ISSN 2077–9879

Eurasian Mathematical Journal

2016, Volume 7, Number 4

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia
the University of Padua

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

EURASIAN MATHEMATICAL JOURNAL

Aims and Scope

The Eurasian Mathematical Journal (EMJ) publishes carefully selected original research papers in all areas of mathematics written by mathematicians, principally from Europe and Asia. However papers by mathematicians from other continents are also welcome.

From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

The contents of EMJ are indexed in Scopus, Web of Science (ESCI), Mathematical Reviews, MathSciNet, Zentralblatt Math (ZMATH), Referativnyi Zhurnal – Matematika, Math-Net.Ru.

The EMJ is included in the list of journals recommended by the Committee for Control of Education and Science (Ministry of Education and Science of the Republic of Kazakhstan) and in the list of journals recommended by the Higher Attestation Commission (Ministry of Education and Science of the Russian Federation).

Information for the Authors

Submission. Manuscripts should be written in LaTeX and should be submitted electronically in DVI, PostScript or PDF format to the EMJ Editorial Office via e-mail (eurasianmj@yandex.kz).

When the paper is accepted, the authors will be asked to send the tex-file of the paper to the Editorial Office.

The author who submitted an article for publication will be considered as a corresponding author. Authors may nominate a member of the Editorial Board whom they consider appropriate for the article. However, assignment to that particular editor is not guaranteed.

Copyright. When the paper is accepted, the copyright is automatically transferred to the EMJ. Manuscripts are accepted for review on the understanding that the same work has not been already published (except in the form of an abstract), that it is not under consideration for publication elsewhere, and that it has been approved by all authors.

Title page. The title page should start with the title of the paper and authors' names (no degrees). It should contain the Keywords (no more than 10), the Subject Classification (AMS Mathematics Subject Classification (2010) with primary (and secondary) subject classification codes), and the Abstract (no more than 150 words with minimal use of mathematical symbols).

Figures. Figures should be prepared in a digital form which is suitable for direct reproduction.

References. Bibliographical references should be listed alphabetically at the end of the article. The authors should consult the Mathematical Reviews for the standard abbreviations of journals' names.

Authors' data. The authors' affiliations, addresses and e-mail addresses should be placed after the References.

Proofs. The authors will receive proofs only once. The late return of proofs may result in the paper being published in a later issue.

Offprints. The authors will receive offprints in electronic form.

Publication Ethics and Publication Malpractice

For information on Ethics in publishing and Ethical guidelines for journal publication see http://www.elsevier.com/publishingethics and http://www.elsevier.com/journal-authors/ethics.

Submission of an article to the EMJ implies that the work described has not been published previously (except in the form of an abstract or as part of a published lecture or academic thesis or as an electronic preprint, see http://www.elsevier.com/postingpolicy), that it is not under consideration for publication elsewhere, that its publication is approved by all authors and tacitly or explicitly by the responsible authorities where the work was carried out, and that, if accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright-holder. In particular, translations into English of papers already published in another language are not accepted.

No other forms of scientific misconduct are allowed, such as plagiarism, falsification, fraudulent data, incorrect interpretation of other works, incorrect citations, etc. The EMJ follows the Code of Conduct of the Committee on Publication Ethics (COPE), and follows the COPE Flowcharts for Resolving Cases of Suspected Misconduct ($http://publicationethics.org/files/u2/New_Code.pdf$). To verify originality, your article may be checked by the originality detection service CrossCheck http://www.elsevier.com/editors/plagdetect.

The authors are obliged to participate in peer review process and be ready to provide corrections, clarifications, retractions and apologies when needed. All authors of a paper should have significantly contributed to the research.

The reviewers should provide objective judgments and should point out relevant published works which are not yet cited. Reviewed articles should be treated confidentially. The reviewers will be chosen in such a way that there is no conflict of interests with respect to the research, the authors and/or the research funders.

The editors have complete responsibility and authority to reject or accept a paper, and they will only accept a paper when reasonably certain. They will preserve anonymity of reviewers and promote publication of corrections, clarifications, retractions and apologies when needed. The acceptance of a paper automatically implies the copyright transfer to the EMJ.

The Editorial Board of the EMJ will monitor and safeguard publishing ethics.

The procedure of reviewing a manuscript, established by the Editorial Board of the Eurasian Mathematical Journal

1. **Reviewing procedure**

1.1. All research papers received by the Eurasian Mathematical Journal (EMJ) are subject to mandatory reviewing.

1.2. The Managing Editor of the journal determines whether a paper fits to the scope of the EMJ and satisfies the rules of writing papers for the EMJ, and directs it for a preliminary review to one of the Editors-in-chief who checks the scientific content of the manuscript and assigns a specialist for reviewing the manuscript.

1.3. Reviewers of manuscripts are selected from highly qualified scientists and specialists of the L.N. Gumilyov Eurasian National University (doctors of sciences, professors), other universities of the Republic of Kazakhstan and foreign countries. An author of a paper cannot be its reviewer.

1.4. Duration of reviewing in each case is determined by the Managing Editor aiming at creating conditions for the most rapid publication of the paper.

1.5. Reviewing is confidential. Information about a reviewer is anonymous to the authors and is available only for the Editorial Board and the Control Committee in the Field of Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan (CCFES). The author has the right to read the text of the review.

1.6. If required, the review is sent to the author by e-mail.

1.7. A positive review is not a sufficient basis for publication of the paper.

1.8. If a reviewer overall approves the paper, but has observations, the review is confidentially sent to the author. A revised version of the paper in which the comments of the reviewer are taken into account is sent to the same reviewer for additional reviewing.

