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YESMUKHANBET SAIDAKHMETOVICH SMAILOV

(to the 70th birthday)



On October 18, 2016 was the 70th birthday of Yesmukhabet Saidakhmetovich Smailov, member of the Editorial Board of the Eurasian Mathematical Journal, director of the Institute of Applied Mathematics (Karaganda), doctor of physical and mathematical sciences (1997), professor (1993), honoured worker of the E.A. Buketov Karaganda State University, honorary professor of the Sh. Valikanov Kokshetau State University, honorary citizen of the Tarbagatai district of the East-Kazakhstan region. In 2011 he was awarded the Order “Kurmet” (= “Honour”).

Y.S. Smailov was born in the Kyzyl-Kesek village (the Aksuat district of the Semipalatinsk region of the Kazakh SSR). He graduated from the S.M. Kirov Kazakh State University (Almaty) in 1968 and in 1971 he completed his postgraduate studies at the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR (Almaty). Starting with 1972 he worked at the E.A. Buketov Karaganda State University (senior lecturer, associate professor, professor, head of the Department of Mathematical Analysis, dean of the Mathematical Faculty; from 2004 director of the Institute of Applied Mathematics).

In 1999 the American Biographical Institute declared professor Smailov “Man of the Year” and published his biography in the “Biographical encyclopedia of professional leaders of the Millennium”.

Professor Smailov is one of the leading experts in the theory of functions and functional analysis and a major organizer of science in the Republic of Kazakhstan. He had a great influence on the formation of the Mathematical Faculty of the E.A. Buketov Karaganda State University and he made a significant contribution to the development of mathematics in Central Kazakhstan. Due to the efforts of Y.S. Smailov, in Karaganda an actively operating Mathematical School on the function theory was established, which is well known in Kazakhstan and abroad.

He has published more than 140 scientific papers, two textbooks for students and one monograph. 10 candidate of sciences and 4 doctor of sciences dissertations have been defended under his supervision.

Research interests of Professor Smailov are quite broad: the embedding theory of function spaces; approximation of functions of real variables; interpolation of function spaces and linear operators; Fourier series for general orthogonal systems; Fourier multipliers; difference embedding theorems.

The Editorial Board of the Eurasian Mathematical Journal congratulates Yesmukhanbet Saidakhmetovich Smailov on the occasion of his 70th birthday and wishes him good health and new achievements in mathematics and mathematical education.

USE OF BUNDLES OF LOCALLY CONVEX SPACES
IN PROBLEMS OF CONVERGENCE
OF SEMIGROUPS OF OPERATORS. II

B. Silvestri

Communicated by V.I. Burenkov

Key words: bundles of locally convex spaces, one-parameter semigroups, spectrum and resolvent.

AMS Mathematics Subject Classification: 55R25, 46E40, 46E10; 47A10, 47D06.

1 Introduction

The present part of the work is dedicated to establish Theorem 2.1 and its corollaries. This result resolves the claim of extending the Kurtz's result to the setting of bundles of Ω -spaces. More exactly we construct an element of the set $\Delta_{\Theta} \langle \mathfrak{V}, \mathfrak{W}, \mathcal{E}, X, \mathbb{R}^+ \rangle$. Roughly $\langle \mathcal{T}, x_{\infty}, \Phi \rangle \in \Delta_{\Theta} \langle \mathfrak{V}, \mathfrak{W}, \mathcal{E}, X, \mathbb{R}^+ \rangle$ if and only if $\mathcal{T}(x)$ is the graph of the infinitesimal generator T_x of a C_0 -semigroup $\mathcal{U}(x)$ on $\mathfrak{E}_x$, for all $x \in X$, [19, equality (1.1)] holds true and

$$\mathcal{U} \in \Gamma^{x_{\infty}}(\rho). \quad (1.1)$$

Thus, according to the way of extending the Kurtz' theorem which we intend to perform in this work and outlined in the Introduction of [19], to find an element in $\Delta_{\Theta} \langle \mathfrak{V}, \mathfrak{W}, \mathcal{E}, X, \mathbb{R}^+ \rangle$ means to find an extension of [19, Theorem 1.1].

There are two strong hypotheses to be satisfied in Theorem 2.1. In constructing a model for hypothesis (ii) one obtains Corollary 3.1, while we establish Corollary 4.3 and Theorem 4.4 as an application of the strategy developed to ensure hypothesis (i). Among the two hypothesis, (i) is the most difficult one to realize. It is the assumption that the $(\Theta, \mathcal{E})$ -structure $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ has the Laplace duality property defined in Definition 2.

Roughly speaking the full Laplace duality property means that the natural action of $\prod_{x \in X} \mathcal{L}(\mathfrak{E}_x)$ over $\prod_{x \in X} \mathfrak{E}_x$, induces, by restriction, an action over $\Gamma(\pi)$ of the Laplace transform of $\Gamma(\rho)$. More exactly

$$(\forall \lambda > 0) (\mathfrak{L}(\Gamma(\rho))(\cdot)(\lambda) \circ \Gamma(\pi) \subseteq \Gamma(\pi)), \quad (\text{LD})$$

where

$$\mathfrak{L}(F)(x)(\lambda) \doteq \int_0^{\infty} e^{-\lambda s} F(x)(s) ds \doteq \int_{\mathbb{R}^+} F(x)(s) d\mu_{\lambda}(s).$$

The implicit assumption is that for all $x \in X$ and $\lambda > 0$

$$\mathfrak{M}_x \subseteq \mathfrak{L}_1(\mathbb{R}^+, \mathcal{L}_{S_x}(\mathfrak{E}_x), \mu_{\lambda}),$$

where μ_λ is the Laplace measure associated with λ and $\mathfrak{L}_1(\mathbb{R}^+, \mathcal{L}_{S_x}(\mathfrak{E}_x), \mu_\lambda)$ is the space of all μ_λ -integrable maps with values in the locally convex space $\mathcal{L}_{S_x}(\mathfrak{E}_x)$. We provide reasonable conditions ensuring the above inclusion in Proposition 4.2.

In Section 4 we investigate a strategy for constructing sets having the full Laplace duality property, result achieved in Corollary 4.2. Although in Section we worked in a wide generality, here we present the applications of interest for the present introduction.

Firstly we note that by construction

$$\Gamma(\pi) \subset \prod_{x \in X} \mathfrak{E}_x,$$

hence the natural duality action to consider over $\Gamma(\pi)$ is the restriction on it of the standard action¹ of

$$\mathcal{L}\left(\prod_{x \in X} \mathfrak{E}_x\right).$$

Secondly we note that the Laplace duality property is described in terms of the action restricted over $\Gamma(\pi)$ of a subspace of $\prod_{x \in X} \mathfrak{L}_1(\mathbb{R}^+, \mathcal{L}_{S_x}(\mathfrak{E}_x); \mu_\lambda)$.

Therefore the idea is to construct what we call in Definition 7 a $\mathbf{U}$ -Space, which is essentially a couple formed by a locally convex space $\mathfrak{G}$ and a linear map Ψ such that

$$\begin{aligned} \mathfrak{G} &\subset \mathcal{L}\left(\prod_{x \in X} \mathfrak{E}_x\right) \text{ as linear spaces} \\ \Psi(\mathfrak{L}_1(\mathbb{R}^+, \mathfrak{G}, \mu_\lambda)) &\subseteq \prod_{x \in X} \mathfrak{L}_1(\mathbb{R}^+, \mathcal{L}_{S_x}(\mathfrak{E}_x); \mu_\lambda), \end{aligned} \tag{1.2}$$

and most importantly such that the following relation between the two actions holds for all $\overline{F} \in \mathfrak{L}_1(\mathbb{R}^+, \mathfrak{G}, \mu_\lambda)$, $x \in X$, $\lambda > 0$ and $v \in \Gamma(\pi)$

$$\left\langle \int \Psi(\overline{F})(x)(s) d\mu_\lambda(s), v(x) \right\rangle_x = \left\langle \int \overline{F}(s) d\mu_\lambda(s), v \right\rangle(x). \tag{1.3}$$

Here $\mathfrak{L}_1(\mathbb{R}^+, \mathfrak{G}, \mu_\lambda)$ is the space of all μ_λ -integrable maps on $\mathbb{R}^+$ and at values in the locally convex space $\mathfrak{G}$, while for any linear space E we denote by $\langle \cdot, \cdot \rangle : \text{End}(E) \times E \rightarrow E$ the standard duality. In Corollary 4.1 we prove the existence of a $\mathbf{U}$ -Space whose topology we assemble in Definition 9 as the final one with respect to a suitable set of linear continuous maps.

Precisely because of (1.3) we can interpret (LD) as a duality problem. More exactly if $\exists \mathcal{F} \subset \bigcap_{\lambda > 0} \mathfrak{L}_1(\mathbb{R}^+, \mathfrak{G}, \mu_\lambda)$ such that $\Psi(\mathcal{F}) = \Gamma(\rho)$ then

$$\text{LD} \Leftrightarrow (\forall \lambda > 0)(\langle \mathcal{A}_\lambda, \Gamma(\pi) \rangle \subseteq \Gamma(\pi)), \tag{1.4}$$

where for all $\lambda > 0$

$$\mathcal{A}_\lambda \doteq \left\{ \int \overline{F}(s) d\mu_\lambda(s) \mid \overline{F} \in \mathcal{F} \right\} \subset \mathcal{L}\left(\prod_{x \in X} \mathfrak{E}_x\right). \tag{1.5}$$

¹namely $(B, v) \mapsto B(v)$.

There are two advantage of decoding the problem of finding the full Laplace duality property into the problem of invariance (1.4). Firstly (1.4) is an example of a classical problem of invariance of a subspace of a linear topological space for the standard action of a subspace of the space of all linear continuous operators on it. Secondly the relatively simple space $\mathfrak{L}_1(\mathbb{R}^+, \mathfrak{G}, \mu_\lambda)$ appears in (1.4) through $\mathbb{A}_\lambda$ while the subspace $\Gamma(\rho)$ of the much more involved space $\prod_{x \in X} \bigcap_{\lambda > 0} \mathfrak{L}_1(\mathbb{R}^+, \mathcal{L}_{S_x}(\mathfrak{E}_x); \mu_\lambda)$ appears in (LD).

The crucial idea behind the construction of the space $\mathfrak{G}$ performed in Definition 9 is the use of the concept of locally convex final topology. Indeed the defining characteristic of this topology allows in Lemma 4.4, to ensure that for all $v \in \Gamma(\pi)$ the evaluation map

$$\mathfrak{G} \ni A \mapsto Av \in \prod_{x \in X} \mathfrak{E}_x \text{ is continuous.} \quad (1.6)$$

And (1.3) is essentially a consequence of (1.6) attained through the two steps Theorem 4.2 and Theorem 4.3. Although we are mainly interested to the equality (1.3), there is an important result strictly determined by the locally convex final topology on $\mathfrak{G}$. Namely Theorem 4.1 ensures that holds the second inclusion in (1.2) and that for all $\bar{F} \in \mathfrak{L}_1(\mathbb{R}^+, \mathfrak{G}, \mu_\lambda)$

$$\int_x \Pr(\Psi(\bar{F}))(s) d\mu_\lambda(s) = \Pr_x \circ \left(\int \bar{F}(s) d\mu_\lambda(s) \right) \circ \iota_x.$$

2 General approximation theorem I

This section is devoted to the proof of the main Theorem 2.1.

Notation 1. We assume the notation in [19, section 2] and that all the vector spaces are over $\mathbb{C}$. Moreover we let lcp stand for the set of locally compact spaces.² For any set A we let $\mathcal{P}(A)$ be the set of all subsets of A . If Y is a topological space and Z is topological vector spaces we let $\mathcal{C}_{cs}(Y, Z)$ denote the linear space of all continuous maps $f : Y \rightarrow Z$ with compact support. For any $\mathfrak{V} \doteq \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \mathfrak{N} \rangle$ full bundle of Ω -spaces and any $\langle \mathcal{T}_0, x_\infty, \Phi \rangle \in Pregraph((\mathfrak{V}, \mathfrak{V}))$, set $X_0 \doteq X - \{x_\infty\}$, and for any $\phi \in \Phi$ $\phi_i(x) \doteq \Pr_i^x(\phi(x))$ for all $x \in X$ and $i = 1, 2$. Moreover let us denote by T_x the operator in $\mathfrak{E}_x$ such that $Graph(T_x) = \mathcal{T}_0(x)$, for all $x \in X_0$, while $\mathcal{T} \in \prod_{x \in X} Graph(\mathfrak{E}_x \times \mathfrak{E}_x)$ so that

$$\begin{cases} \mathcal{T} \upharpoonright X - \{x_\infty\} \doteq \mathcal{T}_0 \\ \mathcal{T}(x_\infty) \doteq \{\phi(x_\infty) \mid \phi \in \Phi\}, \end{cases}$$

in addition set

$$D(T_{x_\infty}) \doteq \Pr_1^{x_\infty}(\mathcal{T}(x_\infty)) = \{\phi_1(x_\infty) \mid \phi \in \Phi\}.$$

Finally for any map $F : A \rightarrow B$ set $\mathcal{R}(F) \doteq F(A)$ the range of F .

²We implicitly consider all the sets involved in this work as elements of a fixed Universe say V . So the set of all the models of a given structure say S , has to be understood as the subset of those elements of V satisfying the request defining S .

Remark 1. Let $\mathfrak{V} \doteq \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \mathfrak{N} \rangle$ be a full bundle of Ω -spaces and $\langle \mathcal{T}_0, x_\infty, \Phi \rangle \in \text{Pregraph}((\mathfrak{V}, \mathfrak{V}))$. By [19, Corollary 4.1] $\forall \phi \in \Phi$

$$\begin{cases} \phi_i \in \Gamma^{x_\infty}(\pi), i = 1, 2 \\ (\forall x \in X_0)(\phi_2(x) = T_x \phi_1(x)). \end{cases} \quad (2.1)$$

Lemma 2.1. Let $\mathfrak{V} \doteq \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \mathfrak{N} \rangle$ be a full bundle of Ω -spaces, where $\mathfrak{N} \doteq \{\nu_j \mid j \in J\}$. Moreover $\langle \mathcal{T}_0, x_\infty, \Phi \rangle \in \text{Pregraph}((\mathfrak{V}, \mathfrak{V}))$. If for all $x \in X_0$, $v_x \in \text{Dom}(T_x)$, $\lambda > 0$ and $j \in J$ we have $\nu_j((\lambda - T_x)v_x) \geq \lambda \nu_j(v_x)$ and $D(T_{x_\infty})$ is dense in $\mathfrak{E}_{x_\infty}$, then

$$\langle \mathcal{T}, x_\infty, \Phi \rangle \in \text{Gr}(\mathfrak{V}, \mathfrak{V})$$

Moreover the following

$$T_{x_\infty} : D(T_{x_\infty}) \ni \phi_1(x_\infty) \mapsto \phi_2(x_\infty) \quad (2.2)$$

is a well-defined linear operator in $\mathfrak{E}_{x_\infty}$ such that $\text{Graph}(T_{x_\infty}) = \mathcal{T}(x_\infty)$ and $\forall v_{x_\infty} \in \text{Dom}(T_{x_\infty})$, $\forall \lambda > 0$ and $\forall j \in J$ we have

$$\nu_j((\lambda - T_{x_\infty})v_{x_\infty}) \geq \lambda \nu_j(v_{x_\infty}).$$

Proof. Clearly $\mathcal{T}(x_\infty) \in \text{Graph}(\mathfrak{E}_{x_\infty} \times \mathfrak{E}_{x_\infty})$ if and only if $\phi_1(x_\infty) = \mathbf{0}_{x_\infty}$ implies $\phi_2(x_\infty) = \mathbf{0}_{x_\infty}$, $\forall \phi \in \Phi$, moreover denoting by T_{x_∞} the corresponding operator we have that $T_{x_\infty} : D(T_{x_\infty}) \rightarrow \mathfrak{E}_{x_\infty}$ is a linear operator. Any real map F defined on a topological space is l.s.c. at a point if and only if $-F$ is u.s.c. at the same point, see [2, §6.2. Chapter 4], thus by [2, Proposition 3 §6.2. Chapter 4] and [2, (13), §5.6. Chapter 4] $F : X \rightarrow \mathbb{R}$ is u.s.c. in $a \in X$ if and only if $\overline{\lim}_{x \rightarrow a} F(x) = F(a)$. Moreover by [2, §6.2. Chapter 4] we know that $F : X \rightarrow \overline{\mathbb{R}}$ is l.s.c. at a if and only if F is continuous at a providing $\overline{\mathbb{R}}$ with the following topology $\{\emptyset, [-\infty, \infty],]a, \infty[\mid a \in \mathbb{R}\}$. Thus for any map $\sigma : Y \rightarrow X$ continuous at b such that $\sigma(b) = a$ we have that $F \circ \sigma$ is l.s.c. at a . Hence because $(-F) \circ \sigma = -(F \circ \sigma)$ we can state that if $F : X \rightarrow \overline{\mathbb{R}}$ is u.s.c. at a then for any map $\sigma : Y \rightarrow X$ continuous at b such that $\sigma(b) = a$ we have that $F \circ \sigma$ is u.s.c. at a . Therefore by using [11, 1.6.(ii)] we have $\forall \sigma \in \Gamma^{x_\infty}(\pi)$ and $\forall j \in J$

$$\nu_j(\sigma(x_\infty)) = \overline{\lim}_{x \rightarrow x_\infty} \nu_j(\sigma(x)). \quad (2.3)$$

