of its existence, one of the inequalities

$$\frac{d|x|^2}{dt} \le 2\varkappa(x,t)|x|^2 \quad \text{or} \quad \frac{d|x|^2}{dt} \ge 2\nu(x,t)|x|^2$$

is satisfied.

The proof of Theorem 1 is given in [3].

Theorem 2. *Let the assumptions of Theorem 1 be satisfied. Moreover, let*

$$1)\ Re\lambda_j(x,t) \le \varphi(t)\mu(|x|) \text{ or } \ 2)\ Re\lambda_j(x,t) \ge \varphi(t)\mu(|x|) \quad (j = \overline{1,n},\ (x,t) \in \Omega \backslash M), \tag{2.2}$$

where the function $\varphi(t)$ is piecewise continuous for $t \geq 0$ and the function $\mu(z)$ is continuous, positive for $0 < z < \delta$ and satisfies the condition

$$\int_0^a \frac{dz}{z\mu(\sqrt{z})} = +\infty \quad (0 < a < \delta^2).$$

Then the solution $x(t) \equiv 0$ of system (2.1) is:
1) asymptotically stable if $\lim\limits_{t\to+\infty} b(t) = -\infty$, where

$$b(t) \equiv \int_0^t \varphi(s)ds,$$

and stable if $\overline{\lim\limits_{t\to+\infty}} b(t) < +\infty$ or
2) unstable if $\overline{\lim\limits_{t\to+\infty}} b(t) = +\infty$.

Proof. Let

$$H(u) = -\int_u^a \frac{dz}{z\mu(\sqrt{z})}, \quad 0 < u < a.$$

Then $H(u)$ is a continuous increasing negative function for all $0 < u < a$ and $H(u) \to -\infty$ as $u \to 0+$. Thus the reverse function $H^{-1}(v)$ is continuous positive increasing function for all $-\infty < v < 0$ and $H^{-1}(v) \to 0$ as $v \to -\infty$.

Further, by Theorem 1 and (2.2) in case 1) for any solution $x(t)$ we get the differential inequality

$$\frac{d|x|^2}{dt} \leq 2\varphi(t)\mu(|x|)|x|^2 \quad \text{for almost all} \quad t \in I.$$

Hence, $0 < |x(0)| < \delta$, the inequality

$$|x(t)|^2 \leq H^{-1}(H(|x(0)|^2) + 2b(t)) \quad \text{for} \quad t \in I$$

directly follows, which immediately implies both statements about stability.

The statement of Theorem 2 about unstability is proved by similar inequalities in the reverse direction. □

Next, we show the possibility of a substantial generalization of Theorem 2. In particular, the results of Theorem 2 can be generalized to systems whose matrices can be presented as the sums of normal matrices (the sum of the normal matrices generally is not a normal matrix).

Let us consider the system

$$\dot{x} = A(x,t)\nabla v(x), \tag{2.3}$$

where $v \in C^1$, $v(0) = 0$ and for some $C > 0$, $c > 0$, $\delta > 0$

$$1)\ 0 < |\nabla v(x)|^2 \leq Cv(x) \quad \text{or} \quad 2)\ 0 < cv(x) \leq |\nabla v(x)|^2 \quad (0 < |x| < \delta). \tag{2.4}$$

Theorem 3. *Let $A(x,t) = \sum\limits_{k=1}^{N} A_k(x,t)$, where $A_k(x,t)$ are square matrices piecewise continuous in Ω and normal in $\Omega\backslash M$ (M is the union of the sets of all points of discontinuity of all matrix-functions $A_k(x,t)$), and*

$$1)\ Re\lambda_{jA_k}(x,t) \leq \varkappa_k(x,t) \quad \text{or} \quad 2)\ Re\lambda_{jA_k}(x,t) \geq \nu_k(x,t),$$