1.9. In the case of a negative review the text of the review is confidentially sent to the author.

1.10. If the author sends a well reasoned response to the comments of the reviewer, the paper should be considered by a commission, consisting of three members of the Editorial Board.

1.11. The final decision on publication of the paper is made by the Editorial Board and is recorded in the minutes of the meeting of the Editorial Board.

1.12. After the paper is accepted for publication by the Editorial Board the Managing Editor informs the author about this and about the date of publication.

1.13. Originals reviews are stored in the Editorial Office for three years from the date of publication and are provided on request of the CCFES.

1.14. No fee for reviewing papers will be charged.

2. **Requirements for the content of a review**

2.1. In the title of a review there should be indicated the author(s) and the title of a paper.

2.2. A review should include a qualified analysis of the material of a paper, objective assessment and reasoned recommendations.

2.3. A review should cover the following topics:

- compliance of the paper with the scope of the EMJ;
- compliance of the title of the paper to its content;

- compliance of the paper to the rules of writing papers for the EMJ (abstract, key words and phrases, bibliography etc.);

- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);

- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

2.4. The final part of the review should contain an overall opinion of a reviewer on the paper and a clear recommendation on whether the paper can be published in the Eurasian Mathematical Journal, should be sent back to the author for revision or cannot be published.

Web-page

The web-page of EMJ is www.emj.enu.kz. One can enter the web-page by typing Eurasian Mathematical Journal in any search engine (Google, Yandex, etc.). The archive of the web-page contains all papers published in EMJ (free access).

Subscription

For Institutions

- US$ 200 (or equivalent) for one volume (4 issues)
- US$ 60 (or equivalent) for one issue

For Individuals

- US$ 160 (or equivalent) for one volume (4 issues)
- US$ 50 (or equivalent) for one issue.

The price includes handling and postage.

The Subscription Form for subscribers can be obtained by e-mail:

eurasianmj@yandex.kz

The Eurasian Mathematical Journal (EMJ)
The Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana
Kazakhstan

YESMUKHANBET SAIDAKHMETOVICH SMAILOV

(to the 70th birthday)

On October 18, 2016 was the 70th birthday of Yesmukhabet Saidakhmetovich Smailov, member of the Editorial Board of the Eurasian Mathematical Journal, director of the Institute of Applied Mathematics (Karaganda), doctor of physical and mathematical sciences (1997), professor (1993), honoured worker of the E.A. Buketov Karaganda State University, honorary professor of the Sh. Valikanov Kokshetau State University, honorary citizen of the Tarbagatai district of the East-Kazakhstan region. In 2011 he was awarded the Order "Kurmet" (= "Honour").

Y.S. Smailov was born in the Kyzyl-Kesek village (the Aksuat district of the Semipalatinsk region of the Kazakh SSR). He graduated from the S.M. Kirov Kazakh State University (Almaty) in 1968 and in 1971 he completed his postgraduate studies at the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR (Almaty). Starting with 1972 he worked at the E.A. Buketov Karaganda State University (senior lecturer, associate professor, professor, head of the Department of Mathematical Analysis, dean of the Mathematical Faculty; from 2004 director of the Institute of Applied Mathematics).

In 1999 the American Biographical Institute declared professor Smailov "Man of the Year" and published his biography in the "Biographical encyclopedia of professional leaders of the Millennium".

Professor Smailov is one of the leading experts in the theory of functions and functional analysis and a major organizer of science in the Republic of Kazakhstan. He had a great influence on the formation of the Mathematical Faculty of the E.A. Buketov Karaganda State University and he made a significant contribution to the development of mathematics in Central Kazakhstan. Due to the efforts of Y.S. Smailov, in Karaganda an actively operating Mathematical School on the function theory was established, which is well known in Kazakhstan and abroad.

He has published more than 140 scientific papers, two textbooks for students and one monograph. 10 candidate of sciences and 4 doctor of sciences dissertations have been defended under his supervision.

Research interests of Professor Smailov are quite broad: the embedding theory of function spaces; approximation of functions of real variables; interpolation of function spaces and linear operators; Fourier series for general orthogonal systems; Fourier multipliers; difference embedding theorems.

The Editorial Board of the Eurasian Mathematical Journal congratulates Yesmukhanbet Saidakhmetovich Smailov on the occasion of his 70th birthday and wishes him good health and new achievements in mathematics and mathematical education.

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 7, Number 4 (2016), 30 – 45

BOUNDS OF THE GROUP OF IA-AUTOMORPHISMS

R.G. Ghumde, S.H. Ghate

Communicated by J.A. Tussupov

Key words: IA-automorphism, finitely generated group.

AMS Mathematics Subject Classification: 20D45, 20D15.

Abstract. In this paper, expressions for the lower and upper bounds on the number of IA-automorphisms of a finitely generated group have been obtained. Using these bounds a few results including the one by Yadav and Vermani on Hasse principle have been derived as simple corollaries. Considering groups of order pq, pqr and p^2q the exact number of IA-automorphisms have been obtained in terms of the distinct primes p, q and r.

1 Introduction

For a group G, we denote the group of all automorphisms on G by $Aut(G)$. Following Bachmuth [3], we call an automorphism α on G an IA-automorphism if and only if it preserves all cosets of G' i.e. $x^{-1}\alpha(x) \in G'$, $\forall\, x \in G$; here G' is the derived subgroup of G. Clearly $Inn(G) \trianglelefteq Aut_c(G) \trianglelefteq IA(G) \trianglelefteq Aut(G)$, where $Inn(G)$, $Aut_c(G)$ and $IA(G)$ denote the groups of inner automorphisms, class preserving automorphisms, and IA-automorphisms of G respectively. In this paper we try to obtain expressions for the bound of $IA(G)$. The paper is divided into two parts. In the first part we consider a finitely generated group to obtain expressions for the bounds of $IA(G)$. With the help of these bounds we also obtain Yadav and Vermani's result on Hasse Principle as a simple corollary. In the second part we consider groups G of the types pq, pqr and p^2q for distinct primes p, q, and r. We obtain the expression for $|IA(G)|$ in terms of these primes.