Let $\psi \in \Phi$ such that $\psi_1(x_\infty) = \mathbf{0}_{x_\infty}$ thus $\forall \phi \in \Phi$, $\forall \lambda > 0$, $\forall x \in X_0$ and $\forall j \in J$ we have by (2.3) and (2.1)

$$\begin{aligned} \nu_j(\lambda \phi_1(x_\infty) - \phi_2(x_\infty) - \lambda \psi_2(x_\infty)) &= \\ \overline{\lim}_{x \rightarrow x_\infty} \nu_j((\lambda - T_x)(\phi_1(x) + \lambda \psi_1(x))) &\geq \\ \overline{\lim}_{x \rightarrow x_\infty} \lambda \nu_j(\phi_1(x) + \lambda \psi_1(x)) &= \lambda \nu_j(\phi_1(x_\infty)), \end{aligned} \quad (2.4)$$

where, the inequality comes by [2, Proposition 11 §5.6. Chapter 4]) and by the hypothesis $\nu_j((\lambda - T_x)(\phi_1(x) + \lambda \psi_1(x))) \geq \lambda \nu_j((\phi_1(x) + \lambda \psi_1(x)))$ for all $x \in X_0$. Now $\lim_{\lambda \rightarrow \infty} v/\lambda = \mathbf{0}_{x_\infty}$ for any $v \in \mathfrak{E}_{x_\infty}$, hence by the fact that $\nu_j^{x_\infty} \doteq \nu_j \upharpoonright \mathfrak{E}_{x_\infty}$ is a continuous seminorm and by (2.4) ($\forall j \in J$) ($\forall \phi \in \Phi$)

$$\nu_j(\phi_1(x_\infty) - \psi_2(x_\infty)) = \lim_{\lambda \rightarrow \infty} \frac{\nu_j(\lambda \phi_1(x_\infty) - \phi_2(x_\infty) - \lambda \psi_2(x_\infty))}{\lambda} \geq \nu_j(\phi_1(x_\infty)). \quad (2.5)$$

By hypothesis $D(T_{x_\infty}) = \{\phi_1(x_\infty) \mid \phi \in \mathcal{T}(x_\infty)\}$ is dense in $\mathfrak{E}_{x_\infty}$ thus $\nu_j(\psi_2(x_\infty)) = 0$ for all $j \in J$. Indeed let $j \in J$ and $v \in \mathfrak{E}_{x_\infty}$ thus $\exists \{\phi^\alpha\}_{\alpha \in D}$ net in Φ such that $\lim_{\alpha \in D} \phi_1^\alpha(x_\infty) = v$ in $\mathfrak{E}_{x_\infty}$. So by the continuity of $\nu_j^{x_\infty}$ and by (2.5) we have $\forall v \in \mathfrak{E}_{x_\infty}$

$$\nu_j(v - \psi_2(x_\infty)) = \lim_{\alpha \in D} \nu_j(\phi_1^\alpha(x_\infty) - \psi_2(x_\infty)) \geq \lim_{\alpha \in D} \nu_j(\phi_1^\alpha(x_\infty)) = \nu_j(v).$$

True in particular for $v = 3\psi_2(x_\infty)$, which implies $\nu_j(\psi_2(x_\infty)) = 0$. Hence $\psi_2(x_\infty) = \mathbf{0}_{x_\infty}$ because of $\mathfrak{E}_{x_\infty}$ is a Hausdorff lcs for which $\{\nu_j^{x_\infty}\}_{j \in J}$ is a generating set of seminorms of its topology. Thus T_{x_∞} is a well-defined (necessarily linear) operator in $\mathfrak{E}_{x_\infty}$ and consequently $\langle \mathcal{T}, x_\infty, \Phi \rangle \in \text{Gr}(\mathfrak{V}, \mathfrak{V})$. Finally $(\forall j \in J)(\forall \phi \in \Phi)(\forall \lambda > 0)$

$$\begin{aligned} \nu_j((\lambda - T_{x_\infty})\phi_1(x_\infty)) &= \\ \nu_j(\lambda\phi_1(x_\infty) - \phi_2(x_\infty)) &= \quad \text{by (2.1), (2.3)} \\ \overline{\lim}_{x \rightarrow x_\infty} \nu_j(\lambda\phi_1(x) - \phi_2(x)) &= \quad \text{by (2.1)} \\ \overline{\lim}_{x \rightarrow x_\infty} \nu_j((\lambda - T_x)\phi_1(x)) &\geq \quad \text{by hypothesis and [2, Proposition 11 §5.6. Chapter 4]} \\ \overline{\lim}_{x \rightarrow x_\infty} \nu_j(\lambda\phi_1(x)) &= \nu_j(\lambda\phi_1(x_\infty)). \end{aligned}$$

□

Lemma 2.2. *In addition to the hypotheses and notation of Lemma 2.1 assume that $(\forall x \in X_0)(\forall \lambda \in \mathbb{R})(\forall j \in J)(\forall v_x \in \text{Dom}(T_x))$*

$$\nu_j((\mathbf{1} - \lambda T_x)v_x) \geq \nu_j(v_x). \quad (2.6)$$

Thus $(\forall \lambda \in \mathbb{R})(\forall j \in J)(\forall v_{x_\infty} \in \text{Dom}(T_{x_\infty}))$

$$\nu_j((\mathbf{1} - \lambda T_{x_\infty})v_{x_\infty}) \geq \nu_j(v_{x_\infty}). \quad (2.7)$$

Moreover $\forall \lambda \in \mathbb{R}$

$$\begin{cases} \exists (\mathbf{1} - \lambda T_{x_\infty})^{-1} \in \mathcal{L}(\mathcal{R}(\mathbf{1} - \lambda T_{x_\infty}), \mathfrak{E}_{x_\infty}), \\ (\forall w \in \mathcal{R}(\mathbf{1} - \lambda T_{x_\infty}))(\forall j \in J)\nu_j((\mathbf{1} - \lambda T_{x_\infty})^{-1}w) \leq \nu_j(w). \end{cases} \quad (2.8)$$

Finally

$$\mathcal{R}(\mathbf{1} - \lambda T_{x_\infty}) \text{ is closed in } \mathfrak{E}_{x_\infty}. \quad (2.9)$$

Proof. $(\forall j \in J)(\forall \phi \in \Phi)(\forall \lambda \in \mathbb{R})$

$$\begin{aligned} \nu_j((\mathbf{1} - \lambda T_{x_\infty})\phi_1(x_\infty)) &= \\ \nu_j(\phi_1(x_\infty) - \lambda\phi_2(x_\infty)) &= \quad \text{by (2.1), (2.3)} \\ \overline{\lim}_{x \rightarrow x_\infty} \nu_j(\phi_1(x) - \lambda\phi_2(x)) &= \quad \text{by (2.1)} \\ \overline{\lim}_{x \rightarrow x_\infty} \nu_j((\mathbf{1} - \lambda T_x)\phi_1(x)) &\geq \quad \text{by (2.6) and [2, Proposition 11 §5.6. Chapter 4]} \\ \overline{\lim}_{x \rightarrow x_\infty} \nu_j(\phi_1(x)) &= \nu_j(\phi_1(x_\infty)). \end{aligned}$$

thus (2.7) follows. Let $\lambda \in \mathbb{R}$, by (2.7) we obtain (2.8), indeed $\forall f, g \in \text{Dom}(T_{x_\infty})$ such that $(\mathbf{1} - \lambda T_{x_\infty})f = (\mathbf{1} - \lambda T_{x_\infty})g$ we have $\forall j \in J$

$$0 = \nu_j((\mathbf{1} - \lambda T_{x_\infty})(f - g)) \geq \nu_j(f - g),$$

so $f = g$ because of by construction $\mathfrak{E}_{x_\infty}$ is Hausdorff. Thus the following is a well-set map

$$(\mathbf{1} - \lambda T_{x_\infty})^{-1} : \mathcal{R}(\mathbf{1} - \lambda T_{x_\infty}) \ni (\mathbf{1} - \lambda T_{x_\infty})f \mapsto f \in \mathfrak{E}_{x_\infty},$$

moreover by (2.7) we obtain the second sentence of (2.8), hence the first one follows by the fact that the inverse map of any linear operator is linear. By (2.8), [2, Proposition 3 §3.1. Chapter 3] and [2, Proposition 11 §3.6. Chapter 2] we deduce that

$$(\exists! B \in \mathcal{L}(\overline{\mathcal{R}(\mathbf{1} - \lambda T_{x_\infty})}, \mathfrak{E}_{x_\infty}))(B \upharpoonright \mathcal{R}(\mathbf{1} - \lambda T_{x_\infty}) = (\mathbf{1} - \lambda T_{x_\infty})^{-1}). \quad (2.10)$$

Let $w \in \overline{\mathcal{R}(\mathbf{1} - \lambda T_{x_\infty})}$ thus $\exists \{f_\alpha\}_{\alpha \in D}$ net in $\text{Dom}(T_{x_\infty})$ such that

$$w = \lim_{\alpha \in D} (\mathbf{1} - \lambda T_{x_\infty})f_\alpha, \quad (2.11)$$

therefore by (2.10)

$$Bw = \lim_{\alpha \in D} f_\alpha, \quad (2.12)$$

while by (2.11) and (2.12)

$$\begin{aligned} w - Bw &= \lim_{\alpha \in D} ((f_\alpha - \lambda T_{x_\infty} f_\alpha) - f_\alpha) \\ &= \lim_{\alpha \in D} -\lambda T_{x_\infty} f_\alpha. \end{aligned}$$

So

$$Bw - w = \lim_{\alpha \in D} \lambda T_{x_\infty} f_\alpha. \quad (2.13)$$

By (2.12), (2.13) and the fact that λT_{x_∞} is closed, we obtain

$$\begin{cases} Bw \in \text{Dom}(T_{x_\infty}), \\ \lambda T_{x_\infty}(Bw) = Bw - w, \end{cases}$$

which means $w = (\mathbf{1} - \lambda T_{x_\infty})Bw$, so $w \in \mathcal{R}(\mathbf{1} - \lambda T_{x_\infty})$ and (2.9) follows. $\square$

Lemma 2.3. *Let us assume the hypotheses of Lemma 2.2, moreover let $\lambda \in \mathbb{R} - \{0\}$, $\{\lambda_n\}_{n \in \mathbb{N}} \subset \mathbb{R} - \{0\}$ such that $\lim_{n \in \mathbb{N}} \lambda_n = \lambda$. Thus*

$$\bigcap_{n \in \mathbb{N}} \mathcal{R}(\mathbf{1} - \lambda_n T_{x_\infty}) \subseteq \mathcal{R}(\mathbf{1} - \lambda T_{x_\infty}).$$

Proof. Set only in this proof $T \doteq T_{x_\infty}$. Let $n \in \mathbb{N}$, by (2.8) $\exists (\mathbf{1} - \lambda_n T)^{-1} : \mathcal{R}(\mathbf{1} - \lambda_n T_{x_\infty}) \rightarrow \text{Dom}(t)$ moreover

$$\begin{cases} \mathbf{1} - \lambda T = \lambda(\lambda^{-1} - T), \\ (\mathbf{1} - \lambda_n T)^{-1} = \lambda_n^{-1}(\lambda_n^{-1} - T)^{-1}. \end{cases}$$

Let $g \in \bigcap_{n \in \mathbb{N}} \mathcal{R}(\mathbf{1} - \lambda_n T_{x_\infty})$ thus

$$\begin{aligned} (\mathbf{1} - \lambda T)(\mathbf{1} - \lambda_n T)^{-1}g - g &= \frac{\lambda}{\lambda_n}(\lambda^{-1} - T)(\lambda_n^{-1} - T)^{-1}g - g \\ &= \frac{\lambda}{\lambda_n}(\lambda^{-1}(\lambda_n^{-1} - T)^{-1}g - \lambda_n^{-1}(\lambda_n^{-1} - T)^{-1}g) \\ &= \frac{\lambda}{\lambda_n}(\lambda^{-1} - \lambda_n^{-1})(\lambda_n^{-1} - T)^{-1}g, \end{aligned}$$

where in the second equality we considered that $-T(\lambda_n^{-1} - T)^{-1}g - g = -\lambda_n^{-1}(\lambda_n^{-1} - T)^{-1}g$ obtained by $(\lambda_n^{-1} - T)(\lambda_n^{-1} - T)^{-1}g = g$. Thus $\forall j \in J$ by (2.8)

$$\nu_j((\mathbf{1} - \lambda T)(\mathbf{1} - \lambda_n T)^{-1}g - g) \leq \left| \frac{\lambda}{\lambda_n} \right| |\lambda^{-1} - \lambda_n^{-1}| \nu_j(g).$$

But $\lim_{n \in \mathbb{N}} |\lambda^{-1} - \lambda_n^{-1}| = 1$ and $\lim_{n \in \mathbb{N}} |\lambda^{-1} - \lambda_n^{-1}| = 0$ so $\nu_j((\mathbf{1} - \lambda T)(\mathbf{1} - \lambda_n T)^{-1}g - g) = 0$, for all $j \in J$. Therefore

$$\lim_{n \in \mathbb{N}} (\mathbf{1} - \lambda T)(\mathbf{1} - \lambda_n T)^{-1}g = g,$$

and the statement follows by (2.9). $\square$

Lemma 2.4. *Under the hypotheses and notation of Lemma 2.1 we have that $\mathbf{1} - \lambda T_{x_\infty}$ is a closed operator.*

Proof. Let $(a, b) \in \overline{\text{Graph}(\mathbf{1} - \lambda T_{x_\infty})}$ closure in the space $\mathfrak{E}_{x_\infty} \times \mathfrak{E}_{x_\infty}$ with the product topology. Thus $(\forall \varepsilon > 0)(\forall j \in J)(\exists v_{(\varepsilon, j)} \in \text{Dom}(T_{x_\infty}))$

$$\begin{cases} \nu_j(a - v_{(\varepsilon, j)}) < \frac{\varepsilon}{2}, \\ \nu_j(b - (\mathbf{1} - \lambda T_{x_\infty})v_{(\varepsilon, j)}) < \frac{\varepsilon}{2}, \end{cases}$$

so

$$\nu_j((b - a) + \lambda T_{x_\infty} v_{(\varepsilon, j)}) \leq \nu_j(b - (\mathbf{1} - \lambda T_{x_\infty})v_{(\varepsilon, j)}) + \nu_j(a - v_{(\varepsilon, j)}) \leq \varepsilon.$$

Therefore $(\forall \varepsilon > 0)(\forall j \in J)(\exists v_{(\varepsilon, j)} \in \text{Dom}(T_{x_\infty}))$

$$\begin{cases} \nu_j(a - v_{(\varepsilon, j)}) < \varepsilon, \\ \nu_j((b - a) - (-\lambda T_{x_\infty} v_{(\varepsilon, j)})) < \varepsilon, \end{cases}$$

which means $(a, (b - a)) \in \overline{\text{Graph}(-\lambda T_{x_\infty})}$. Moreover $-\lambda T_{x_\infty}$ is a closed operator thus $b - a = -\lambda T_{x_\infty} a$ or equivalently $(a, b) \in \text{Graph}(\mathbf{1} - \lambda T_{x_\infty})$. $\square$

Remark 2. By (2.1) we have $\forall \phi \in \Phi$ that $\phi_1(x_\infty) = \lim_{z \rightarrow x_\infty} \phi_1(z)$ and $\phi_2(x_\infty) = \lim_{z \rightarrow x_\infty} \phi_2(z) = \lim_{z \rightarrow x_\infty} T_x \phi_1(z)$, hence

$$\begin{cases} \phi_1(x_\infty) = \lim_{z \rightarrow x_\infty} \phi_1(z) \\ T_{x_\infty} \phi_1(x_\infty) = \lim_{z \rightarrow x_\infty} T_z \phi_1(z). \end{cases}$$

Definition 1. Let $\lambda \in \mathbb{R}^+$ set

$$\mu_\lambda : \mathcal{C}_{cs}(\mathbb{R}^+, \mathbb{R}) \ni f \mapsto \int_{\mathbb{R}^+} e^{-s\lambda} f(s) ds,$$

where the integral is with respect to the Lebesgue measure on $\mathbb{R}^+$.