where $j=\overline{1,n}$, $(x,t)\in\Omega\backslash M$, $\varkappa_k$ *and* ν_k *are continuous functions in* Ω $(k=\overline{1,N})$, $\varkappa(x,t)=\sum\limits_{k=1}^{N}\varkappa_k(x,t)$, $\nu(x,t)=\sum\limits_{k=1}^{N}\nu_k(x,t)$. *Suppose further that one of the inequalities*

$$1)\ \varkappa(x,t)\le\varphi(t)\mu(\sqrt{v(x)})\quad\text{or}\quad 2)\ \nu(x,t)\ge\varphi(t)\mu(\sqrt{v(x)}),$$

is satisfied, where the functions $\varphi(t)$ *and* $\mu(z)$ *are the same as in Theorem 2.*

Then the solution $x(t)\equiv 0$ *of system (2.3) is:*
1) asymptotically stable if $\lim\limits_{t\to+\infty}b(t)=-\infty$, *where*

$$b(t)=\int_0^t\varphi(s)ds,$$

and stable if $\overline{\lim\limits_{t\to+\infty}}\,b(t)<+\infty$ *or*
2) unstable if $\overline{\lim\limits_{t\to+\infty}}\,b(t)=+\infty$.

Proof. In the case 1), as in the proof of Theorem 2, for any solution $x(t)=x(t,x^0)$ $(x(0)=x(0,x^0)=x^0)$ of system (2.3) , we get the differential inequality

$$\frac{d}{dt}v(x)\le\varphi(t)\mu(\sqrt{v(x)})|\nabla v(x)|^2.$$

Since $|\nabla v(x)|^2\le Cv(x)$ (see inequality 1) in (6)), we get

$$\frac{d}{dt}v(x)\le C\varphi(t)\mu(\sqrt{v(x)})v(x).$$

Hence, for $0<|x(0)|<\delta$, we have the inequality

$$v(x(t;x^0))\le H^{-1}(H(v(x^0))+Cb(t))\quad\text{for}\quad t\in I \tag{2.5}$$

(here $H(u)$ and $H^{-1}(\xi)$ are the same as in the proof of Theorem 2).

If $\overline{\lim\limits_{t\to+\infty}}\,b(t)<+\infty$, it immediately follows by (2.5) and the properties of the functions $H(u)$ and $H^{-1}(\xi)$ that

$$0\le\lim_{x^0\to0}v(x(t;x^0))\le\lim_{x^0\to0}H^{-1}(H(v(x^0))+Cb(t))=0$$

is uniformly in $t\ge0$ and therefore $\lim\limits_{x^0\to0}x(t;x^0)=0$ uniformly in $t\ge0$, since $v(x)$ is continuous and positive definite for $|x|<\delta$ (that is $v(0)=0$ and $v(x)>0$ for $0<|x|<\delta$). The last equality means, by definition, the stability of the solution $x(t)\equiv0$.

If $\lim\limits_{t\to+\infty}b(t)=-\infty$, then $(x^0\ne0)$

$$0\le\lim_{t\to+\infty}v(x(t;x^0))\le\lim_{t\to+\infty}H^{-1}(H(v(x^0))+Cb(t))=0,$$

therefore $\lim\limits_{t\to+\infty}x(t;x^0)=0$ and $x(t)\equiv0$ is an asymptotically stable solution.

In the case 2) $v(x)$ satisfies inequality 2) in (6) and we have

$$\frac{d}{dt}v(x) \geq c\varphi(t)\mu(\sqrt{v(x)})v(x).$$

Therefore $(0 < |x^0| < \delta)$

$$v(x(t;x^0)) \geq H^{-1}(H(v(x^0)) + cb(t)), \quad t \in I. \tag{2.6}$$

Since $\overline{\lim_{t\to+\infty}} b(t) = +\infty$ and the function $H^{-1}(\xi)$ is defined for $-\infty < \xi < 0$, then inequality (2.6) can not be satisfied, if $I = [0, +\infty)$. This means that $x(t) \equiv 0$ is an unstable solution of system (2.3). □

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