2 $|IA(G)|$ of finitely generated groups

Let G be a finite p-group of order p^n. Let $\{x_1, x_2 \ldots, x_d\}$ be any minimal generating set for G. Let $\alpha \in IA(G)$. Since $\alpha(x_i) \in G'x_i$ for $1 \le i \le d$, there are at the most $|G'|$ choices for the image of x_i under α. Hence,

$$|IA(G)| \le \prod_{i=1}^{d} |G'| = |G'|^d. \tag{2.1}$$

Let $|G'| = p^m$. Since G' is contained in $\phi(G)$ (where $\phi(G)$ is the Frattini subgroup of G), then by the Burnside Basis Theorem $d \le n - m$. Hence,

$$|IA(G)| \le p^{md} \le (p^m)^{n-m} = p^{m(n-m)}. \tag{2.2}$$

Thus, $p^{m(n-m)}$ is an upper bound of $IA(G)$ for the p-group G.

Also, as every inner automorphism is an IA-automorphism, it follows that $|G/Z(G)|$ is a lower bound of $|IA(G)|$. Thus,

$$|G/Z(G)| \leq |IA(G)| \leq p^{m(n-m)}. \tag{2.3}$$

Obviously, for an abelian group G, both the lower and upper bounds for $IA(G)$ are the same, and are given by equal to 1.

Here we consider the examples where one or both of these upper and lower bounds are attained and also where strict inequality follows at both the ends.

i. For the semidihedral group,

$$SD_{16} = \langle x, y | x^8 = y^2 = 1, y^{-1}xy = x^3 \rangle,$$

$|G/Z(G)| = 8,\ |IA(G)| = |G'|^2 = 16$. Hence, $|Inn(G)| < |IA(G)| = |G'|^2$.

ii. For the dihedral group D_8, $|G/Z(G)| = |IA(G)| = |G'|^2 = 4$. Hence $|Inn(G)| = |IA(G)| = |G'|^2$.

iii. For the group

$$G = \langle x, y, z | x^{p^3} = y^{p^2} = z^p = 1, yxy^{-1} = x^{1+p}, zxz^{-1} = x, zyz^{-1} = yx^{p^2} \rangle,$$

$|G/Z(G)| = |IA(G)| = p^5,\ |G'|^3 = p^6$. Hence $|Inn(G)| = |IA(G)| < |G'|^3$.

iv. For the group $G = D_8 \times D_{16}$, where $D_8 = \langle x, y | x^2 = y^2 = 1, (xy)^4 = 1 \rangle$ and $D_{16} = \langle z, w | z^8 = w^2 = 1, w^{-1}zw = z^{-1} \rangle$, $|G/Z(G)| = 2^5, |IA(G)| = 2^9,\ |G'|^4 = 2^{12}$. Hence $|Inn(G)| < |IA(G)| < |G'|^4$.

We now prove a result containing the expression for the upper bound in a different form for the group of IA-automorphisms.

Theorem 2.1. *Let G be a non-trivial p-group having order p^n. Then*

$$|IA(G)| \leq \begin{cases} p^{\frac{n^2}{4}} & \text{if } n \text{ is even} \\ p^{\frac{n^2-1}{4}} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. If G is abelian, then the theorem holds trivially.
Now, consider G to be a non-abelian group and $|G'| = p^m$. Let $|\phi(G)| = p^t$. Since $G' \subseteq \phi(G)$, $m \leq t$. By the Basis Theorem of Burnside, it follows that from any generating set for G one can select $n-t$ elements such that these $n-t$ elements generate G. The number $n-t$ becomes maximum for $t = m$.

But $|G'| = p^m$. So we have $1 \leq m \leq n-2$.
Thus, all possible values of $m(n-m)$ are

$$\{n-1, 2(n-2), 3(n-3), \ldots, n^2/4\} \qquad \text{if } n \text{ is even, and}$$
$$\{n-1, 2(n-2), 3(n-3), \ldots, (n-1)(n+1)/4\} \qquad \text{if } n \text{ is odd.}$$

Clearly the maximum value of $m(n-m)$ is $\frac{n^2}{4}$ when n is even, and $\frac{(n-1)(n+1)}{4}$ when n is odd. Putting these values in inequality (2.2), we get

$$|IA(G)| \leq \begin{cases} p^{\frac{n^2}{4}} & \text{if } n \text{ is even} \\ p^{\frac{n^2-1}{4}} & \text{if } n \text{ is odd.} \end{cases}$$

□

Using the order inequality (2.1), we can obtain the following finiteness condition of $IA(G)$ for a finitely generated group G.

Theorem 2.2. *Let G be a finitely generated group. Then $IA(G)$ is finite if and only if $Inn(G)$ is finite.*

Proof. Since G is a finitely generated group, assume that the number of generators is d.
By inequality (2.1),

$$|IA(G)| \leq |G'|^d.$$

If $Inn(G)$ is finite, then $|G/Z(G)|$ is finite, and hence by the Schur Theorem G' is finite. Therefore, by inequality (2.1), $|IA(G)| \leq |G'|^d$. The converse is obviously true. □

In [5], Sh. Fouladi proved that, for a non-cyclic p-group G of maximal class and order p^n, $|Aut_\phi(G)| = p^{2n-4}$ if and only if G is metabelian, where $Aut_\phi(G)$ is the collection of automorphisms which preserve the cosets of $\phi(G)$.