Definition 2. Let $\mathfrak{W} \doteq \langle \langle \mathfrak{M}, \gamma \rangle, \rho, X, \mathfrak{A} \rangle$ and $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ be a $(\Theta, \mathcal{E})$ -structure such that

$$\mathfrak{M}_x \subseteq \bigcap_{\lambda > 0} \mathfrak{L}_1(\mathbb{R}^+, \mathcal{L}_{S_x}(\mathfrak{E}_x); \mu_\lambda), \quad \forall x \in X. \quad (2.14)$$

About S_x and $\mathfrak{E}_x$ see [19, Definition 6] Let $x \in X$, $\mathcal{O} \subseteq \Gamma(\rho)$. and $\mathcal{D} \subseteq \Gamma(\pi)$. By recalling the notation in [19, equality (2.1)] we say that $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ has the *Laplace duality property on $\mathcal{O}$ and $\mathcal{D}$ at x* , shortly $\text{LD}_x(\mathcal{O}, \mathcal{D})$ if

$$(\forall \lambda > 0)(\mathfrak{L}(\Gamma_{\mathcal{O}}^x(\rho))_\lambda \circ \Gamma_{\mathcal{D}}^x(\pi) \subseteq \Gamma^x(\pi)).$$

Moreover we say that $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ has the *full Laplace duality property on $\mathcal{O}$ and $\mathcal{D}$* , shortly $\text{LD}(\mathcal{O}, \mathcal{D})$ if

$$(\forall \lambda > 0)(\mathfrak{L}(\mathcal{O})_\lambda \circ \mathcal{D} \subseteq \Gamma(\pi)).$$

Finally LD is for $\text{LD}(\Gamma(\rho), \Gamma(\pi))$. Here $\mathfrak{L} : \prod_{x \in X} \mathfrak{M}_x \rightarrow \prod_{x \in X} \mathcal{L}_{S_x}(\mathfrak{E}_x)^{\mathbb{R}^+}$ such that $(\forall x \in X)(\forall \lambda \in \mathbb{R}^+)$

$$\mathfrak{L}(F)(x)(\lambda) \doteq \int_0^\infty e^{-\lambda s} F(x)(s) ds,$$

where we recall that the integration is with respect to the Lebesgue measure on $\mathbb{R}^+$ and with respect to the locally convex topology on $\mathcal{L}_{S_x}(\mathfrak{E}_x)$. Finally we used the notation in [19, equality (2.1)]

Remark 3. Under the notation of Definition 2 and by letting $\mathfrak{n}$ be the Lebesgue measure on $\mathbb{R}^+$, (2.14) follows if the following holds

$$\mathfrak{M}_x \subseteq \mathfrak{L}_1(\mathbb{R}^+, \mathcal{L}_{S_x}(\mathfrak{E}_x); \mathfrak{n}), \quad \forall x \in X.$$

Moreover under assumptions of Definition 2 we have

$$\begin{cases} \mathfrak{L}(\Gamma_{\mathcal{O}}^x(\rho)) \subseteq \Gamma_{\mathcal{O}}^x(\rho) \\ (\forall t > 0)(\Gamma_{\mathcal{O}}^x(\rho)_t \circ \Gamma_{\mathcal{D}}^x(\pi) \subseteq \Gamma^x(\pi)) \end{cases} \Rightarrow \langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle \text{ has the } \text{LD}_x(\mathcal{O}, \mathcal{D}).$$

Similarly

$$\begin{cases} \mathfrak{L}(\mathcal{O}) \subseteq \mathcal{O} \\ (\forall t > 0)(\mathcal{O}_t \circ \mathcal{D} \subseteq \Gamma(\pi)) \end{cases} \Rightarrow \langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle \text{ has the } \text{LD}(\mathcal{O}, \mathcal{D}).$$

A useful property is the following one

Proposition 2.1. *Let $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ be a $(\Theta, \mathcal{E})$ -structure satisfying (2.14), $x_\infty \in X$. Set $S_z = \{B_l^z \mid l \in L\}$, then $\forall z \in X$, $\forall G \in \mathfrak{L}_1(\mathbb{R}^+, \mathcal{L}_{S_z}(\mathfrak{E}_z); \mu_\lambda)$ and $\forall w_z \in \bigcup_{l \in L} B_l^z$*

$$\left(\int_0^\infty e^{-\lambda s} G(s) ds \right) w_z = \int_0^\infty e^{-\lambda s} G(s) w_z ds. \quad (2.15)$$

Here in the second member the integration is with respect to the locally convex topology on $\mathfrak{E}_z$, while in the first member the integration is with respect to the locally convex topology on $\mathcal{L}_{S_z}(\mathfrak{E}_z)$.

Proof. Let $z \in X$ and $v \in \bigcup_{l \in L} B_l^z = \mathfrak{E}_z$ then map $\mathcal{L}_{S_z}(\mathfrak{E}_z) \ni A \mapsto Av \in \mathfrak{E}_z$ is linear and continuous. Indeed let $l(v) \in L$ such that $v \in B_{l(v)}^z$, thus we have $\nu_j^z(Av) \leq \sup_{w \in B_{l(v)}^z} \nu_j^z(Aw) \doteq p_{j, l(v)}^z(A)$. Hence by a well-known result in vector valued integration we have (2.15). $\square$

Remark 4. Let $\mathfrak{V} \doteq \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \mathfrak{N} \rangle$ be a bundle of Ω -spaces and $\mathcal{E} \subseteq \prod_{x \in X} \mathfrak{E}_x$. Set for all $v \in \prod_{x \in X} \mathfrak{E}_x$

$$\begin{cases} B_v : X \ni x \mapsto \{v(x)\}, \\ \Theta \doteq \{B_w \mid w \in \mathcal{E}\} \end{cases} \quad (2.16)$$

Thus $\Theta \subset \prod_{x \in X} \text{Bounded}(\mathfrak{E}_x)$ and $\forall v \in \mathcal{E}$

$$\mathcal{E} \cap \prod_{x \in X} B_v(x) = \{v\}. \quad (2.17)$$

Therefore for all $v \in \mathcal{E}$, and for all $x \in X$ with the notation of [19, Definition 6]

$$\begin{cases} D(B_v, \mathcal{E}) = \{v\}, \\ \mathcal{B}_{B_v}^x = \{v(x)\}, \\ S_x = \{\{w(x)\} \mid w \in \mathcal{E}\}, \\ \mathcal{E}(\Theta) = \mathcal{E}. \end{cases}$$

Recall that since the Dupre' theorem any Banach bundle over a completely regular topological space is full.

Definition 3. Let $\mathfrak{V} \doteq \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \|\cdot\| \rangle$ be a full Banach bundle. Let $x_\infty \in X$ and $\mathcal{U}_0 \in \prod_{x \in X_0} \mathcal{C}(\mathbb{R}^+, B_s(\mathfrak{E}_x))$ be such that $\mathcal{U}_0(x)$ is a (C_0) -semigroup of contractions (respectively of isometries) on $\mathfrak{E}_x$ for all $x \in X_0$. Moreover let us denote by T_x the infinitesimal generator of the semigroup $\mathcal{U}_0(x)$ for any $x \in X_0$ and set

$$\begin{cases} \mathcal{T}_0(x) \doteq \text{Graph}(T_x), x \in X_0 \\ \Phi \doteq \{\phi \in \Gamma^{x_\infty}(\pi_{\mathfrak{E}^\oplus}) \mid (\forall x \in X_0)(\phi(x) \in \mathcal{T}_0(x))\} \\ \mathcal{E} \doteq \{v \in \Gamma(\pi) \mid (\exists \phi \in \Phi)(v(x_\infty) = \phi_1(x_\infty))\} \\ \Theta \doteq \{B_w \mid w \in \mathcal{E}\}, \end{cases} \quad (2.18)$$

where $\langle \langle \mathfrak{E}(\mathfrak{E}^\oplus), \tau(\mathfrak{E}^\oplus, \mathcal{E}^\oplus) \rangle, \pi_{\mathfrak{E}^\oplus}, X, \mathfrak{n}^\oplus \rangle$ is the bundle direct sum of the family $\{\mathfrak{V}, \mathfrak{V}\}$.

The following is a direct generalization to our context of the definition given in [14, Lm. 2.11]

Definition 4. Let $\mathfrak{V} \doteq \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \mathfrak{N} \rangle$ be a bundle of Ω -spaces, where $\mathfrak{N} \doteq \{\nu_j \mid j \in J\}$. Moreover let Y be a topological space, $s_0 \in Y$, $f \in \prod_{x \in X} \mathfrak{E}_x^Y$ and $\{z_n\}_{n \in \mathbb{N}} \subset X$. Then we say that $\{f(z_n)\}_{n \in \mathbb{N}}$ is *bounded* if $\sup_{(n,s) \in \mathbb{N} \times Y} \nu_j(f(z_n)(s)) < \infty$ for all $j \in J$. $\{f(z_n)\}_{n \in \mathbb{N}}$ is *equicontinuous at s_0* if for all $j \in J$ and for all $\varepsilon > 0$ there exists a neighbourhood U of s_0 such that for all $s \in U$ we have $\sup_{n \in \mathbb{N}} \nu_j(f(z_n)(s) - f(z_n)(s_0)) \leq \varepsilon$. Finally $\{f(z_n)\}_{n \in \mathbb{N}}$ is *equicontinuous* if $\{f(z_n)\}_{n \in \mathbb{N}}$ is equicontinuous at s for every $s \in Y$.

Proposition 2.2. *Let us assume the notation of Definition 3. Thus $\{v(x_\infty) \mid v \in \mathcal{E}\} = \{\phi_1(x_\infty) \mid \phi \in \Phi\}$.*

Proof. By definition follows the inclusion $\subseteq$. $\mathfrak{V}$ being full we have $(\forall \phi \in \Phi)(\exists v \in \Gamma(\pi))(v(x_\infty) = \phi_1(x_\infty))$. Thus $(\forall \phi \in \Phi)(\exists v \in \mathcal{E})(v(x_\infty) = \phi_1(x_\infty))$ hence the inclusion $\supseteq$. $\square$

Theorem 2.1 (MAIN 1). *Let $\mathfrak{V} \doteq \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \|\cdot\| \rangle$ be a Banach bundle where X is a completely regular space for which there exists $x_\infty \in X$ such that its filter of neighbourhoods admits a countable basis. Let $\mathcal{U}_0 \in \prod_{x \in X_0} \mathcal{C}(\mathbb{R}^+, B_s(\mathfrak{E}_x))$ be such that $\mathcal{U}_0(x)$ is a (C_0) -semigroup of contractions (respectively of isometries) on $\mathfrak{E}_x$ for all $x \in X_0$.*

If $D(T_{x_\infty})$ is dense in $\mathfrak{E}_{x_\infty}$ and $\exists \lambda_0 > 0$ (respectively $\exists \lambda_0 > 0, \lambda_1 < 0$) such that the range $\mathcal{R}(\lambda_0 - T_{x_\infty})$ is dense in $\mathfrak{E}_{x_\infty}$, (respectively the ranges $\mathcal{R}(\lambda_0 - T_{x_\infty})$ and $\mathcal{R}(\lambda_1 - T_{x_\infty})$ are dense in $\mathfrak{E}_{x_\infty}$), then

$$\langle \mathcal{T}, x_\infty, \Phi \rangle \in \text{Gr}(\mathfrak{V}, \mathfrak{V}),$$

and T_{x_∞} in (2.2) is the generator of a C_0 -semigroup of contractions (respectively of isometries) on $\mathfrak{E}_{x_\infty}$.

Moreover assume that $\{v(x) \mid v \in \mathcal{E}\}$ is dense in $\mathfrak{E}_x$ for all $x \in X_0$, by taking the notation in (2.18), let $\mathfrak{W} \doteq \langle \langle \mathfrak{M}, \gamma \rangle, \rho, X, \mathfrak{R} \rangle$ and $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ be a $(\Theta, \mathcal{E})$ -structure³ such that (2.14) holds. Assume $\bigcup_{\|\cdot\|_{B(\mathfrak{E}_z)}} (\mathcal{L}_{S_z}(\mathfrak{E}_z)) \subseteq \mathfrak{M}_z$ (respectively $\bigcup_{is} (\mathcal{L}_{S_z}(\mathfrak{E}_z)) \subseteq \mathfrak{M}_z$) for all $z \in X$ ⁴ and that there exists $F \in \Gamma(\rho)$ such that $F(x_\infty) = \mathcal{U}(x_\infty)$ and

i $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ has the $\text{LD}_{x_\infty}(\{F\}, \mathcal{E})$; **or** it has the $\text{LD}(\{F\}, \mathcal{E})$;

ii $(\forall v \in \mathcal{E})(\exists \phi \in \Phi)$ s.t. $\phi_1(x_\infty) = v(x_\infty)$ and $(\forall \{z_n\}_{n \in \mathbb{N}} \subset X \mid \lim_{n \in \mathbb{N}} z_n = x_\infty)$ we have that $\{\mathcal{U}(z_n)(\cdot)\phi_1(z_n) - F(z_n)(\cdot)v(z_n)\}_{n \in \mathbb{N}}$ is a bounded equicontinuous sequence.

Then $(\forall v \in \mathcal{E})(\forall K \in \text{Comp}(\mathbb{R}^+))$

$$\lim_{z \rightarrow x_\infty} \sup_{s \in K} \|\mathcal{U}(z)(s)v(z) - F(z)(s)v(z)\| = 0, \quad (2.19)$$

and

$$\mathcal{U} \in \Gamma^{x_\infty}(\rho). \quad (2.20)$$

In particular

$$\{\langle \mathcal{T}, x_\infty, \Phi \rangle\} \in \Delta_\Theta \langle \mathfrak{V}, \mathfrak{W}, \mathcal{E}, X, \mathbb{R}^+ \rangle. \quad (2.21)$$

Here $\mathcal{T}$ and $D(T_{x_\infty})$ are defined as in Notation 1 with $\mathcal{T}_0$ and Φ given in (2.18), while $\mathcal{U} \in \prod_{x \in X} \mathfrak{M}_x$ such that $\mathcal{U} \upharpoonright X_0 \doteq \mathcal{U}_0$ and $\mathcal{U}(x_\infty)$ is the semigroup on $\mathfrak{E}_{x_\infty}$ generated by T_{x_∞} .

³Well set indeed by Proposition 2.2, the density assumptions and Remark 4 we have that S_x is dense in $\mathfrak{E}_x$ for all $x \in X$.

⁴See Proposition 4.2 for models of $\mathfrak{M}$ satisfying (2.14) and $\bigcup_{\|\cdot\|_{B(\mathfrak{E}_z)}} (\mathcal{L}_{S_z}(\mathfrak{E}_z)) \subseteq \mathfrak{M}_z$.

Proof. Since the Dupre' theorem, see for example [11, Corollary 2.10], we obtain that $\mathfrak{V}$ is full. By Lemma 2.1, [14, Lemmas (2.8) – (2.9)], and the Hille-Yosida theorem, see [14, Theorem (1.2)], we have the first sentence of the statement for the case of semigroup of contractions. By [5, Corollary 3.1.19.] applied to T_x , for any $x \in X_0$, and by (2.7) we have $(\forall \lambda \in \mathbb{R})(\forall v_{x_\infty} \in \text{Dom}(T_{x_\infty}))$

$$\|(\mathbf{1} - \lambda T_{x_\infty})v_{x_\infty}\|_{x_\infty} \geq \|v_{x_\infty}\|_{x_\infty}. \quad (2.22)$$

Hence by [5, Corollary 3.1.19.], T_{x_∞} will be a generator of a strongly continuous semigroup of isometries if we show that $\forall \lambda \in \mathbb{R} - \{0\}$

$$\mathcal{R}(\mathbf{1} - \lambda T_{x_\infty}) = \mathfrak{E}_{x_\infty}. \quad (2.23)$$

Let us set

$$\rho_0(T_{x_\infty}) \doteq \{\lambda \in \mathbb{R} - \{0\} \mid \mathcal{R}(\mathbf{1} - \lambda T) = \mathfrak{E}_x\}.$$

By (2.8) $\rho_0(T_{x_\infty}) = \rho(T_{x_\infty}) \cap (\mathbb{R} - \{0\})$, where $\rho(T_{x_\infty})$ is the resolvent set of T_x . By [9, Lemma 7.3.2] $\rho(T_{x_\infty})$ is open in $\mathbb{C}$ so $\rho_0(T_{x_\infty})$ is open in $\mathbb{R} - \{0\}$ with respect to the topology on $\mathbb{R} - \{0\}$ induced by that on $\mathbb{C}$. By Lemma 2.3 we deduce that $\rho_0(T_{x_\infty})$ is also closed in $\mathbb{R} - \{0\}$, therefore $\rho_0(T_{x_\infty}) = \mathbb{R} - \{0\}$ and (2.23) follows as well that T_{x_∞} is a generator of a strongly continuous semigroup of isometries.