But whenever G is a p-group of maximal class with $|G| = p^n$, then $G' = \phi(G)$, and $|G'| = p^{n-2}$. Thus, by inequality (2.2) the upper bound for $IA(G)$ is $p^{2(n-2)} = p^{2n-4}$. Hence, using this result, we can restate the Fouladi result as follows.

Proposition 2.1. *In a p-group G of maximal class and of order p^n, $IA(G)$ attains the upper bound if and only if G is metabelian.*

Consider a p-group G with a cyclic maximal subgroup. Then, by [8], either
(i) If p is odd and G is isomorphic to

$$M(p^n) = \langle x, y | x^{p^{n-1}} = 1 = y^p, y^{-1}xy = x^{1+p^{n-2}} \rangle.$$

(ii) If $p = 2$ and G is isomorphic to $M(2^n)(n \geq 4)$, or the dihedral group D_{2^n}, or the generalized quaternion group Q_{2^n}, or the quasi-dihedral group S_{2^n} which has the following representation

$$S_{2^n} = \langle x, y | x^{2a} = y^2 = 1,\ y^{-1}xy = x^{-1+a} \rangle,$$

where $a = 2^{n-2}$ and $n \geq 4$.

Notice that in the groups $D_{2^m}, Q_{2^m}, S_{2^m}$, the class of G is $m-1$; $\phi(G) = G'$, and G' is cyclic. Thus, for this case G is metabelian.

In [7] Yadav and Vermani proved that every non-abelian finite p-group of order p^n having a maximal subgroup which is cyclic enjoys "*Hasse Principle*" i.e. $Aut_c(G) = Inn(G)$.

In analogy with Hasse Principle, here we find some conditions on G when $IA(G)$ equals $Inn(G)$.

Theorem 2.3. *Let G be a p-group with a cyclic maximal subgroup. Then $IA(G)$ coincides with $Inn(G)$ if and only if G is of the type $M(p^n)$ or $|G| = 2^3$.*

Proof. Let G be a (non-abelian) p-group with a cyclic maximal subgroup.

Then G is in one of the following classes:
(1) $M(p^n)$,
(2) $p = 2$ and G is a 2-group of maximal class.

In case (1), $G' = \langle x^{p^{n-1}} \rangle$ which has order p, and $Z(G) = \langle x^p \rangle$ which has order p^{n-2}.

In this case, the maximum order of $IA(G)$ is p^2 and $Inn(G) = G/Z(G)$ also has order p^2. Hence, $IA(G) = Inn(G)$.

If $|G| = 2^3$, then the non abelian group G is isomorphic with D_8 or Q_8. In both of these situations, $IA(G) = Inn(G)$.

Conversely, let $|IA(G)| = |Inn(G)|$. Hence, in case (2), G is a metabelian $2-$group of maximal class. By Proposition 2.1, $IA(G)$ attains upper bound. Hence, the order of $IA(G)$ is $2^{2(n-2)}$. But, $|Inn(G)| = 2^{n-1}$.

So, $|Inn(G)| = |IA(G)|$ implies $2^{n-1} = 2^{2(n-2)}$. That is, $n = 3$.

□

Lemma 2.1. [9] *If G is an extraspecial p-group, then the order of G is p^{2n+1} and G is a central product of*

i. n-groups isomorphic to $E(p^3)$, or

ii. $M(p^3)$ and $n-1$ groups isomorphic to $E(p^3)$, where $E(p^3) = \langle x, y | x^p = y^p = z^p = 1 = [z, x] = [z, y] \rangle$ with $z = [x, y]$, or

iii. n dihedral groups of order 8, or

iv. $n-1$ dihedral groups of order 8 and a quaternion group of order 8.

We can use this characterization to prove the following result.

Theorem 2.4. *For every extraspecial p-group G, $Inn(G) = IA(G)$.*

Proof. From the above characterization of extraspecial p-group, it is clear that extraspecial p-group is generated by $2n$ elements say $x_1, x_2, \ldots\ldots x_n, y_1, y_2, \ldots\ldots, y_n$, and $|G'| = |Z(G)| = p$. Thus, the group G/G' is an elementary abelian p-group of order p^{2n}, and generated by the images x_i' and y_i' of the above generators.

Let $G' = \langle z \rangle = c_p$. Then $x_i \longrightarrow x_i z^{r_i}$ and $y_i \longrightarrow y_i z^{s_i}$ defines an IA-automorphism of G for any $r_i, s_i \in \{0, 1, \ldots p-1\}$.

Hence, $|IA(G)| = p^{2n}$ and $|Inn(G)| = |G/Z(G)| = p^{2n}$. Thus, the result holds true. □

The main result of Yadav and Vermani[7] now follows as a simple corollary.

Corollary 2.1. *Every extraspecial p-group enjoys "Hasse Principle".*

Proof. By the above theorem, $Inn(G) = IA(G)$. But $Inn(G) \leq Aut_c(G) \leq IA(G)$. Hence the result follows directly. □

Proposition 2.2. *Let G be a finite 2-generated p-group of class 2. Then $IA(G)$ attains the upper bound.*

Proof. Since G is 2-generated group of class 2, G' is cyclic.

Let the order of G' be p^c, and hence assume that $G/G' = C_{p^a} \times C_{p^b}$. Therefore, $|G| = p^{a+b+c}$.