Now we shall apply [19, Lemma 5.1] in order to show the remaining part of the statement. Let $v \in \mathcal{E}$ be fixed then by (2.18), $(\exists \phi \in \Phi)(v(x_\infty) = \phi_1(x_\infty))$ thus by (2.1) and [19, Corollary 3.1]

$$\lim_{z \rightarrow x_\infty} \|v(z) - \phi_1(z)\| = 0. \quad (2.24)$$

Now let $F \in \Gamma(\rho)$ of which in hypothesis so in particular

$$F(x_\infty) = \mathcal{U}(x_\infty), \quad (2.25)$$

moreover $\forall s \in \mathbb{R}^+$ and $z \in X$

$$\begin{aligned} & \|\mathcal{U}(z)(s)v(z) - F(z)(s)v(z)\| \leq \\ & \|\mathcal{U}(z)(s)v(z) - \mathcal{U}(z)(s)\phi_1(z)\| + \|\mathcal{U}(z)(s)\phi_1(z) - F(z)(s)v(z)\| \leq \\ & \|v(z) - \phi_1(z)\| + \|\mathcal{U}(z)(s)\phi_1(z) - F(z)(s)v(z)\|. \end{aligned} \quad (2.26)$$

For any $\lambda > 0$ let us set

$$g_\infty^\lambda \doteq (\lambda - T_{x_\infty})^{-1}\phi_1(x_\infty)$$

thus $g_\infty^\lambda \in \text{Dom}(T_{x_\infty})$ hence by Remark 2 and the construction of T_{x_∞} $\exists \psi^\lambda \in \Phi$ such that

$$\begin{cases} g_\infty^\lambda = \psi_1^\lambda(x_\infty) = \lim_{z \in x_\infty} \psi_1^\lambda(z) \\ T_{x_\infty} g_\infty^\lambda = \lim_{z \rightarrow x_\infty} T_z \psi_1^\lambda(z). \end{cases} \quad (2.27)$$

By (2.15) and (2.17) for all $z \in X$ and for all $w_z \in \bigcup_{v \in \mathcal{E}} v(z)$

$$\left(\int_0^\infty e^{-\lambda s} F(z)(s) ds \right) w_z = \int_0^\infty e^{-\lambda s} F(z)(s) w_z ds. \quad (2.28)$$

Moreover by the fact that $\mathfrak{V}$ is full we have that for all $\phi \in \Phi$ there exists a $v \in \Gamma(\pi)$ such that $v(x_\infty) = \phi_1(x_\infty)$, thus by construction of $\mathcal{E}$

$$(\forall \phi \in \Phi)(\exists v \in \mathcal{E})(v(x_\infty) = \phi_1(x_\infty)). \quad (2.29)$$

Hence by (2.28), (2.29) and (2.25) for all $\phi \in \Phi$

$$\left(\int_0^\infty e^{-\lambda s} F(x_\infty)(s) ds \right) \phi_1(x_\infty) = \int_0^\infty e^{-\lambda s} \mathcal{U}(x_\infty)(s) \phi_1(x_\infty) ds. \quad (2.30)$$

Now set

$$\xi \doteq \mathfrak{L}(F),$$

thus by hypothesis (i) we have for all $\lambda > 0$

$$\xi(\cdot)(\lambda)v(\cdot) \in \Gamma^{x_\infty}(\pi). \quad (2.31)$$

Moreover

$$\begin{aligned} \xi(x_\infty)(\lambda)v(x_\infty) &= \xi(x_\infty)(\lambda)\phi_1(x_\infty) \\ &= \int_0^\infty e^{-\lambda s} \mathcal{U}(x_\infty)(s) \phi_1(x_\infty) ds \text{ by (2.30)} \\ &= (\lambda - T_{x_\infty})^{-1} \phi_1(x_\infty) \text{ by [14, (1.3)]} \\ &\doteq g_\infty^\lambda = \psi_1^\lambda(x_\infty) \text{ by (2.27)}. \end{aligned} \quad (2.32)$$

By the fact that $\mathfrak{V}$ is full, by (2.31), the fact that $\psi_1^\lambda \in \Gamma^{x_\infty}(\pi)$ by (2.1), by (2.32) and by [19, Corollary 3.2] we have $\forall \lambda > 0$

$$\lim_{z \rightarrow x_\infty} \|\psi_1^\lambda(z) - \xi(z)(\lambda)v(z)\| = 0. \quad (2.33)$$

Now $(\forall \lambda > 0)(\forall z \in X)$ set

$$w^\lambda(z) \doteq (\lambda \mathbf{1} - T_z) \psi_1^\lambda(z),$$

thus

$$\begin{aligned} &\left\| \int_0^\infty e^{-\lambda s} (\mathcal{U}(z)(s) \phi_1(z) - F(z)(s) v(z)) ds \right\| \leq \\ &\left\| \int_0^\infty e^{-\lambda s} \mathcal{U}(z)(s) (\phi_1(z) - w^\lambda(z)) ds \right\| + \left\| \int_0^\infty e^{-\lambda s} (\mathcal{U}(z)(s) w^\lambda(z) - F(z)(s) v(z)) ds \right\| \leq \\ &\quad \frac{1}{\lambda} \|\phi_1(z) - w^\lambda(z)\| + \|\psi_1^\lambda(z) - \xi(z)(\lambda)v(z)\|. \end{aligned} \quad (2.34)$$

Here we consider that by hypothesis and by the first part of the statemet $\|\mathcal{U}(z)\| \leq 1$ for all $z \in X$, moreover we applied the Hille-Yosida formula [14, (1.3)]. Now

$$\begin{aligned} \|\phi_1(z) - w^\lambda(z)\| &= \quad (2.35) \\ \|\phi_1(z) - (\lambda \mathbf{1} - T_z) \psi_1^\lambda(z)\| &\leq \\ \|\phi_1(z) - v(z)\| + \|v(z) - \lambda \xi(z)(\lambda)v(z) + \lambda \xi(z)(\lambda)v(z) - (\lambda \mathbf{1} - T_z) \psi_1^\lambda(z)\| &\leq \\ \|\phi_1(z) - v(z)\| + \lambda \|\xi(z)(\lambda)v(z) - \psi_1^\lambda(z)\| + \|T_z \psi_1^\lambda(z) - (\lambda \xi(z)(\lambda)v(z) - v(z))\|. \end{aligned}$$

By (2.27) $T_{x_\infty}\psi_1^\lambda(x_\infty) = T_{x_\infty}g_\infty^\lambda$ moreover

$$\begin{aligned} T_{x_\infty}g_\infty^\lambda &= -(\lambda - T_{x_\infty})g_\infty^\lambda + \lambda g_\infty^\lambda \\ &= -(\lambda - T_{x_\infty})(\lambda - T_{x_\infty})^{-1}\phi_1(x_\infty) + \lambda g_\infty^\lambda \\ &= \lambda g_\infty^\lambda - \phi_1(x_\infty) = \lambda \xi(x_\infty)(\lambda)v(x_\infty) - v(x_\infty), \end{aligned} \quad (2.36)$$

where in the last equality we used (2.32) and the construction of ϕ . By (2.27) we have that $(X \ni z \mapsto T_z\psi_1^\lambda(z)) \in \Gamma^{x_\infty}(\pi)$, hence by (2.36), the fact that $\lambda\xi(\cdot)(\lambda)v(\cdot) - v \in \Gamma^{x_\infty}(\pi)$ by (2.31), we deduce by the fact that $\mathfrak{V}$ is full and by [19, Corollary 3.2] that $\forall \lambda > 0$

$$\lim_{z \rightarrow x_\infty} \|T_z\psi_1^\lambda(z) - (\lambda\xi(z)(\lambda)v(z) - v(z))\| = 0. \quad (2.37)$$

Therefore by (2.35), (2.24), (2.33) and (2.37)

$$\lim_{z \rightarrow x_\infty} \|\phi_1(z) - w^\lambda(z)\| = 0.$$

By this one along with (2.33) we can state by using (2.34) that $\forall \lambda > 0$

$$\lim_{z \rightarrow x_\infty} \left\| \int_0^\infty e^{-\lambda s} (\mathcal{U}(z)(s)\phi_1(z) - F(z)(s)v(z)) ds \right\| = 0.$$

Therefore $\forall \lambda > 0$ and $(\forall \{z_n\}_{n \in \mathbb{N}} \subset X \mid \lim_{n \in \mathbb{N}} z_n = x_\infty)$

$$\lim_{n \in \mathbb{N}} \left\| \int_0^\infty e^{-\lambda s} (\mathcal{U}(z_n)(s)\phi_1(z_n) - F(z_n)(s)v(z_n)) ds \right\| = 0. \quad (2.38)$$

By (2.38), hypothesis (ii) and [14, Lemma (2.11)] we have $(\forall \{z_n\}_{n \in \mathbb{N}} \subset X \mid \lim_{n \in \mathbb{N}} z_n = x_\infty)$ and $\forall K \in \text{Comp}(\mathbb{R}^+)$

$$\limsup_{n \in \mathbb{N}} \sup_{s \in K} \|\mathcal{U}(z_n)(s)\phi_1(z_n) - F(z_n)(s)v(z_n)\| = 0.$$

Therefore since the hypothesis on x_∞ we obtain $\forall K \in \text{Comp}(\mathbb{R}^+)$

$$\lim_{z \rightarrow x_\infty} \sup_{s \in K} \|\mathcal{U}(z)(s)\phi_1(z) - F(z)(s)v(z)\| = 0. \quad (2.39)$$

In conclusion by (2.39), (2.24) and (2.26) we obtain $\forall K \in \text{Comp}(\mathbb{R}^+)$

$$\lim_{z \rightarrow x_\infty} \sup_{s \in K} \|\mathcal{U}(z)(s)v(z) - F(z)(s)v(z)\| = 0, \quad (2.40)$$

hence (2.19). By (2.17) and (2.40) we obtain [19, equality (5.8)]. Thus (2.20) and (2.21) follow by [19, Lemma 5.1] by (2.17) and by the following one $\forall K \in \text{Comp}(\mathbb{R}^+)$ and $\forall v \in \mathcal{E}$

$$\sup_{z \in X} \sup_{s \in K} \|\mathcal{U}(z)(s)v(z)\| \leq \sup_{z \in X} \|v(z)\| < \infty.$$

where we considered that by construction $\|\mathcal{U}(z)(s)\| \leq 1$, for all $s \in \mathbb{R}^+$ and $z \in X$ and that $v \in \Gamma(\pi)$. $\square$

Remark 5. If $\mathfrak{W}$ is full $(\exists F \in \Gamma(\rho))(F(x_\infty) = \mathcal{U}(x_\infty))$, so hypotheses reduce.

3 Corollary I. Construction of equicontinuous sequences

By providing conditions ensuring the bounded equicontinuity of which in hypothesis (ii) of Theorem 2.1 we obtain the following

Corollary 3.1. *Let us assume the hypotheses of Theorem 2.1 except (ii) replaced by the following one*

$$(\exists G \in \prod_{z \in X} \mathfrak{L}_1(\mathbb{R}^+, \mathcal{L}_{S_x}(\mathfrak{E}_x)) (\exists H \in \prod_{z \in X}^b \mathcal{L}(\mathfrak{E}_z)) (\exists F \in \Gamma(\rho))$$

such that $F(x_\infty) = \mathcal{U}(x_\infty)$ and $\forall s > 0$

$$\begin{cases} \sup_{x \in X} \sup_{s > 0} \|F(x)(s)\| < \infty \\ (\forall s_1 > 0)(\exists a > 0)(\sup_{u \in [s_1, s]} \sup_{z \in X} \|G(z)(u)\| \leq a|s - s_1|) \\ (\forall z \in X)(F(z)(s) = H(z) + \int_0^s G(z)(u) du), \end{cases} \quad (3.1)$$

where the integration is with respect to the Lebesgue measure on $[0, s]$ and with respect to the locally convex topology on $\mathcal{L}_{S_z}(\mathfrak{E}_z)$. Then holds the statement of Theorem 2.1.

Proof. Let $v \in \mathcal{E}$ thus $(\exists \phi \in \Phi)(v(x_\infty) = \phi_1(x_\infty))$ so $(\forall \{z_n\}_{n \in \mathbb{N}} \subset X \mid \lim_{n \in \mathbb{N}} z_n = x_\infty)$ we have

$$\sup_{n \in \mathbb{N}} \sup_{s > 0} \|\mathcal{U}(z_n)(s)\phi_1(z_n) - F(z_n)(s)v(z_n)\| \leq \sup_{n \in \mathbb{N}} \|\phi_1(z_n)\| + M \sup_{n \in \mathbb{N}} \|v(z_n)\| < \infty.$$

Here in the first inequality we used $\|\mathcal{U}(z)(s)\| \leq 1$ for all $z \in X$ and $s > 0$ by construction, and $M \doteq \sup_{z \in X} \sup_{s > 0} \|F(z)(s)\| < \infty$ by hypothesis, while in the second inequality we used the fact that $v \in \prod_{x \in X}^b \mathfrak{E}_x$, by construction and that $\sup_{n \in \mathbb{N}} \|\phi_1(z_n)\| < \infty$ because of $\exists \overline{\lim}_{n \in \mathbb{N}} \|\phi_1(z_n)\| \in \mathbb{R}$ by Remark 2 and by construction $\|\cdot\|$ is *u.s.c.* Moreover by [14, (1.4)], (3.1) and $S_x = \{\{w(x)\} \mid w \in \mathcal{E}\}$ for all $x \in X$ we have

$$\begin{aligned} \mathcal{U}(z_n)(s)\phi_1(z_n) - F(z_n)(s)v(z_n) &= \int_0^s (\mathcal{U}(z_n)(u)T_{z_n}\phi_1(z_n) - G(z_n)(u)v(z_n)) du + \\ &+ \phi_1(z_n) - H(z_n)v(z_n). \end{aligned}$$

Thus for any $s_1, s_2 \in \mathbb{R}^+$

$$\begin{aligned} \sup_{n \in \mathbb{N}} \|(\mathcal{U}(z_n)(s_1)\phi_1(z_n) - F(z_n)(s_1)v(z_n)) - (\mathcal{U}(z_n)(s_2)\phi_1(z_n) - F(z_n)(s_2)v(z_n))\| &\leq \\ |s_1 - s_2| \sup_{n \in \mathbb{N}} \sup_{u \in [s_1, s_2]} \|\mathcal{U}(z_n)(u)T_{z_n}\phi_1(z_n) - G(z_n)(u)v(z_n)\| &\leq \\ |s_1 - s_2| \sup_{n \in \mathbb{N}} (\|T_{z_n}\phi_1(z_n)\| - a\|v(z_n)\|) &\leq J|s_1 - s_2|. \end{aligned}$$

Here in the second inequality we used $\|\mathcal{U}(z)(u)\| \leq 1$ by construction and the hypothesis, in the third one the fact that $\sup_{n \in \mathbb{N}} \|T_{z_n}\phi_1(z_n)\| < \infty$ as well $\sup_{n \in \mathbb{N}} \|v(z_n)\| < \infty$, because of $\exists \overline{\lim}_{n \in \mathbb{N}} \|T_{z_n}\phi_1(z_n)\| \in \mathbb{R}$ and $\exists \overline{\lim}_{n \in \mathbb{N}} \|v(z_n)\| \in \mathbb{R}$ due to the fact that $\|\cdot\|$ is *u.s.c.* by construction and Remark 2 for the first limit and the continuity of v for the second one. Therefore hypothesis (ii) of Theorem 2.1 is satisfied, hence the statement follows by Theorem 2.1. $\square$

4 Corollaries II. Construction of $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ with the LD

In section 4.1 we develop a general strategy to establish the Laplace duality properties. When this procedure is applied to fulfill hypothesis (i) of Theorem 2.1, we obtain Corollary 4.3 and Theorem 4.4. Let us start with the following simple result about the relation among full and Laplace duality property.

Proposition 4.1. *Let $\mathfrak{W} \doteq \langle \langle \mathfrak{M}, \gamma \rangle, \rho, X, \mathfrak{R} \rangle$ and $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ be a $(\Theta, \mathcal{E})$ -structure such that $\mathfrak{V}$ is a Banach bundle and $x_\infty \in X$. Assume that*

1. $\mathfrak{V}$ and $\mathfrak{W}$ are full;
2. $\mathcal{E} = \Gamma(\pi)$ and Θ is given in (2.16);
3. $(\forall F \in \Gamma^{x_\infty}(\rho))(M(F) \doteq \sup_{x \in X} \sup_{s \in \mathbb{R}^+} \|F(x)(s)\| < \infty)$;
4. $(\forall \sigma \in \Gamma(\rho))(\sup_{x \in X} \sup_{s \in \mathbb{R}^+} \|\sigma(x)(s)\| < \infty)$;
5. the filter of neighbourhoods of x_∞ admits a countable basis.

If $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ has the LD then it has the LD_{x_∞} .

Proof. Let $F \in \Gamma^{x_\infty}(\rho)$ and $w \in \Gamma^{x_\infty}(\pi)$ thus by hypothesis (2) and [19, Corollary 3.1] there exist $\sigma \in \Gamma(\rho)$ and $v \in \Gamma(\pi)$ such that $\sigma(x_\infty) = F(x_\infty)$, $v(x_\infty) = w(x_\infty)$, and $\forall K \in \text{Comp}(\mathbb{R}^+), \forall v \in \mathcal{E}$

$$\begin{cases} \lim_{z \rightarrow x_\infty} \|w(z) - v(z)\| = 0 \\ \lim_{z \rightarrow x_\infty} \sup_{s \in K} \|(F(z)(s) - \sigma(z)(s))v(z)\| = 0. \end{cases} \quad (4.1)$$

Moreover $\forall \lambda > 0$

$$\begin{aligned} & \left\| \int_0^\infty e^{\lambda s} F(z)(s)w(z) ds - \int_0^\infty e^{\lambda s} \sigma(z)(s)v(z) ds \right\| \leq \\ & \left\| \int_0^\infty e^{\lambda s} F(z)(s)(w(z) - v(z)) ds \right\| + \left\| \int_0^\infty e^{\lambda s} (F(z)(s) - \sigma(z)(s))v(z) ds \right\| \leq \\ & \frac{1}{\lambda} M(F) \|v(z) - w(z)\| + \int_0^\infty e^{\lambda s} \|(F(z)(s) - \sigma(z)(s))v(z)\| ds. \end{aligned} \quad (4.2)$$

By the hypotheses (3–4) $\sup_{z \in X} \sup_{s \in \mathbb{R}^+} \|(F(z)(s) - \sigma(z)(s))v(z)\| < \infty$ hence $\forall \{z_n\}_{n \in \mathbb{N}} \subset X$ such that $\lim_{n \in \mathbb{N}} z_n = x_\infty$ we have by (4.1), (4.2) and a well-known theorem on convergence of sequences of integrals that $\forall \lambda > 0$

$$\lim_{n \in \mathbb{N}} \left\| \int_0^\infty e^{\lambda s} F(z_n)(s)w(z_n) ds - \int_0^\infty e^{\lambda s} \sigma(z_n)(s)v(z_n) ds \right\| = 0.$$

Thus $\forall \lambda > 0$ by hypothesis (5)

$$\lim_{z \rightarrow x_\infty} \left\| \int_0^\infty e^{\lambda s} F(z)(s)w(z) ds - \int_0^\infty e^{\lambda s} \sigma(z)(s)v(z) ds \right\| = 0,$$

hence the statement by [19, Corollary 3.1]. □

Now we shall see that in the case of a bundle of normed space we can choose for all x a simple space $\mathfrak{M}_x$ satisfying (2.14).