By Proposition 5.4 of [2], $|Z(G)| = p^{a+b-c}$. Thus,

$$|G/Z(G)| = \frac{p^{a+b+c}}{p^{a+b-c}} = (p^c)^2 = |G'|^2 = |IA(G)|.$$

Hence, the result follows. □

For a given $x \in G$, let $C_G(x) = \{g \in G : gx = xg\}$ be the centralizer of x in G.

We now quote two results in [6] without giving their proofs.

Lemma 2.2. *Let G be a finite group and N a normal subgroup. Then for any $x \in G$,*

$$|C_G(x)| \geq |C_{G/N}(xN)|.$$

Lemma 2.3. *Let G be a p-group of maximal class with $|G| = p^n \geq p^3$. Then*

1. *$Z_i(G)$ is the unique normal subgroup of G of order p^i $(i = 1, 2, \ldots, n-2)$,*
2. *G has exactly $p+1$ normal subgroups of order p^{n-1} (i.e. maximal subgroups).*

As a corollary of Lemma 2.3, in a p-group G of maximal class, the upper and lower central series coincide. So, if the upper and lower central series are

$$1 = Z_0(G) < Z_1(G) < Z_2(G) < \ldots < Z_{n-2}(G) < Z_{n-1}(G) < Z_n = G,$$

$$1 = \gamma_{n+1}(G) < \gamma_n(G) < \gamma_{n-1}(G) < \ldots < \gamma_2(G) = G' < G,$$

then

$$Z_1(G) = \gamma_n(G),\ Z_2(G) = \gamma_{n-1}(G), \ldots$$

Lemma 2.4. *If G is a p-group of maximal class with $|G| \geq p^4$, then there is a unique maximal subgroup whose center is bigger than the center of G, and the centers of other maximal subgroups coincide with the center of G.*

Proof. As G is of maximal class, its center has order p, and for a maximal subgroup of M, $Z(G)$ must be contained in $Z(M)$.

In the upper central series of G, we have

$$|Z_1(G)| = p,\ |Z_2(G)| = p^2,\ |Z_3(G)| = p^3, \ldots$$

For $x \in Z_2(G)\backslash Z_1(G)$ the element $xZ_1(G)$ will be central in $G/Z_1(G)$ (since $Z_2(G)/Z_1(G)$ is, by definition, the center of $G/Z_1(G)$). Hence the centralizer of $xZ_1(G)$ will be the full group $G/Z_1(G)$ which has order p^{n-1}. By Lemma 2.2, the centralizer of x in G has order at least p^{n-1}. Since $x \notin Z_1(G) = Z(G)$, the centralizer of x must be a proper subgroup. It follows that the centralizer of x must be of order p^{n-1}, i.e. it is a maximal subgroup; call it M_0. Then, in M_0, x is a central element, and as noted above $Z(G) \subseteq Z(M_0)$. Thus, the center of M_0 contains $\langle x, Z(G)\rangle$, which has order p^2 (since $x \notin Z(G)$).

Note that $|Z_2(G)| = p^2$, and $x \in Z_2(G)\backslash Z_1(G)$. Thus, $\langle x, Z_1(G)\rangle = Z_2(G)$, and hence the center of M_0 contains $Z_2(G)$.

If M_1 is another maximal subgroup such that $Z(M_1)$ has order at least p^2, then $Z(M_1)$ will contain $Z_2(G)$ (by Lemma 2.3) i.e. $Z(M_1) \supseteq \langle x, Z_1(G)\rangle$, i.e. x is central in both M_0 and M_1.

Since M_0, M_1 are distinct maximal subgroups, $\langle M_0, M_1\rangle = G$, and x is central in $\langle M_0, M_1\rangle = G$, i.e. $x \in Z(G) = Z_1(G)$, a contradiction.

Thus, the centers of other maximal subgroups must have order p, and thus, the centers coincide with the center of G. □

Theorem 2.5. *Let G be a p-group of maximal class, and $IA(G)$ attain the upper bound $|G'|^2$. Then $IA(G) = Aut_c(G)$ if and only if $|G| = p^3$.*

Proof. Let G be a p-group of maximal class.

Let $|G| = p^3$, then $|Z(G)| = |G'| = p$, and $G/Z(G) \cong C_p \times C_p$. Thus, the $p+1$ maximal subgroups of G have order p^2, and so they are abelian. Let $M_0, M_1, \ldots, M_p$ be the (abelian) maximal subgroups, and take $x_i \in M_i\backslash Z(G)$. Then the conjugacy class of x_i is x_iG'. Thus, it is easy to see that an automorphism preserves conjugacy classes of G if and only if it preserves cosets of G', i.e. $Aut_c(G) = IA(G)$.

Consider now the case $|G| > p^3$.

Here G has exactly $p+1$ maximal subgroups, say $M_0, M_1, \ldots, M_p$, and one of these maximal subgroups has the center bigger than the center of G, say M_0, and the rest having the centers equal to the center of G. Note that G' has index p, in all M_i. However, for $x_0 \in M_0\backslash G'$, its conjugacy class is not the whole coset x_0G'. This is verified by the following arguments.

Since the center of M_0 contains at least $Z_2(G)$ (which has order p^2), the centralizer of x_0 contains $\langle x_0, Z_2(G)\rangle$ (note that $|G| > p^3$ so $Z_2(G)$ is proper subgroup). Thus, the centralizer of x_0 has order at least p^3, so the index of the centralizer of x_0 is at most p^{n-3} which is strictly less than the order of G' (equal to p^{n-2}). This means that the conjugacy class of x_0 is not the full coset x_0G'. In other words, x_0G' is a union of more than one conjugacy classes.