Proposition 4.2. *Let $\mathfrak{W} \doteq \langle \langle \mathfrak{M}, \gamma \rangle, \rho, X, \mathfrak{R} \rangle$ and $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ be a $(\Theta, \mathcal{E})$ -structure such that for all $x \in X$, $\mathfrak{E}_x$ is a reflexive Banach space, $S_x \subseteq \mathcal{P}_\omega(\mathfrak{E}_x)$ and*

$$\mathfrak{M}_x \subseteq \left\{ F \in \mathcal{C}(\mathbb{R}^+, \mathcal{L}_{S_x}(\mathfrak{E}_x)) \mid (\forall \lambda > 0) \left(\int_{\mathbb{R}^+}^* e^{-\lambda s} \|F(s)\|_{B(\mathfrak{E}_x)} ds < \infty \right) \right\}.$$

Thus

$$\mathfrak{M}_x \subset \bigcap_{\lambda > 0} \mathfrak{L}_1(\mathbb{R}^+, \mathcal{L}_{S_x}(\mathfrak{E}_x); \mu_\lambda). \quad (4.3)$$

In particular (4.3), and $\mathcal{U}_{\|\cdot\|_{B(\mathfrak{E}_x)}}(\mathcal{L}_{S_x}(\mathfrak{E}_x)) \subseteq \mathfrak{M}_x$ hold if for any $x \in X$

$$\mathfrak{M}_x = \left\{ F \in \mathcal{C}_c(\mathbb{R}^+, \mathcal{L}_{S_x}(\mathfrak{E}_x)) \mid \sup_{s \in \mathbb{R}^+} \|F(s)\|_{B(\mathfrak{E}_x)} < \infty \right\}.$$

Proof. The first sentence follows by [18, Corollary 2.6.], while the second sentence comes by the first one. $\square$

4.1 U-spaces

Aim of this section is to establish a procedure ensuring the full Laplace duality property, result achieved in Corollary 4.2. The core concept is that of U-Space provided in Definition 7, whose existence is established in Corollary 4.1 by mean of a special locally convex final topology constructed in Definition 9. Theorem 4.1, Theorem 4.2 and Theorem 4.3 represent the steps to obtain Corollary 4.1.

Let us recall and introduce some notation. For any W, Z topological vector spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ we denote by $\mathcal{L}(W, Z)$ the $\mathbb{K}$ -linear space of all continuous linear map on W and with values in Z and set $\mathcal{L}(Z) \doteq \mathcal{L}(Z, Z)$ and $Z^* \doteq \mathcal{L}(Z, \mathbb{K})$. If Y is a topological space we let $\mathcal{C}_{cs}(Y, Z)$ denote the linear space of all continuous maps $f : Y \rightarrow Z$ with compact support. If $Z \in Hlcs$ and Y is locally compact we denote by $\mathfrak{L}_1(Y, Z, \mu)$ the linear space of all maps on Y and with values in Z which are essentially μ -integrable in the sense described in [4, Chapter 6]. Moreover for any family $\{Z_x\}_{x \in X}$ of linear spaces and for all $x \in X$ set $\text{Pr}_x : \prod_{y \in X} Z_y \ni f \mapsto f(x) \in Z_x$ and $\iota_x : Z_x \rightarrow \prod_{y \in X} Z_y$ such that for all $x \neq y$ and $z_x \in Z_x$ $\text{Pr}_y \circ \iota_x(z_x) = \mathbf{0}_y$, while $\text{Pr}_x \circ \iota_x = \text{Id}_x$. Let us set

$$\langle \cdot, \cdot \rangle : \text{End}(\mathcal{H}) \times \mathcal{H} \ni (A, v) \mapsto A(v) \in \mathcal{H},$$

and for all $x \in X$

$$\langle \cdot, \cdot \rangle_x : \text{End}(\mathfrak{E}_x) \times \mathfrak{E}_x \ni (A, v) \mapsto A(v) \in \mathfrak{E}_x.$$

Definition 5. We call $\mathfrak{Q}$ a consistent class of data if $\mathfrak{Q} = \langle X, Y, \mu, \{\mathfrak{E}_x\}_{x \in X}, \{\tau_x\}_{x \in X}, \{\mathfrak{M}_x\}_{x \in X}, \{Q_x\}_{x \in X}, \langle \mathcal{H}, \mathfrak{T} \rangle \rangle$ where

1. X is a set, Y is a locally compact space and μ is a Radon measure on Y ;
2. $\{\mathfrak{E}_x\}_{x \in X}$ is a family of $Hlcs$;

3. $\{\tau_x\}_{x \in X}$ is a family of topologies such that $\langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle \in Hlcs, \forall x \in X$;
4. $\{\mathfrak{N}_x\}_{x \in X}$ is a family such that $\mathfrak{N}_x \doteq \{\nu_{j_x}^x \mid j_x \in J_x\}$ is a fundamental set of seminorms of $\mathfrak{E}_x, \forall x \in X$;
5. $\{Q_x\}_{x \in X}$ is a family such that $Q_x \doteq \{q_{\alpha_x}^x \mid \alpha_x \in A_x\}$ is a fundamental set of seminorms of $\langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle, \forall x \in X$;
6. $\langle \mathcal{H}, \mathfrak{T} \rangle \in Hlcs$ such that
 - $\mathcal{H} \subseteq \prod_{x \in X} \mathfrak{E}_x$ as linear spaces;
 - $\iota_x(\mathfrak{E}_x) \subset \mathcal{H}$, for all $x \in X$;
 - $\text{Pr}_x \in \mathcal{L}(\langle \mathcal{H}, \mathfrak{T} \rangle, \mathfrak{E}_x)$ and $\iota_x \in \mathcal{L}(\mathfrak{E}_x, \langle \mathcal{H}, \mathfrak{T} \rangle)$, for all $x \in X$;
 - $\exists \mathbb{A} \subseteq \prod_{x \in X} \mathcal{L}(\mathfrak{E}_x)$ linear space such that
 - (a) $\theta(\mathbb{A}) \upharpoonright \mathcal{H} \subseteq \mathcal{L}(\langle \mathcal{H}, \mathfrak{T} \rangle)$,
 - (b) $\iota_x(\mathcal{L}(\mathfrak{E}_x)) \subseteq \mathbb{A}$ for all $x \in X$;

where θ is defined in Definition 8. We call X the base of $\mathfrak{Q}$, Y the locally compact space of $\mathfrak{Q}$ and μ the Radon measure of $\mathfrak{Q}$. Moreover we call $\{\mathfrak{E}_x\}_{x \in X}$ the primary family underlying $\mathfrak{Q}$, while we call $\{\tau_x\}_{x \in X}$ the secondary family underlying $\mathfrak{Q}$. We call $\mathfrak{Q}$ entire if $\mathcal{H} = \prod_{x \in X} \mathfrak{E}_x$.

In the present section let $\mathfrak{Q} = \langle X, Y, \mu, \{\mathfrak{E}_x\}_{x \in X}, \{\tau_x\}_{x \in X}, \{\mathfrak{N}_x\}_{x \in X}, \{Q_x\}_{x \in X}, \langle \mathcal{H}, \mathfrak{T} \rangle \rangle$ be a fixed consistent class of data.

Definition 6. Let $\mathfrak{W} \doteq \langle \langle \mathfrak{M}, \gamma \rangle, \rho, X, \mathfrak{R} \rangle$ be a bundle of Ω -spaces such that for all $x \in X$

$$\mathfrak{M}_x \subseteq \mathfrak{L}_1(Y, \langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle; \mu).$$

Set

$$\begin{cases} \square_\mu : \prod_{x \in X} \mathfrak{L}_1(Y, \langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle; \mu) \times \prod_{x \in X} \mathfrak{E}_x \rightarrow \prod_{x \in X} \mathfrak{E}_x \\ \square_\mu(H, v)(x) \doteq \left\langle \int_{\mathbb{R}^+} H(x)(s) d\mu(s), v(x) \right\rangle_x \in \mathfrak{E}_x. \end{cases} \quad (4.4)$$

Remark 6. Let $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ be a $(\Theta, \mathcal{E})$ -structure satisfying (2.14) and $\mathcal{O} \subseteq \Gamma(\rho)$, $\mathcal{D} \subseteq \Gamma(\pi)$. Then

$$\text{LD}(\mathcal{O}, \mathcal{D}) \Leftrightarrow (\forall \lambda > 0) (\square_{\mu_\lambda}(\mathcal{O}, \mathcal{D}) \subseteq \Gamma(\pi)). \quad (4.5)$$

Similarly for all $x \in X$

$$\text{LD}_x(\mathcal{O}, \mathcal{D}) \Leftrightarrow (\forall \lambda > 0) (\square_{\mu_\lambda}(\Gamma_{\mathcal{O}}^x(\rho), \Gamma_{\mathcal{D}}^x(\pi)) \subseteq \Gamma^x(\pi)). \quad (4.6)$$

Definition 7. [U-Spaces] $\mathfrak{G}$ is a U-space with respect to $\{\langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle\}_{x \in X}$, $\mathfrak{T}$ and D if and only if

1. $\mathfrak{G} \in Hlcs$;
2. $\mathfrak{G} \subset \mathcal{L}(\langle \mathcal{H}, \mathfrak{T} \rangle)$ as linear spaces;
3. $D \subseteq \mathcal{H}$;

$$4. (\forall T \in lcp)(\exists \Psi_T \in \text{End}(\text{End}(\mathcal{H})^T, \prod_{x \in X} \text{End}(\mathfrak{E}_x)^Y))(\forall \nu \in \text{Radon}(T))$$

$$\Psi_T(\mathfrak{L}_1(T, \mathfrak{G}, \nu)) \subseteq \prod_{x \in X} \mathfrak{L}_1(T, \langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle; \nu),$$

$$\text{and } \forall \bar{F} \in \mathfrak{L}_1(T, \mathfrak{G}, \nu), \forall v \in D. \forall x \in X$$

$$\left\langle \int \Psi_T(\bar{F})(x)(s) d\nu(s), v(x) \right\rangle_x = \left\langle \int \bar{F}(s) d\nu(s), v \right\rangle(x) \quad (4.7)$$

The reason of introducing the concept of U-spaces will be clarified by the following

Proposition 4.3. *Let $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ be a $(\Theta, \mathcal{E})$ -structure satisfying (2.14), and let $\mathfrak{G}$ be a U-space with respect to $\{\mathcal{L}_{S_x}(\mathfrak{E}_x)\}_{x \in X}$, $\mathfrak{T}$ and $\mathcal{D}$. Then $\forall \lambda > 0, \bar{F} \in \mathfrak{L}_1(\mathbb{R}^+, \mathfrak{G}, \mu_\lambda), v \in \mathcal{D}$*

$$\square_{\mu_\lambda}(\Psi_{\mathbb{R}^+}(\bar{F}), v) = \left\langle \int \bar{F}(s) d\mu_\lambda(s), v \right\rangle. \quad (4.8)$$

Moreover if $\exists \mathcal{F} \subset \bigcap_{\lambda > 0} \mathfrak{L}_1(\mathbb{R}^+, \mathfrak{G}, \mu_\lambda)$ such that $\Psi_{\mathbb{R}^+}(\mathcal{F}) = \mathcal{O}$ then

$$\boxed{\text{LD}(\mathcal{O}, \mathcal{D}) \Leftrightarrow (\forall \lambda > 0)(\langle \mathcal{B}_\lambda, \mathcal{D} \rangle \subseteq \Gamma(\pi)).} \quad (4.9)$$

Here

$$\mathcal{B}_\lambda \doteq \left\{ \int \bar{F}(s) d\mu_\lambda(s) \mid \bar{F} \in \mathcal{F} \right\}.$$

Proof. (4.8) follows by (4.7), while (4.9) follows by (4.8) and Remark 6. $\square$

Remark 7. In particular if $\exists \mathcal{F} \subset \bigcap_{\lambda > 0} \mathfrak{L}_1(\mathbb{R}^+, \mathfrak{G}, \mu_\lambda)$ such that $\Psi_{\mathbb{R}^+}(\mathcal{F}) = \mathcal{O}$ then

$$\langle \mathfrak{G}, \mathcal{D} \rangle \subseteq \Gamma(\pi) \Rightarrow \text{LD}(\mathcal{O}, \mathcal{D}).$$

More in general if $\exists \mathfrak{G}_0$ complete subspace of $\mathfrak{G}$ and $\exists \mathcal{F} \subset \{\bar{F} \in \bigcap_{\lambda > 0} \mathfrak{L}_1(\mathbb{R}^+, \mathfrak{G}, \mu_\lambda) \mid \bar{F}(\mathbb{R}^+) \subseteq \mathfrak{G}_0\}$ such that $\Psi_{\mathbb{R}^+}(\mathcal{F}) = \mathcal{O}$ then

$$\langle \mathfrak{G}_0, \mathcal{D} \rangle \subseteq \Gamma(\pi) \Rightarrow \text{LD}(\mathcal{O}, \mathcal{D}).$$

Thus the U property expressed by (4.7) is an important tool for ensuring the satisfaction of the LD. For this reason the remaining of the present section will be dedicated to the construction of a space $\mathfrak{G}$, Definition (9), which is a U-space, see Theorem 4.3 and Corollary 4.2 for the $\text{LD}(\mathcal{O}, \mathcal{D})$.

Definition 8. Set

$$\begin{cases} \chi_{\mathcal{H}} : \text{End}(\mathcal{H}) \rightarrow \prod_{x \in X} \text{End}(\mathfrak{E}_x), \\ (\forall x \in X)(\forall w \in \text{End}(\mathcal{H}))((\text{Pr}_x \circ \chi_{\mathcal{H}})(w) = \text{Pr}_x \circ w \circ \iota_x), \\ \chi \doteq \chi_{\prod_{x \in X} \mathfrak{E}_x}. \end{cases}$$

Well defined indeed by construction $\iota_x(\mathfrak{E}_x) \subset \mathcal{H}$, for all $x \in X$. Finally set

$$\begin{cases} \theta : \prod_{x \in X} \text{End}(\mathfrak{E}_x) \rightarrow \text{End}(\prod_{x \in X} \mathfrak{E}_x), \\ (\forall x \in X)(\forall u \in \prod_{x \in X} \text{End}(\mathfrak{E}_x))(\text{Pr}_x \circ \theta(u) = \text{Pr}_x(u) \circ \text{Pr}_x), \\ \theta_{\mathcal{H}} : \text{Im}(\chi_{\mathcal{H}}) \ni u \mapsto \theta(u) \upharpoonright \mathcal{H}. \end{cases}$$

Well-posed by applying [1, Proposition 4, n°5, §1, Chapter 2].

Remark 8. $(\forall x \in X)(\forall u \in \prod_{x \in X} \text{End}(\mathfrak{E}_x))$ we have $(\text{Pr}_x \circ \theta(u) \circ \iota_x = \text{Pr}_x(u))$.

Proposition 4.4. *The space $\prod_{x \in X} \mathfrak{E}_x$ with the product topology satisfies the request for the space $\langle \mathcal{H}, \mathfrak{T} \rangle$ in Definition 5 with the choice $\mathbb{A} = \prod_{x \in X} \mathcal{L}(\mathfrak{E}_x)$.*

Proof. $\text{Pr}_x \in \mathcal{L}(\prod_{y \in X} \mathfrak{E}_y, \mathfrak{E}_x)$ by definition of the product topology, moreover $\iota_x \in \mathcal{L}(\mathfrak{E}_x, \prod_{y \in X} \mathfrak{E}_y)$. Indeed ι_x is clearly linear and by considering that for any net $\{f^\alpha\}_{\alpha \in D}$ and any f in $\prod_{y \in X} \mathfrak{E}_y$, $\lim_{\alpha \in D} f^\alpha = f$ if and only if $\lim_{\alpha \in D} f^\alpha(y) = f(y)$ for all $y \in X$, we deduce that for any net $\{f_x^\alpha\}_{\alpha \in D}$ and any f_x in $\mathfrak{E}_x$ such that $\lim_{\alpha \in D} f_x^\alpha = f_x$ we have $\lim_{\alpha \in D} \iota_x(f_x^\alpha) = \iota_x(f_x)$, so ι_x is continuous. Let $x \in X$ and $u \in \prod_{x \in X} \mathcal{L}(\mathfrak{E}_x)$ so $\text{Pr}_x(u) \circ \text{Pr}_x \in \mathcal{L}(\prod_{y \in X} \mathfrak{E}_y, \mathfrak{E}_x)$, so (6a) follows by the definition of θ and [2, Proposition 4, №3, §2]. Finally (6b) is trivial. $\square$

The following is the main structure of the present section. For the definition and properties of locally convex final topologies see [3, №4, §4].

Definition 9. Set for all $x \in X$

$$\begin{cases} G \doteq \theta(\mathbb{A}) \upharpoonright \mathcal{H}, \\ g_x : \mathcal{L}(\mathfrak{E}_x) \ni f_x \mapsto \iota_x \circ f_x \circ \text{Pr}_x \in \text{End}(\prod_{y \in X} \mathfrak{E}_y) \\ h_x : \mathcal{L}(\mathfrak{E}_x) \ni f_x \mapsto g_x(f_x) \upharpoonright \mathcal{H}. \end{cases}$$

We shall denote by $\mathfrak{G}$ and call the locally convex space relative to the consistent class of data $\mathfrak{Q}$, the lcs G provided with the locally convex final topology of the family of topologies $\{\tau_x\}_{x \in X}$ of the family $\{\mathcal{L}(\mathfrak{E}_x)\}_{x \in X}$, for the family of linear mappings $\{h_x\}_{x \in X}$.