Since $IA(G)$ attains the upper bound, which means, given any $x \in G\backslash G'$, for every $t \in G'$, there is an IA-automorphism of G which takes x to xt, i.e. $IA(G)$ permutes the elements of the coset xG' transitively.

However, the coset x_0G' is a union of more than one conjugacy class. Thus, taking $t \in G'$ such that x_0 is not conjugate to x_0t, there exists an IA-automorphism of G which takes x_0 to x_0t, and so this IA-automorphism is not preserving the conjugacy class of x_0. Hence the result follows. □

Theorem 2.6. *Let G be a p-group and N a normal subgroup such that $N \leq G'$. Moreover, let N be invariant under all the IA-automorphisms of G and $\bar{G} = G/N$. If $IA(G)$ attains the upper bound, then $IA(\bar{G})$ also attains the upper bound.*

Proof. Let $\{x_1, x_2, \ldots, x_d\}$ be a minimal generating set of G. Then $\{\bar{x_1}, \ldots, \bar{x_d}\}$ is a minimal generating set for $\bar{G}$.

Saying '$IA(G)$ attains upper bound' is equivalent to saying that, given any $t_1, t_2, \ldots, t_d$ in G', there is an IA-automorphism of G taking $(x_1, \ldots, x_d)$ to $t_1x_1, \ldots, t_dx_d$.

N is a normal subgroup of G with $N \leq G'$. Thus, the derived subgroup of $\bar{G} = G/N$ is equal to G'/N.

Now, consider arbitrary $\bar{t_1}, \ldots, \bar{t_d}$ in the derived subgroup of $\bar{G}$. Suppose that $IA(G)$ attains the upper bound. Then for $t_1, t_2, \ldots, t_d$ in G', there is an IA-automorphism σ of G taking $(x_1, \ldots, x_d)$ to $(t_1x_1, \ldots, t_dx_d)$. Since N is invariant under σ, σ induces an automorphism of G/N, and also it takes $(\bar{x_1}, \ldots, \bar{x_d})$ to $(\bar{x_1}\bar{t_1}, \ldots, \bar{x_d}\bar{t_d})$. This means that $IA(\bar{G})$ attains the upper bound. □

Definition 1. *A group G is called a central product of its normal subgroups H and K if*

1. *$HK = G$,*
2. *$H \cap K \subseteq Z(G)$,*
3. *every element of H commutes with every element of K.*

The following result follows directly by the above definition.

Proposition 2.3. *If G is a central product of H and K then,*

i. *$Z(G) = Z(H)Z(K)$,*

ii. *$G' = H'K'$.*

If G is a central product of two groups H and K, then there are some interesting relations between some natural subgroups of $Aut(G)$ with subgroups of $Aut(H)$ and $Aut(K)$. As an example, it follows by the Theorem of Yadav and Vermani [7] that if G is the central product of H and K, then $Aut_c(G) = Inn(G)$ if and only if $Aut_c(H) = Inn(H)$ and $Aut_c(K) = Inn(K)$.

The question that arises naturally is whether such a relation also holds for IA-automorphisms. At this stage it seems difficult to get any relation of this kind on IA-automorphisms. One can, however, show that a one way implication holds for IA^z-automorphisms as stated below where IA^z-automorphisms mean those IA-automorphisms which preserve the center elementwise.

Proposition 2.4. *Let G be central product of its subgroups H and K. If $IA^z(G) = Inn(G)$ then $IA^z(H) = Inn(H)$ and $IA^z(K) = Inn(K)$.*

Proof. Let $f \in IA^z(H)$, define $F : G \longrightarrow G$ by $F(hk) = f(h)k,\ h \in H$ *and* $k \in K$.

We first show that F is well defined. To this end, let $hk = h_1k_1$, for $h, h_1 \in H$, and $k, k_1 \in K$. We have to show that $F(hk) = F(h_1k_1)$, *i.e.* $f(h)k = f(h_1)k_1$, *i.e.* $f(h_1^{-1}h) = k_1k^{-1}$. But $H \cap K \subseteq Z(G)$ and $f \in IA^z(G)$).Thus,

$$hk = h_1k_1 \Rightarrow h_1^{-1}h = k_1k^{-1} \Rightarrow h_1^{-1}h \in Z(G) \Rightarrow h_1^{-1}h \in Z(H) \Rightarrow k_1k^{-1} = h_1^{-1}h = f(h_1^{-1}h).$$

Hence F is well defined. It is easy to see that F is an isomorphism, and the extension of f also.

Further,

$$F(hkG') = F(hkH'K')\ = F(hH')F(kK')\ = hH'kK'\ = hkH'K' = hkG'.$$

Therefore, F is an IA-automorphism of G. Since f is the identity on $Z(H)$, and, by definition, it is obvious that F is the identity on $Z(K)$. Therefore, F is the identity on $Z(H)Z(K) = Z(G)$, i.e. F is in $IA^z(G)$. This shows that every f in $IA^z(H)$ extends to an F in $IA^z(G)$.

By the hypothesis, F is an inner automorphism, say by the conjugation by $h'k'$. k' acts trivially on H by conjugation. So, F acts on H as the conjugation by h'.

By the definition of F, it is clear that the restriction of F on H is f. Hence, f is an inner automorphism, namely conjugation by h'. It means every element of $IA^z(H)$ is inner. Similarly we can show that $IA^z(K) = Inn(K)$. □

The following example shows that the converse of the above proposition is not true.