Definition 10. Set in $\prod_{x \in X} \text{End}(\mathfrak{E}_x)$ the following binary operation $\circ$. For all $x \in X$ we set $\text{Pr}_x(f \circ h) \doteq f(x) \circ h(x)$.

It is easy to verify that $\langle \prod_{x \in X} \text{End}(\mathfrak{E}_x), +, \circ \rangle$ is an algebra over $\mathbb{K}$ as well as $\langle \prod_{x \in X} \mathcal{L}(\mathfrak{E}_x), +, \circ \rangle$.

Lemma 4.1. *$G \subset \mathcal{L}(\langle \mathcal{H}, \mathfrak{T} \rangle)$, moreover θ is a morphism of algebras. Finally if $\mathbb{A}$ is a subalgebra of $\prod_{x \in X} \mathcal{L}(\mathfrak{E}_x)$ then G is a subalgebra of $\mathcal{L}(\langle \mathcal{H}, \mathfrak{T} \rangle)$.*

Proof. The first sentence is immediate by (6a) in Definition 5. Let $u, v \in \prod_{x \in X} \mathcal{L}(\mathfrak{E}_x)$ thus for all $x \in X$

$$\begin{aligned} \text{Pr}_x \circ \theta(u \circ v) &= (u(x) \circ v(x)) \circ \text{Pr}_x \\ &= u(x) \circ \text{Pr}_x \circ \theta(v) \\ &= \text{Pr}_x \circ \theta(u) \circ \theta(v), \end{aligned}$$

so $\theta(u \circ v) = \theta(u) \circ \theta(v)$, similarly we can show that θ is linear by the linearity of Pr_x for all $x \in X$. Thus θ is a morphism of algebras, so the last sentence of the statement follows by the first one and the fact that $\mathbb{A}$ is an algebra. $\square$

Proposition 4.5. $\theta_{\mathcal{H}} \circ \chi_{\mathcal{H}}(w) = w \circ \iota_x \circ \text{Pr}_x \upharpoonright \mathcal{H}$ for all $w \in \text{End}(\mathcal{H})$, Moreover $\theta_{\mathcal{H}}(\text{Im}(\chi_{\mathcal{H}})) \subset \text{Dom}(\chi_{\mathcal{H}})$ and $\chi_{\mathcal{H}} \circ \theta_{\mathcal{H}} = \text{Id} \upharpoonright \text{Im}(\chi_{\mathcal{H}})$.

Proof. Let $w \in \text{End}(\mathcal{H})$ thus for all $x \in X$ we have $(\text{Pr}_x \circ \theta_{\mathcal{H}} \circ \chi_{\mathcal{H}})(w) = \text{Pr}_x(\chi_{\mathcal{H}}(w)) \circ \text{Pr}_x \upharpoonright \mathcal{H} = \text{Pr}_x \circ w \circ \iota_x \circ \text{Pr}_x \upharpoonright \mathcal{H}$ and the first sentence of the statement follows. By the first sentence and the assumption that $\iota_x(\mathfrak{E}_x) \subset \mathcal{H}$ we have $\theta(\text{Im}(\chi_{\mathcal{H}})) \upharpoonright \mathcal{H} \subset \text{End}(\mathcal{H})$ so $\chi_{\mathcal{H}} \circ \theta_{\mathcal{H}}$ is well set. Moreover for all $x \in X$ and $u \in \text{Im}(\chi_{\mathcal{H}})$ we have $\text{Pr}_x(\chi_{\mathcal{H}}(\theta(u) \upharpoonright \mathcal{H})) = \text{Pr}_x \circ \theta(u) \circ \iota_x = \text{Pr}_x(u) \circ \text{Pr}_x \circ \iota_x = \text{Pr}_x(u)$. $\square$

Proposition 4.6. Let $x \in X$, then

1. $g_x = \theta \circ \iota_x$ so $\text{Im}(h_x) \subseteq G$;
2. $h_x \in \text{End}(\mathcal{L}(\mathfrak{E}_x), G)$;
3. $\exists h_x^{-1} : \text{Im}(h_x) \rightarrow \mathcal{L}(\mathfrak{E}_x)$ and

$$\begin{cases} h_x^{-1} = \text{Pr}_x \circ \chi_{\mathcal{H}} \upharpoonright \text{Im}(h_x), \\ \text{Im}(h_x) = \{\theta(\iota_x(f_x)) \upharpoonright \mathcal{H} \mid f_x \in \mathcal{L}(\mathfrak{E}_x)\}. \end{cases}$$

Proof. $\forall y \in X$ we have

$$\text{Pr}_y \circ \theta(\iota_x(f_x)) = \text{Pr}_y(\iota_x(f_x)) \circ \text{Pr}_y = \begin{cases} \mathbf{0}_y, x \neq y \\ f_x \circ \text{Pr}_x, x = y. \end{cases}$$

Moreover

$$\text{Pr}_y \circ g_x(f_x) = \text{Pr}_y \circ \iota_x \circ f_x \circ \text{Pr}_x = \begin{cases} \mathbf{0}_y, x \neq y \\ f_x \circ \text{Pr}_x, x = y. \end{cases}$$

So the first sentence of statement (1) follows. Thus $h_x(\mathcal{L}(\mathfrak{E}_x)) = g_x(\mathcal{L}(\mathfrak{E}_x)) \upharpoonright \mathcal{H} = \theta(\iota_x(\mathcal{L}(\mathfrak{E}_x))) \upharpoonright \mathcal{H}$ so by (6b) of Definition 5 the second sentence of statement (1) follows. Statement (2) follows by the trivial linearity of g_x and by the second sentence of statement (1). Let $f_x \in \mathcal{L}(\mathfrak{E}_x)$ and $w = \iota_x \circ f_x \circ \text{Pr}_x \upharpoonright \mathcal{H}$. Then by the assumption (6) we have that $w \in \text{End}(\mathcal{H})$, and $\chi_{\mathcal{H}}(w) = \iota_x(f_x)$, indeed $\text{Pr}_x(\chi_{\mathcal{H}}(w)) = \text{Pr}_x \circ \iota_x \circ f_x \circ \text{Pr}_x \circ \iota_x = f_x = \text{Pr}_x(\iota(f_x))$. Thus $\iota_x(f_x) \in \text{Im}(\chi_{\mathcal{H}})$ so by Proposition 4.5 $\theta(\iota_x(f_x)) \upharpoonright \mathcal{H} \in \text{Dom}(\chi_{\mathcal{H}})$ and h_x^{-1} is well set. Moreover

$$\begin{aligned} (\text{Pr}_x \circ \chi_{\mathcal{H}}) \circ h_x(f_x) &= \text{Pr}_x \circ \chi_{\mathcal{H}} \circ \theta_{\mathcal{H}}(\iota_x(f_x)) \\ &= \text{Pr}_x(\iota_x(f_x)) = f_x, \end{aligned}$$

where the first equality comes by statement (1) and by $\iota_x(f_x) \in \text{Im}(\chi_{\mathcal{H}})$, while the second by Proposition 4.5. Finally

$$\begin{aligned} g_x \circ \text{Pr}_x \circ \chi_{\mathcal{H}}(\theta(\iota_x(f_x))) &= g_x \circ \text{Pr}_x(\iota_x(f_x)) \\ &= g_x(f_x) = \theta(\iota_x(f_x)). \end{aligned}$$

Thus statement (3) follows. $\square$

Lemma 4.2. *If $\langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle$ is a topological algebra for all $x \in X$ and $\mathbb{A}$ is an algebra then $\mathfrak{G}$ is a topological algebra.*

Proof. Let us set for all $F \in \mathfrak{G}$ $L_F : \mathfrak{G} \ni H \mapsto F \circ H \in \mathfrak{G}$, well set $\mathfrak{G}$ being an algebra by Lemma 4.1. Thus for all $x \in X$, $f \in \mathbb{A}$ and $l_x \in \mathcal{L}(\mathfrak{E}_x)$

$$\begin{aligned} (L_{\theta(f)} \circ h_x)l_x &= L_{\theta(f)}(\theta(\iota_x(l_x)) \upharpoonright \mathcal{H}) = \theta(f \circ \iota_x(l_x)) \upharpoonright \mathcal{H} \\ &= (\theta \circ \iota_x(f(x) \circ l_x)) \upharpoonright \mathcal{H} = (g_x(f(x) \circ l_x)) \upharpoonright \mathcal{H} \\ &= h_x(f(x) \circ l_x) = (h_x \circ L_{f(x)})l_x, \end{aligned}$$

where $L_{f_x} : \mathcal{L}(\mathfrak{E}_x) \ni s_x \mapsto f_x \circ s_x \in \mathcal{L}(\mathfrak{E}_x)$ for all $f_x \in \mathcal{L}(\mathfrak{E}_x)$. Here the first and fourth equality follow by Proposition 4.6, the second one by Lemma 4.1. Moreover by hypothesis $L_{f(x)}$ is continuous, while h_x is continuous by [3, Proposition 5, №4, §4 Chapter 2], so $L_{\theta(f)} \circ h_x$ is linear and continuous. Therefore $L_{\theta(f)}$ is linear and continuous by [3, Proposition 5, №4, §4 Chapter 2]. Similarly R_F is linear and continuous, where $R_F : \mathfrak{G} \ni H \mapsto H \circ F \in \mathfrak{G}$, thus the statement. $\square$

Definition 11. Set

$$\begin{cases} \Psi_Y^{\mathcal{H}} : \text{End}(\mathcal{H})^Y \rightarrow \prod_{x \in X} \text{End}(\mathfrak{E}_x)^Y, \\ (\text{Pr}_x \circ \Psi_Y^{\mathcal{H}})(\overline{F})(s) = (\text{Pr}_x \circ \chi_{\mathcal{H}})(\overline{F}(s)). \end{cases}$$

Moreover set

$$\begin{cases} \Lambda : \prod_{x \in X} \text{End}(\mathfrak{E}_x)^Y \rightarrow (\text{End}(\prod_{x \in X} \mathfrak{E}_x))^Y, \\ \Lambda(F)(s) = \theta(F(\cdot)(s)). \end{cases}$$

$\forall \overline{F} \in \text{End}(\mathcal{H})^Y$, $\forall F \in \prod_{x \in X} \text{End}(\mathfrak{E}_x)^Y$, $\forall x \in X$ and $\forall s \in Y$, where $F(\cdot)(s) \in \prod_{y \in X} \text{End}(\mathfrak{E}_x)$ such that $\text{Pr}_x(F(\cdot)(s)) = F(x)(s)$.

Finally set

$$\Lambda_{\mathbb{A}}^Y \doteq \Lambda \upharpoonright \{F \in \prod_{x \in X} \mathcal{L}(\mathfrak{E}_x)^Y \mid (\forall s \in Y)(F(\cdot)(s) \in \mathbb{A})\}.$$

Proposition 4.7. *Let $x \in X$ and $s \in Y$, then for all $\overline{F} \in \text{End}(\mathcal{H})^Y$*

1. $(\text{Pr}_x \circ \Psi_Y^{\mathcal{H}})(\overline{F})(s) = \text{Pr}_x \circ \overline{F}(s) \circ \iota_x$;
2. $\Psi_Y^{\mathcal{H}} \circ \Lambda_{\mathbb{A}}^Y = \text{Id}$;
3. $\text{Im}(\Lambda_{\mathbb{A}}^Y) \subset G^Y$.

Proof. Statements (1) and (3) are trivial. Let $F \in \text{Dom}(\Lambda_{\mathbb{A}}^Y)$ so

$$\begin{aligned} (\text{Pr}_x \circ \Psi_Y^{\mathcal{H}} \circ \Lambda_{\mathbb{A}}^Y)(F)(s) &= (\text{Pr}_x \circ \chi_{\mathcal{H}})(\Lambda_{\mathbb{A}}^Y(F)(s)) = \text{Pr}_x \circ \Lambda_{\mathbb{A}}^Y(F)(s) \circ \iota_x \\ &= \text{Pr}_x \circ \theta(F(\cdot)(s)) \circ \iota_x = \text{Pr}_x(F(\cdot)(s)) \circ \text{Pr}_x \circ \iota_x \\ &= F(x)(s) = \text{Pr}_x(F)(s), \end{aligned}$$

and statement (2) follows. $\square$

Proposition 4.8. $(\forall x \in X)(\forall s \in Y)(\forall \bar{F} \in G^Y)$ we have

$$(\Pr \circ \Psi_Y^{\mathcal{H}})(\bar{F})(s) \circ \Pr = \Pr \circ (\bar{F}(s))$$

Proof. Let $\bar{F} \in G^Y$ thus $\exists U \in \mathbb{A}^Y$ such that $\bar{F}(s) = \theta(U(s)) \upharpoonright \mathcal{H}$, hence for all $x \in X, s \in Y$

$$\begin{aligned} (\Pr \circ \Psi_Y^{\mathcal{H}})(\bar{F})(s) \circ \Pr &= \Pr(\Psi_Y^{\mathcal{H}}(\bar{F}))(s) \circ \Pr \\ &= \Pr \circ \bar{F}(s) \circ \iota_x \circ \Pr, && \text{by Proposition 4.7} \\ &= (\Pr \circ \theta(U(s))) \upharpoonright \mathcal{H} \circ \iota_x \circ \Pr \\ &\doteq (\Pr(U(s)) \circ \Pr) \upharpoonright \mathcal{H} \circ \iota_x \circ \Pr \\ &= \Pr(U(s)) \circ \Pr \upharpoonright \mathcal{H} \\ &\doteq \Pr \circ \theta(U(s)) \upharpoonright \mathcal{H} \\ &= \Pr \circ (\bar{F}(s)). \end{aligned}$$

□

Definition 12. Let $x \in X$

$$\begin{cases} I_x : \text{Hom}(\mathcal{L}(\mathfrak{E}_x), \mathbb{K}) \rightarrow \text{Hom}\left(\prod_{y \in X} \mathcal{L}(\mathfrak{E}_y), \mathbb{K}\right), \\ I_x(t_x) \doteq t_x \circ \Pr_x. \end{cases}$$

Lemma 4.3. Let $x \in X$ thus

1. $(\forall t_x \in \text{Hom}(\mathcal{L}(\mathfrak{E}_x), \mathbb{K}))(\forall y \in X)$ we have

$$\begin{cases} I_x(t_x) \circ \chi_{\mathcal{H}} \circ h_y = t_x, x = y \\ I_x(t_x) \circ \chi_{\mathcal{H}} \circ h_y = \mathbf{0}, x \neq y; \end{cases}$$

2. $(\forall t_x \in \langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle^*)(I_x(t_x) \circ \chi_{\mathcal{H}} \upharpoonright \mathfrak{G} \in \mathfrak{G}^*)$

Proof. Let $x \in X$ and $t_x \in \text{Hom}(\mathcal{L}(\mathfrak{E}_x), \mathbb{K})$ thus for all $y \in X$ and $f_y \in \mathcal{L}(\mathfrak{E}_y)$ we have

$$\begin{aligned} I_x(t_x) \circ \chi_{\mathcal{H}} \circ h_y(f_y) &= t_x \circ \Pr \circ \chi_{\mathcal{H}}(\iota_y \circ f_y \circ \Pr \upharpoonright \mathcal{H}) \\ &= t_x \circ (\Pr \circ \iota_y \circ f_y \circ \Pr \circ \iota_x), \end{aligned}$$

and statement (1) follows. Statement (2) follows by statement (1) and [3, Proposition 5, №4, §4 Chapter 2]. □

The following is the first main result of this section.

Theorem 4.1. We have

1. $\Psi_Y^{\mathcal{H}} \in \text{Hom}(\mathfrak{L}_1(Y, \mathfrak{G}, \mu), \prod_{x \in X} \mathfrak{L}_1(Y, \langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle, \mu));$

$$2. (\forall x \in X)(\forall s \in Y)(\forall \bar{F} \in \mathfrak{L}_1(Y, \mathfrak{G}, \mu))$$

$$\int \Pr_x(\Psi_Y^{\mathcal{H}}(\bar{F}))(s) d\mu(s) = \Pr_x \circ \left(\int \bar{F}(s) d\mu(s) \right) \circ \iota_x.$$

Proof. Let $x \in X$, set

$$\Delta_x : G \ni f \mapsto \Pr_x \circ f \circ \iota_x \in \mathcal{L}(\mathfrak{E}_x).$$

Δ_x is well-defined by Lemma 4.1. By applying [3, Proposition 5, №3, §4 Chapter 2] $\Delta_x \in \mathcal{L}(\mathfrak{G}, \langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle)$ if and only if $(\forall y \in X)(\Delta_x \circ h_y \in \mathcal{L}(\langle \mathcal{L}(\mathfrak{E}_y), \tau_y \rangle, \langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle))$. Moreover $\forall y \in X$ and $\forall f_y \in \mathcal{L}(\mathfrak{E}_y)$ we have

$$(\Delta_x \circ h_y)(f_y) = \Pr_x \circ \iota_y \circ f_y \circ \Pr_y \circ \iota_x,$$

so

$$\begin{cases} \Delta_x \circ h_y = Id, x = y \\ \Delta_x \circ h_y = \mathbf{0}, x \neq y. \end{cases}$$

In any case $\Delta_x \circ h_y \in \mathcal{L}(\langle \mathcal{L}(\mathfrak{E}_y), \tau_y \rangle, \langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle)$, thus

$$\Delta_x \in \mathcal{L}(\mathfrak{G}, \langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle)$$

hence

$$(\forall t_x \in \langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle^*)(t_x \circ \Delta_x \in \mathfrak{G}^*). \quad (4.10)$$

Therefore

$$\begin{aligned} t_x \left(\Pr_x \circ \left(\int \bar{F}(s) d\mu(s) \right) \circ \iota_x \right) &= (t_x \circ \Delta) \left(\int \bar{F}(s) d\mu(s) \right) \\ &= \int (t_x \circ \Delta)(\bar{F}(s)) d\mu(s) \\ &= \int t_x (\Pr_x \circ \bar{F}(s) \circ \iota_x) d\mu(s) \\ &= \int t_x \left((\Pr_x \circ \Psi_Y^{\mathcal{H}})(\bar{F})(s) \right) d\mu(s), \end{aligned}$$

where the second equality comes by (4.10) and [4, Proposition 1, №1, §1, Chapter 6], while the last one comes by Proposition 4.7. $\square$

Definition 13. Let Z be a topological vector space set

$$\begin{cases} \mathbf{ev}_Z \in Hom(Z, Hom(\mathcal{L}(Z), Z)), \\ (\forall v \in Z)(\forall f \in \mathcal{L}(Z))(\mathbf{ev}_Z(v)(f)) \doteq f(v). \end{cases}$$

Moreover set $\eta \doteq \mathbf{ev}_{\mathcal{H}}$ and $\forall x \in X$ set $\varepsilon_x \doteq \mathbf{ev}_{\mathfrak{E}_x}$.