Example 1. By the definition of the central product, it is clear that the direct product is a special case of the central product. Consider groups H and K given by

$$H = \langle x, y \colon x^4 = y^2 = 1, yxy^{-1} = x^{-1}\rangle,\ K = \langle z, w \colon z^4 = w^2 = 1, wzw^{-1} = z^{-1}\rangle.$$

Here $G' = \langle x^2, z^2\rangle$ and $IA^z(H) = Inn(H)$ and $IA^z(K) = Inn(K)$. Consider the map

$$f : H \times K \longrightarrow H \times K,$$

which we define on generators as follows

$$x \longrightarrow xz^2,\ y \longrightarrow yz^2,\ z \longrightarrow zx^2,\ w \longrightarrow wx^2.$$

Obviously f is an IA^z-automorphism which is not an inner automorphism.

3 $|IA(G)|$ for groups G of order pq, or pqr, or p^2q

In this section we consider groups G of orders pq, pqr and p^2q, where p, q, r are distinct primes, and try to find the order of the group $IA(G)$ in terms of p, q, r. They are considered in Subsections 3.1, 3.2, 3.3 respectively.

3.1 $|IA(G)|$ for groups of order pq

It is well known (see Alperin[1]) that for each pair of primes $p,\ q$ satisfying the condition $q|p-1$, there is a unique, up to an isomorphism, a non-abelian group of order pq having the representation:

$$G = \langle x, y | x^p = y^q = 1, y^{-1}xy = x^u \rangle, \tag{3.1}$$

where u is an element of order q in the multiplicative group Z_p^*.

Theorem 3.1. *For a non-abelian group of order pq satisfying the condition $q|p-1$, $|IA(G)| = p(p-1)$.*

Proof. By representation (3.1), it is clear that $G' = \langle x \rangle$. Let α be an IA-automorphism of G. Then we have $\alpha(x) = x^i\ (1 \le i < p),\ \alpha(y) = yx^j\ (0 \le j < p)$.

On the other hand, for every choice of i, j in these equalities, if we denote by x_1 the element x^i and by y_1 the element yx^j, then it is easy to see that x_1 and y_1 generate G and satisfy the same relations as x, y, i.e.

$$x_1^p = 1,\ y_1^q = 1,\ y_1 x_1 y_1^{-1} = x_1^u.$$

Thus any IA-automorphism of G is uniquely determined for every pair of integers i, j satisfying the conditions $1 \le i < p$ and $0 \le j < p$. Thus $|IA(G)| = p(p-1)$. □

If $q \nmid p-1$ then G is a cyclic group in which case $|IA(G)| = 1$.

3.2 $|IA(G)|$ for groups of order pqr

Let G be a non-abelian group of order pqr, with $p > q > r$. Then G always has a normal subgroup P of order p. Since $|P|$ and $|G/P|$ are relatively prime, by the Schur-Zassenhaus Theorem, P has a complement in G, i.e. there exists a subgroup H of order equal to $|G/P| = qr$ such that $P \cap H = 1$ and $PH = G$.

Let $P = \langle x \rangle$ for some $x \in G$. The subgroup H acts on P by conjugation, hence it induces a homomorphism from H to $Aut(P)$:

$$\phi : H \to Aut(P);\ \phi(h) = (x \to hxh^{-1}).$$

Since $ker\phi$ is a subgroup of H and $|H| = qr$, the order of $ker\phi$ can be

$$1,\ q,\ r,\ qr.$$

Case 1: $ker\ \phi$ **is trivial.** This means that ϕ is injective. Since P is cyclic, $Aut(P)$ is abelian (cyclic), and thus H must also be cyclic.

Let $H = \langle y \rangle \cong C_{qr}$. Then, G has the representation

$$G = \langle x, y : x^p = y^{qr} = 1; yxy^{-1} = x^i,\ \text{for some positive integer i},\ i \ne\ 1 \rangle. \tag{3.2}$$

Clearly, $yxy^{-1} = x^i$, implies $i^{qr} \equiv 1(mod\ p)$.

It is easy to see that i^q or i^r can not be 1 modulo p, otherwise y^q or $y^r \in ker\phi$.

By (3.2), it is clear that $G' = \langle x \rangle$. Hence, any IA-automorphism of G will be of the form

$$x \to x^a;\ y \to x^b y;\ \text{with } 1 \leq a < p \text{ and } 0 \leq b < p-1. \tag{3.3}$$

Let $x_1 = x^a$ and $y_1 = x^b y$. It is easy to see that $x_1^p = 1$ and $y_1^{qr} = 1$. Clearly, the order of y_1 is neither q nor r. Thus, the order of y_1 is qr.

So, for the above choices of integers a and b, the elements x_1 and y_1 satisfy the same relations as the ones satisfied by x and y. Hence map (3.3) is an IA-automorphism of G, and $|IA(G)| = (p-1)p$.

Case 2: $ker\ \phi$ **is of order** q. In this case, $ker\phi$ together with P forms an abelian (cyclic) subgroup of order pq, denote it by H. Clearly, H is normal in G. Therefore, if z is an element of order pq and y is an element of order r, then $G = \langle z, y \rangle$, and $yzy^{-1} = z^i, i \neq 1$.

So, the following three cases arise:

(A) y centralizes elements of order p but no element of order q in $\langle z \rangle$.

(B) y centralizes elements of order q but no element of order p in $\langle z \rangle$.

(C) y does not centralize any element of order p as well as of order q in $\langle z \rangle$.

In case (A), the subgroup of order p becomes central, hence G is of the form

$$G = C_p \times (C_q \rtimes C_r).$$

Here, $Z(G) = C_p, G' = C_q$; and any IA-automorphism of G is nothing but an IA-automorphism of $C_q \rtimes C_r$. By Theorem 3.1, it is clear that the order of the group of IA-automorphisms of $C_q \rtimes C_r$ is given by $q(q-1)$. Hence $|IA(G)| = q(q-1)$.