Lemma 4.4. Let $D \subseteq \mathcal{H}$ thus $(A) \Rightarrow (B)$, where

$$(A) \quad (\forall x \in X)(\forall v_x \in \Pr_x(D))(\varepsilon_x(v_x) \in \mathcal{L}(\langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle, \mathfrak{E}_x));$$

(B) $(\forall v \in D)(\eta(v) \in \mathcal{L}(\mathfrak{G}, \langle \mathcal{H}, \mathfrak{T} \rangle))$.

Proof. Let $y \in X$ thus for all $v \in \mathcal{H}$

$$\eta(v) \circ h_y = \iota_y \circ \varepsilon_y(\text{Pr}(v)).$$

Hence by (A) and the fact that by construction ι_y is continuous with respect to the topology $\mathfrak{T}$ we have for all $v \in D$

$$\eta(v) \circ g_y \in \mathcal{L}(\langle \mathcal{L}(\mathfrak{E}_y), \tau_y \rangle, \langle \mathcal{H}, \mathfrak{T} \rangle).$$

Thus (B) follows by the universal property of any locally final topology, cf. [3, (ii) of Proposition 5, №4, §4 Chapter 2]. $\square$

The following is the second main result of the section

Theorem 4.2. *Let $D \subseteq \mathcal{H}$ and assume (A) of Lemma 4.4. Then $(\forall \bar{F} \in \mathfrak{L}_1(Y, \mathfrak{G}, \mu))(\forall x \in X)(\forall v \in D)$*

$$\int \left\langle \text{Pr}_x(\Psi_Y^{\mathcal{H}}(\bar{F}))(s), v(x) \right\rangle_x d\mu(s) = \left\langle \int \bar{F}(s) d\mu(s), v \right\rangle(x). \quad (4.11)$$

Here the integral in the left-side is with respect to the μ and the locally convex topology on $\mathfrak{E}_x$, while the integral in the right-side is with respect to the μ and the locally convex topology on $\mathfrak{G}$.

Proof. $(\forall \bar{F} \in \mathfrak{L}_1(Y, \mathfrak{G}, \mu))(\forall x \in X)(\forall v \in D)$ we have

$$\begin{aligned} \text{Pr}_x \circ \left(\int \bar{F}(s) d\mu(s) \right)(v) &= (\text{Pr}_x \circ \eta(v)) \left(\int \bar{F}(s) d\mu(s) \right) \\ &= \int (\text{Pr}_x \circ \eta(v))(\bar{F}(s)) d\mu(s) \\ &= \int (\text{Pr}_x \circ \bar{F})(s)(v) d\mu(s) \\ &= \int \text{Pr}_x(\Psi_Y^{\mathcal{H}}(\bar{F}))(s)(v(x)) d\mu(s). \end{aligned}$$

Here in the second equality we applied [4, Proposition 1, №1, §1, Chapter 6] and the fact that $\text{Pr}_x \circ \eta(v) \in \mathcal{L}(\mathfrak{G}, \mathfrak{E}_x)$ because of Lemma 4.4 and the linearity and continuity of Pr_x with respect to the topology $\mathfrak{T}$. Finally in the last equality we used Proposition 4.8. $\square$

The following is the main result of this section

Theorem 4.3. *Let $D \subseteq \mathcal{H}$ and assume (A) of Lemma 4.4. Then $(\forall \bar{F} \in \mathfrak{L}_1(Y, \mathfrak{G}, \mu))(\forall x \in X)(\forall v \in D)$*

$$\boxed{\left\langle \int \text{Pr}_x(\Psi_Y^{\mathcal{H}}(\bar{F}))(s) d\mu(s), v(x) \right\rangle_x = \left\langle \int \bar{F}(s) d\mu(s), v \right\rangle(x)}. \quad (4.12)$$

Equivalently $\mathfrak{G}$ is a $\mathcal{U}$ -space with respect to $\{\langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle\}_{x \in X}$, $\mathfrak{T}$ and D . Here the integral in the left-side is with respect to the μ and the locally convex topology on $\langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle$.

Proof. By (A) of Lemma 4.4, statement (1) of Theorem 4.1 and [4, Proposition 1, №1, §1, Chapter 6] we have $(\forall \bar{F} \in \mathfrak{L}_1(Y, \mathfrak{G}, \mu))(\forall x \in X)(\forall v \in D)$

$$\int \left\langle \Pr_x(\Psi_Y^{\mathcal{H}}(\bar{F}))(s), v(x) \right\rangle_x d\mu(s) = \left\langle \int_x \Pr_x(\Psi_Y^{\mathcal{H}}(\bar{F}))(s) d\mu(s), v(x) \right\rangle_x,$$

hence the statement follows by Theorem 4.2. $\square$

Remark 9. By (4.12) and statement (2) of Theorem 4.1 $(\forall \bar{F} \in \mathfrak{L}_1(Y, \mathfrak{G}, \mu))(\forall x \in X)(\forall v \in D)$

$$\left\langle \int \bar{F}(s) d\mu(s), v \right\rangle(x) = \left\langle \int \bar{F}(s) d\mu(s), \iota_x(v(x)) \right\rangle(x).$$

Thus for all $v, w \in D$ and $x \in X$

$$v(x) = w(x) \Rightarrow \left\langle \int \bar{F}(s) d\mu(s), v \right\rangle(x) = \left\langle \int \bar{F}(s) d\mu(s), w \right\rangle(x).$$

Corollary 4.1. Let $\mathcal{S} \in \prod_{x \in X} \mathcal{P}(\text{Bounded}(\mathfrak{E}_x))$ and $\mathcal{D}$ such that

$$\begin{cases} N(x) \doteq \bigcup_{l_x \in L_x} B_{l_x}^x \text{ is total in } \mathfrak{E}_x, \forall x \in X, \\ \mathcal{D} \subseteq \mathcal{H} \cap \prod_{x \in X} N(x), \end{cases} \quad (4.13)$$

where $\mathcal{S}(x) = \{B_{l_x}^x \mid l_x \in L_x\}$. Assume that for all $x \in X$ the topology τ_x is generated by the set of seminorms $\{p_{(l_x, j_x)}^x \mid (l_x, j_x) \in L_x \times J_x\}$, where ⁵

$$p_{(l_x, j_x)}^x : \mathcal{L}(\mathfrak{E}_x) \ni f_x \mapsto \sup_{w \in B_{l_x}^x} \nu_{j_x}^x(f_x w) \in \mathbb{R}^+. \quad (4.14)$$

Then

1. (A) of Lemma 4.4 for $D = \mathcal{D}$;
2. (4.11) holds and $\mathfrak{G}$ is a $\mathcal{U}$ -space with respect to $\{\mathcal{L}_{\mathcal{S}(x)}(\mathfrak{E}_x)\}_{x \in X}$, $\mathfrak{T}$ and $\mathcal{D}$.

Proof. By request (4.13) we have that the lcs $\langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle$ is Hausdorff so the position is well-set. By construction $(\forall x \in X)(\forall v_x \in D(x))(\exists \bar{l}_x \in L_x)(v_x \in B_{\bar{l}_x}^x)$, so $(\forall f_x \in \mathcal{L}(\mathfrak{E}_x))(\forall j_x \in J_x)$

$$\begin{aligned} \nu_{j_x}^x(\varepsilon_x(v_x)f_x) &= \nu_{j_x}^x(f_x(v_x)) \\ &\leq p_{(\bar{l}_x, j_x)}^x(f_x), \end{aligned}$$

hence statement (1) by [3, Proposition 5, №4, §1 Ch 2]. Statement (2) follows by statement (1), Theorem 4.2 and Theorem 4.3 respectively. $\square$

⁵In others words $\langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle = \mathcal{L}_{\mathcal{S}_x}(\mathfrak{E}_x)$, see [19, Notation 2] and [19, Definition 4]

Corollary 4.2 ($\text{LD}(\mathcal{O}, \mathcal{D})$). *Let $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ be a $(\Theta, \mathcal{E})$ -structure satisfying (2.14) and $\Gamma(\pi) \cap \mathcal{H} \cap \prod_{x \in X} \mathcal{B}_B^x \neq \emptyset$. Set*

$$\begin{cases} \mathcal{O} \subseteq \Gamma(\rho) \\ \mathcal{D} \subseteq \Gamma(\pi) \cap \mathcal{H} \cap \prod_{x \in X} \mathcal{B}_B^x \end{cases} \quad (4.15)$$

If $\exists \mathcal{F} \subset \bigcap_{\lambda > 0} \mathfrak{L}_1(\mathbb{R}^+, \mathfrak{G}, \mu_\lambda)$ such that $\Psi_{\mathbb{R}^+}^{\mathcal{H}}(\mathcal{F}) = \mathcal{O}$ then (4.9) holds.

In particular if $\exists \mathcal{F} \subset \bigcap_{\lambda > 0} \mathfrak{L}_1(\mathbb{R}^+, \mathfrak{G}, \mu_\lambda)$ such that $\Psi_{\mathbb{R}^+}^{\mathcal{H}}(\mathcal{F}) = \mathcal{O}$ then

$$\langle \mathfrak{G}, \mathcal{D} \rangle \subseteq \Gamma(\pi) \Rightarrow \text{LD}(\mathcal{O}, \mathcal{D}).$$

Here $\mathcal{B}_B^x$, for all $x \in X$, is defined in [19, equality (5.3)]

Proof. By statement (2) of Corollary 4.1, Proposition 4.3 and Remark 7. $\square$

Remark 10. Note that if $\mathcal{E} \subset \Theta$, as for example for the positions taken in Remark 4, we have $\mathcal{E} \subset \prod_{x \in X} \mathcal{B}_B^x$. Hence if $\mathcal{E} \subseteq \mathcal{H}$ we have $\mathcal{E} \subseteq \Gamma(\pi) \cap \mathcal{H} \cap \prod_{x \in X} \mathcal{B}_B^x$.

Corollary 4.3. *Let us assume the hypotheses of Theorem 2.1 made exception for the (i) replaced by the following one: $\mathcal{E} \subseteq \mathcal{H}$ and $\exists \mathcal{F} \subset \bigcap_{\lambda > 0} \mathfrak{L}_1(\mathbb{R}^+, \mathfrak{G}, \mu_\lambda)$ such that $\Psi_{\mathbb{R}^+}^{\mathcal{H}}(\mathcal{F}) = \Gamma(\rho)$ and*

$$\langle \mathfrak{G}, \mathcal{E} \rangle \subseteq \Gamma(\pi).$$

Then all the statements of Theorem 2.1 hold true.

Proof. Since Remark 10, Corollary 4.2 and Theorem 2.1. $\square$

4.2 Uniform convergence over $\mathcal{K} \in \text{Comp}(\langle \mathcal{H}, \mathfrak{T} \rangle)$.

In this section we assume given the following data

1. a Banach bundle $\mathfrak{V}$, a $(\Theta, \mathcal{E})$ -structure $\langle \mathfrak{V}, \mathfrak{M}, X, Y \rangle$ where Θ is defined in (2.16), where we denote $\mathfrak{W} \doteq \langle \langle \mathfrak{M}, \gamma \rangle, \rho, X, \mathfrak{R} \rangle$ and $\mathfrak{V} \doteq \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \{\|\cdot\|\} \rangle$;
2. a Banach space $\langle \mathcal{H}, \|\cdot\|_{\mathcal{H}} \rangle$ such that $\langle \mathcal{H}, \mathfrak{T} \rangle$ satisfies (6) of Definition 5, where $\mathfrak{T}$ is the topology induced by the norm $\|\cdot\|_{\mathcal{H}}$ and τ_x is such that $\langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle = \mathcal{L}_{S_x}(\mathfrak{E}_x)$ for every $x \in X$;
3. $\mathbb{A}$ as in (6) of Definition 5;
4. $\mathfrak{G}$, $\Psi_Y^{\mathcal{H}}$ and $\Lambda_{\mathbb{A}}^Y$ as defined in Definition 9 and Definition 11 respectively.

The proof of the following Lemma is an adaptation to the present framework of the proof of [8, Proposition 5.13].

Lemma 4.5. *Let $\mathcal{U} \in \prod_{x \in X} \mathfrak{M}_x$ and $x_\infty \in X$ moreover assume that*

1. $\mathcal{E} \subseteq \mathcal{H} \subseteq \prod_{x \in X}^b \mathfrak{E}_x$ such that $(\exists a > 0)(\forall f \in \mathcal{H})(\|f\|_{\sup} \leq a\|f\|_{\mathcal{H}})$, where $\|f\|_{\sup} \doteq \sup_{x \in X} \|f(x)\|_x$;
2. $\exists F \in \Gamma(\rho)$ such that $F(x_\infty) = \mathcal{U}(x_\infty)$ and $\{F(\cdot)(s) \mid s \in Y\} \subseteq \mathbb{A}$

3. $\{\mathcal{U}(\cdot)(s) \mid s \in Y\} \subseteq \mathbb{A}$;
4. $\{\overline{F}(s) \mid s \in Y\}$ and $\{\overline{\mathcal{U}}(s) \mid s \in Y\}$ are equicontinuous as subsets of $\mathcal{L}(\langle \mathcal{H}, \|\cdot\|_{\mathcal{H}} \rangle)$, where $\overline{\mathcal{U}} \doteq \Lambda_{\mathbb{A}}^Y(\mathcal{U})$. and $\overline{F} \doteq \Lambda_{\mathbb{A}}^Y(F)$.

Then $(A) \Leftrightarrow (B)$ where

(A) $\mathcal{U} \in \Gamma^{x_\infty}(\rho)$;

(B) For all $\mathcal{K} \in \text{Comp}(\mathcal{H})$ such that $\mathcal{K} \subseteq \mathcal{E}$ and for all $K \in \text{Comp}(Y)$

$$\lim_{z \rightarrow x_\infty} \sup_{s \in K} \sup_{v \in \mathcal{K}} \|\mathcal{U}(z)(s)v(z) - F(z)(s)v(z)\| = 0.$$

Proof. We shall prove only $(A) \Rightarrow (B)$, indeed the other implication follows by $(3) \Rightarrow (4)$ of [19, Lemma 5.1]. So assume (A) to be true. In this proof let us set $B(\mathcal{H}) \doteq \mathcal{L}(\langle \mathcal{H}, \|\cdot\|_{\mathcal{H}} \rangle)$, moreover $\Psi \doteq \Psi_Y^{\mathcal{H}}$ and $\Lambda \doteq \Lambda_{\mathbb{A}}^Y$, moreover set $\overline{F} \doteq \Lambda_{\mathbb{A}}^Y(F)$ for every $F \in \Gamma(\rho)$; thus by statement (2) of Proposition 4.7 $\Psi(\overline{F}) = F$ and $\Psi(\overline{\mathcal{U}}) = \mathcal{U}$. Hence by Proposition 4.8 for all $v \in \mathcal{E}$ $F \in \Gamma(\rho)$, $z \in X$ and $s \in Y$

$$\mathcal{U}(z)(s)v(z) = (\overline{\mathcal{U}}v)(z), \quad F(z)(s)v(z) = (\overline{F}v)(z). \quad (4.16)$$

By (A) and implication $(4) \Rightarrow (3)$ of [19, Lemma 5.1] we have for all $K \in \text{Comp}(Y)$ and $v \in \mathcal{E}$

$$\lim_{z \rightarrow x_\infty} \sup_{s \in K} \|\mathcal{U}(z)(s)v(z) - F(z)(s)v(z)\| = 0. \quad (4.17)$$

Fix $\mathcal{K} \in \text{Comp}(\mathcal{H})$ such that $\mathcal{K} \subseteq \mathcal{E}$, $f \in \mathcal{K}$ and $\varepsilon > 0$, thus by (4.17) and (4.16) there exists U neighbourhood of x_∞ such that

$$\sup_{s \in K} \sup_{z \in U} \|(\overline{\mathcal{U}}(s) - \overline{F}(s))f(z)\| \leq \varepsilon/2. \quad (4.18)$$

Define

$$\begin{cases} M \doteq \max\{\sup_{s \in Y} \|\overline{F}(s)\|_{B(\mathcal{H})}, \sup_{s \in Y} \|\overline{\mathcal{U}}(s)\|_{B(\mathcal{H})}\} \\ \eta \doteq \varepsilon/4aM \\ \mathfrak{U}(f) \doteq \{g \in \mathcal{K} \mid \|f - g\|_{\mathcal{H}} < \eta\}. \end{cases}$$

Thus for all $g \in \mathfrak{U}(f)$

$$\begin{aligned} & \sup_{z \in U} \sup_{s \in K} \|\mathcal{U}(z)(s)g(z) - F(z)(s)g(z)\| = \\ & \sup_{s \in K} \sup_{z \in U} \|(\overline{\mathcal{U}}(s) - \overline{F}(s))g(z)\| \leq \\ & \sup_{s \in K} \sup_{z \in U} \|(\overline{\mathcal{U}}(s) - \overline{F}(s))f(z)\| + \sup_{s \in K} \sup_{z \in U} \|\overline{\mathcal{U}}(s)(g - f)(z)\| + \sup_{s \in K} \sup_{z \in U} \|F(s)(g - f)(z)\| \leq \\ & \varepsilon/2 + a \sup_{s \in K} \|\overline{\mathcal{U}}(s)(g - f)\|_{\mathcal{H}} + a \sup_{s \in K} \|F(s)(g - f)\|_{\mathcal{H}} \leq \\ & \varepsilon/2 + 2aM \|g - f\|_{\mathcal{H}} < \varepsilon. \end{aligned}$$

Therefore (B) follows by considering that $\{\mathfrak{U}(f) \mid f \in \mathcal{K}\}$ is an open cover of the compact $\mathcal{K}$. Indeed let for example $\{\mathfrak{U}(f_i) \mid i = 1, \dots, n\}$ a finite subcover of $\mathcal{K}$ thus by setting $W \doteq \bigcap_{i=1}^n U_i$ with obvious meaning of U_i , we have

$$\sup_{z \in W} \sup_{s \in K} \sup_{g \in \mathcal{K}} \|\mathfrak{U}(z)(s)g(z) - F(z)(s)g(z)\| < \varepsilon.$$

□

Remark 11. We can set $\mathcal{H} = \prod_{x \in X}^b \mathfrak{E}_x$ with the usual norm $\|\cdot\|_{\sup}$.

Theorem 4.4 ($\mathcal{K}$ -Uniform Convergence). *Let $\mathfrak{V} \doteq \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \|\cdot\| \rangle$ be a Banach bundle. Let $x_\infty \in X$ and $\mathcal{U}_0 \in \prod_{x \in X_0} \mathcal{C}(\mathbb{R}^+, B_s(\mathfrak{E}_x))$ be such that $\mathcal{U}_0(x)$ is a (C_0) -semigroup of contractions (respectively of isometries) on $\mathfrak{E}_x$ for all $x \in X_0$. Assume that*

1. $D(T_{x_\infty})$ is dense in $\mathfrak{E}_{x_\infty}$;
2. $\mathfrak{V}$ and $\mathfrak{W}$ satisfy (2.18);
3. $\exists \lambda_0 > 0$ (respectively $\exists \lambda_0 > 0, \lambda_1 < 0$) such that the range $\mathcal{R}(\lambda_0 - T_{x_\infty})$ is dense in $\mathfrak{E}_{x_\infty}$, (respectively the ranges $\mathcal{R}(\lambda_0 - T_{x_\infty})$ and $\mathcal{R}(\lambda_1 - T_{x_\infty})$ are dense in $\mathfrak{E}_{x_\infty}$);
4. $\mathcal{U}_{\|\cdot\|_{B(\mathfrak{E}_z)}}(\mathcal{L}_{S_z}(\mathfrak{E}_z)) \subseteq \mathfrak{M}_z$ (respectively $\mathcal{U}_{is}(\mathcal{L}_{S_z}(\mathfrak{E}_z)) \subseteq \mathfrak{M}_z$) for all $z \in X$;
5. $\mathcal{E} \subseteq \mathcal{H} \subseteq \prod_{x \in X}^b \mathfrak{E}_x$
6. X is completely regular and the filter of neighbourhoods of x_∞ admits a countable basis;
7. $\exists \mathcal{F} \subset \bigcap_{\lambda > 0} \mathfrak{L}_1(\mathbb{R}^+, \mathfrak{E}, \mu_\lambda)$ such that $\Psi_{\mathbb{R}^+}^{\mathcal{H}}(\mathcal{F}) = \Gamma(\rho)$;
8. $(\exists F \in \Gamma(\rho))(F(x_\infty) = \mathcal{U}(x_\infty))$ such that
 - (a) $\langle \int \overline{F}(s) d\mu_\lambda(s), \mathcal{E} \rangle \subseteq \Gamma(\pi)$, for all $\lambda > 0$;
 - (b) $(\forall v \in \mathcal{E})(\exists \phi \in \Phi)$ s.t. $\phi_1(x_\infty) = v(x_\infty)$ and $(\forall \{z_n\}_{n \in \mathbb{N}} \subset X \mid \lim_{n \in \mathbb{N}} z_n = x_\infty)$ we have that $\{\mathcal{U}(z_n)(\cdot)\phi_1(z_n) - F(z_n)(\cdot)v(z_n)\}_{n \in \mathbb{N}}$ is a bounded equicontinuous sequence.

Then

$$\mathcal{U} \in \Gamma^{x_\infty}(\rho). \quad (4.19)$$

Furthermore if

1. $(\exists a > 0)(\forall f \in \mathcal{H})(\|f\|_{\sup} \leq a\|f\|_{\mathcal{H}})$,
2. $\{F(\cdot)(s) \mid s \in \mathbb{R}^+\} \subseteq \mathbb{A}$ and $\{\mathcal{U}(\cdot)(s) \mid s \in \mathbb{R}^+\} \subseteq \mathbb{A}$;
3. $\{\overline{F}(s) \mid s \in \mathbb{R}^+\}$ and $\{\overline{\mathcal{U}}(s) \mid s \in \mathbb{R}^+\}$ are equicontinuous as subsets of $\mathcal{L}(\langle \mathcal{H}, \|\cdot\|_{\mathcal{H}} \rangle)$.

Then for all $\mathcal{K} \in \text{Comp}(\mathcal{H})$ such that $\mathcal{K} \subseteq \mathcal{E}$ and for all $K \in \text{Comp}(\mathbb{R}^+)$

$$\lim_{z \rightarrow x_\infty} \sup_{s \in K} \sup_{v \in \mathcal{K}} \|\mathcal{U}(z)(s)v(z) - F(z)(s)v(z)\| = 0. \quad (4.20)$$

Here $D(T_{x_\infty})$ is defined as in Notation 1 with $\mathcal{T}_0$ and Φ given in (2.18). While $\mathcal{U} \in \prod_{x \in X} \mathfrak{M}_x$ such that $\mathcal{U} \upharpoonright X_0 \doteq \mathcal{U}_0$ and $\mathcal{U}(x_\infty)$ is the semigroup on $\mathfrak{E}_{x_\infty}$ generated by T_{x_∞} operator defined in (2.2). Moreover $\|f\|_{\sup} \doteq \sup_{x \in X} \|f(x)\|_x$, while $\overline{\mathcal{U}} \doteq \Lambda_{\mathbb{A}}^Y(\mathcal{U})$ and $\overline{F} \doteq \Lambda_{\mathbb{A}}^Y(F)$.

Proof. By hypothesis (7) and statement (1) of Theorem 4.1, (2.14) follows. Moreover (4.15) follows by hypothesis (5), and Remark 10. Hence by hyps. (7–8a), and Corollary 4.2 follows the LD($\{F\}, \mathcal{E}$). Then (4.19) follows by Theorem 2.1. (4.20) follows by (4.19) and Lemma 4.5. $\square$

Remark 12. By statement (2) of Proposition 4.7 hypothesis (7) is equivalent to the following one $\Lambda_{\mathbb{A}}^{\mathbb{R}^+}(\Gamma(\rho)) \subseteq \bigcap_{\lambda > 0} \mathfrak{L}_1(\mathbb{R}^+, \mathfrak{G}, \mu_\lambda)$. In any case the form in hypothesis (7) has the advantage to be considered as a tool for constructing $\Gamma(\rho)$. Finally note that

$$\langle \mathfrak{G}, \mathcal{E} \rangle \subseteq \Gamma(\pi) \Rightarrow (8a).$$

4.3 $\langle \mathcal{H}, \mathfrak{T} \rangle$ as direct integral of a continuous field of left-Hilbert and associated left-von Neumann algebras

Assume that $\mathfrak{V} = \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \mathfrak{N} \rangle$ is a continuous field of left-Hilbert algebras on X . Let $\mathcal{H}$ be the direct integral of $\mathfrak{V}$ with respect to some finite Radon measure on X and $\mathcal{B} \subset \mathcal{H}$ a linear space, set

$$\mathbb{A}(\mathcal{B}) \doteq \{X \ni x \mapsto L_{a(x)} \mid a \in \mathcal{B}\},$$

where $L_{a_x} \in B(\mathfrak{E}_x)$ for any $a_x \in \mathfrak{E}_x$, is the left multiplication on the left-Hilbert algebra $\mathfrak{E}_x$. Then $\mathcal{H}$ and $\mathbb{A}(\mathcal{B})$ satisfies the requirements in Definition 5, moreover

$$G(\mathcal{B}) \doteq \theta(\mathbb{A}(\mathcal{B})) \upharpoonright \mathcal{H} = L_{\mathcal{B}} \quad (4.21)$$

where $L_a \in B(\mathcal{H})$ for any $a \in \mathcal{H}$, is the left multiplication on the left-Hilbert algebra $\mathcal{H}$. If every $\mathfrak{E}_x$ is unital then $\mathcal{H}$ is unital, thus $L_{(\cdot)}$ is an injective (isometric) map of $\mathcal{H}$ into $B(\mathcal{H})$. Therefore under this additional requirement we can take the following identification

$$G(\mathcal{B}) \simeq \mathcal{B} \text{ as linear spaces.}$$

Let $\mathbf{H} \doteq \{H^i \in \prod_{x \in X} \mathfrak{E}_x\}_{i=0}^2$ such that H_x^0 is a left Hilbert subalgebra of $\mathfrak{E}_x$, while H_x^k is a linear subspace of H_x^0 , for all $k = 1, 2$ and $x \in X$. Set

$$\begin{cases} \Gamma(\pi, \mathbf{H}) \doteq \{\sigma \in \mathcal{H} \mid (\forall x \in X)(\sigma(x) \in H_x^0)\} \\ \mathcal{D}_{\mathbf{H}} \doteq \{\sigma \in \mathcal{H} \mid (\forall x \in X)(\sigma(x) \in H_x^1)\} \\ \mathcal{B}_{\mathbf{H}} \doteq \{\sigma \in \mathcal{H} \mid (\forall x \in X)(\sigma(x) \in H_x^2)\}. \end{cases} \quad (4.22)$$

Thus $\Gamma(\pi, \mathbf{H})$ is a left Hilbert subalgebra of $\mathcal{H}$ and $\mathcal{B}_{\mathbf{H}}, \mathcal{D}_{\mathbf{H}}$ are linear subspaces of $\Gamma(\pi, \mathbf{H})$, so

$$L_{\mathcal{B}_{\mathbf{H}}}(\mathcal{D}_{\mathbf{H}}) \subseteq \Gamma(\pi, \mathbf{H}). \quad (4.23)$$

By (4.23) and (4.21) follows that for all $\sigma \in \mathcal{B}_{\mathbf{H}}, \eta \in \mathcal{D}_{\mathbf{H}}$ and $y \in X$

$$\begin{cases} \langle G(\mathcal{B}_{\mathbf{H}}), \mathcal{D}_{\mathbf{H}} \rangle \subseteq \Gamma(\pi, \mathbf{H}), \\ \langle \theta(x \mapsto L_{\sigma(x)}), \eta \rangle(y) = \sigma(y)\eta(y). \end{cases} \quad (4.24)$$

Let us consider now the continuous field of left-von Neumann algebras associated with the fixed field of Hilbert algebras, and by abusing of language, let us denote it with the same

symbol $\mathfrak{V} = \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \mathfrak{N} \rangle$, as well as $\mathcal{H}$ will denote the associated direct integral with respect to some finite Radon measure on X . Let Δ_x be the modular operator associated with the Hilbert algebra $\mathfrak{E}_x$ and σ_x the corresponding modular group. Thus we can set

$$\begin{cases} \mathbb{A}_\Delta \doteq \{S_t : X \ni x \mapsto \sigma_x(t) \in \text{Aut}(\mathfrak{E}_x) \mid t \in \mathbb{R}\} \\ G_\Delta \doteq \theta(\mathbb{A}_\Delta) \upharpoonright \mathcal{H} \\ \Sigma_t \doteq \theta(S_t) \upharpoonright \mathcal{H}, t \in \mathbb{R}. \end{cases}$$

Note that for every $t \in \mathbb{R}$, $v \in \mathcal{H}$ and $x \in X$

$$\Sigma_t(v)(x) = \sigma_x(t)(v(x)).$$

Now if we set

$$\Gamma(\pi) \doteq \mathcal{H}$$

for any linear subspace $\mathcal{D}$ of $\mathcal{H}$ we have

$$\langle G_\Delta, \mathcal{D} \rangle \subseteq \Gamma(\pi).$$

Finally note that to $\mathbb{A}_\Delta$ we can associate the following map

$$\bar{\Sigma} : \mathbb{R}^+ \ni t \mapsto \Sigma_t \in G_\Delta,$$

for which we have for all $x \in X$

$$\Psi_{\mathbb{R}}^{\mathcal{H}}(\bar{\Sigma})(x) = \sigma_x.$$

In the previous example we consider the extreme case in which $\Gamma(\pi) = \mathcal{H}$. In order to have a model where $\Gamma(\pi) \subset \mathcal{H}$ we have to get a more detailed structure, namely the half-side modular inclusion. So for any $x \in X$ let $\langle \mathfrak{N}_x \subset \mathfrak{E}_x, \Omega_x \rangle$ be a *hsmi*⁺ and V_x the Wiesbrock one-parameter semigroup of unitarities associated with it so $V_x \in \text{Hstr}(\mathfrak{E}_x)^+$ such that $\mathfrak{N}_x = \text{Ad}(V_x(1))\mathfrak{E}_x$. Therefore what we are interested in is that for all $t \in \mathbb{R}^+$

$$\begin{cases} \text{Ad}(V_x(t))(\mathfrak{E}_x) \subseteq \mathfrak{E}_x, \\ \text{Ad}(V_x(t))(\mathfrak{N}_x) \subseteq \mathfrak{N}_x. \end{cases} \quad (4.25)$$

By using the first inclusion in (4.25) we can set

$$\begin{cases} \mathbb{A}_V \doteq \{V_t : X \ni x \mapsto \text{Ad}(V_x(t)) \upharpoonright \mathfrak{E}_x \in \text{Aut}(\mathfrak{E}_x) \mid t \in \mathbb{R}\} \\ G_V \doteq \theta(\mathbb{A}_V) \upharpoonright \mathcal{H} \\ \bar{V}_t \doteq \theta(V_t) \upharpoonright \mathcal{H}, t \in \mathbb{R}. \end{cases}$$

Hence for all $x \in X$ and $t \in \mathbb{R}$

$$\begin{cases} \bar{V}_t(v)(x) = \text{Ad}(V_x(t))v(x) \\ \Psi_{\mathbb{R}}^{\mathcal{H}}(\bar{V})(x)(t) = \text{Ad}(V_x(t)) \end{cases}$$

Therefore if we set $\mathcal{D}$ and $\Gamma(\pi)$ such that

$$\mathcal{D} \subseteq \Gamma(\pi) \doteq \int^{\oplus} \mathfrak{N}_x d\mu(x) \subset \mathcal{H}$$

then by using the second inclusion in (4.25) we have

$$\langle G_V, \mathcal{D} \rangle \subseteq \Gamma(\pi).$$

For any semi-finite von Neumann algebra $\mathfrak{N}$ and any $\phi \in \mathbf{N}_{\mathfrak{N}}$ faithful we have that the Tomita-Takesaki modular group $\sigma_{\mathfrak{N}}^{\phi}$ is inner (see [21, Theorem 8.3.14]) i.e. it is implemented by a strongly continuous group morphism $V : \mathbb{R} \rightarrow U(\mathfrak{N})$, where $U(\mathfrak{N}) \doteq \{U \in \mathfrak{N} \mid U^{-1} = U^*\}$, so in particular

$$V(\mathbb{R}) \subset \mathfrak{N}. \quad (4.26)$$

Now let $\langle H_{\phi}, \pi_{\phi}, \Omega_{\phi} \rangle$ be a cyclic representation associated with ϕ and $\mathfrak{N}_{\phi} \doteq \pi(\mathfrak{N}_{\phi})$ which is a von Neumann algebra ϕ being normal, then by (4.26) immediatedly we have

$$\pi_{\phi}(V(\mathbb{R})) \subset \mathfrak{N}_{\phi}. \quad (4.27)$$

By the invariance $\phi = \phi \circ \sigma_{\mathfrak{N}}^{\phi}$, and the cited unitary implementation we obtain that there exists W_{ϕ} unitary action on H_{ϕ} such that

$$\begin{cases} Ad(W_{\phi}(t)) \circ \pi_{\phi} = Ad(\pi_{\phi}(V(t))) \circ \pi_{\phi}, \\ W_{\phi}(t) = \Delta_{\phi}^{it}, \end{cases} \quad (4.28)$$

where the second sentence comes by [21, Theorem 8.1.2], with Δ_{ϕ} the modular operator associated with $\langle \mathfrak{N}_{\phi}, \Omega_{\phi} \rangle$.

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