Case (B) is similar to case (A), where G will be of the form $C_q \times (C_p \rtimes C_r)$, and hence $|IA(G)| = p(p-1)$.

Now, we consider case (C). Suppose that y does not centralize any element of order p or q in $\langle x \rangle$. Hence $Z(G) = 1$, G has the representation,

$$G = \langle z, y : z^{pq} = y^r = 1, yzy^{-1} = z^i \rangle; \text{and } i^r \equiv 1(mod\ pq).$$

Here $G' = \langle z \rangle$, and $\langle z \rangle = \langle z^q \rangle \times \langle z^p \rangle \cong C_p \times C_q$. Hence, an IA-automorphism of G has the form

$$z \to z^a, y \to yx^b.$$

where $1 \leq a < pq$ and a is not divisible by p as well by q, and $0 \leq b < pq$. Clearly, every choice of a, and b gives an IA-automorphism of G. Hence,

$$|IA(G)| = (p-1)(q-1)pq.$$

Case 3: $ker\ \phi$ **is of order r.** This means that a Sylow-r subgroup acts trivially on Sylow-p subgroup; hence G contains a cyclic subgroup of order pr. Clearly, this subgroup of order pr is normal in G. Thus, G is of the form

$$G = C_{pr} \rtimes C_q = (C_p \times C_r) \rtimes C_q.$$

Now C_q acts on C_{pr} by conjugation and since C_r is the unique subgroup of order r inside the cyclic group C_{pr}, C_q acts by conjugation on C_r. But since $r < q$, the action of C_q on C_r must be trivial, which means that C_r commutes with C_q. Therefore, the Sylow-r subgroup in G is central, and hence G is of the form

$$G = C_r \times (C_p \rtimes C_q).$$

Here C_r is the center of G and C_p is the commutator subgroup of G. Therefore, any IA-automorphism of G is exactly the IA-automorphism of $C_p \rtimes C_q$. So,

$$|IA(G)| = (p-1)p.$$

Case 4: *ker* ϕ **is of order qr.** In this case, the subgroup of order qr acts trivially on the Sylow-p subgroup by conjugation i.e. Sylow-p subgroup is in the center of G. Thus, G is of the form

$$G = C_p \times H,$$

where H is a subgroup of order qr. Since G is non-abelian, H must be non-abelian, and so

$$H = C_p \times (C_q \rtimes C_r).$$

Again as in Case 3, $Z(G) = C_p, G' = C_q$ and hence an IA-automorphism of G is nothing but an IA-automorphism of $C_q \rtimes C_r$. Thus, $|IA(G)| = (q-1)q$.

By using the arguments of IA-automorphism of group of order pqr, we can prove the following important theorem.

Theorem 3.2. *If G is a group of square-free order then $|IA(G)| = \phi(|G'|).|G'|$ (where ϕ is Euler's phi function).*

Proof. It is well known that whenever G is a group of square free order, then G is a split metacyclic group, i.e. G has following representation

$$G = C_m \rtimes C_n,$$

where, C_m is a maximal subgroup among all cyclic normal subgroups. Since, G is of square free order this implies the center of G is inside C_m. Now, consider the conjugation action of C_n on C_m. So, the fixed point set under the conjugation action by C_n is precisely the center of G. Hence the centre becomes a direct abelian factor of G. Hence, G has the following representation

$$G = A \times (C_r \rtimes C_n),$$

where A is a direct abelian/cyclic factor. Here the action of cyclic group C_n on C_r has no fixed point except the identity. This implies that $Z(G) = A,\ G' = C_r$.

Since $Z(G) \cap G' = 1$, any IA-automorphism fixes $Z(G)$ elementwise. Therefore, $IA(G)$ is nothing but an IA-automorphisms of the group $H = C_r \rtimes C_n$.

In the group H, C_n acts on C_r by conjugation and has no fixed points. This implies that $H' = C_r$. Thus, H has the representation

$$H = \langle x, y | x^r = y^n = 1,\ yxy^{-1} = x^i \rangle,$$

where $i^n \equiv 1\ (mod\ r)$. Hence any IA-automorphism of H has the form $x \to x^a,\ y \to x^b y$, where $1 \leq a < r$ and is relatively prime to r, $0 \leq b < r - 1$.

It is easy to see that every choice of a, b in above conditions, gives an automorphisms of G, and it is obviously an IA-automorphism. Thus, $|IA(G)| = |IA(H)| = \phi(|H'|), |H'| = \phi(|G'|).|G'|$. □

3.3 $|IA(G)|$ for group of order p^2q

It is well known that in groups of order p^2q, a Sylow-p or a Sylow-q subgroup is normal. Therefore, if H_p and H_q denote some Sylow-p and Sylow-q subgroups of G, then G is of the form

$$G = H_p \rtimes H_q \text{ or } G = H_q \rtimes H_p.$$

Here, either $H_p \cong C_p \times C_p$ or C_{p^2} and $H_q \cong C_q$.

1. $G = C_{p^2} \rtimes C_q$.

In this case G has the representation

$$G = \langle x, y : x^{p^2} = y^q = 1, yxy^{-1} = x^i \rangle, \qquad i^q \equiv 1\ mod(p^2),\ i \neq 1.$$

One can easily note that the automorphism group of C_{p^2} is cyclic, hence it has at most one subgroup of order q, hence there is at most one action of C_q on C_{p^2} by conjugation (via automorphism of order q).

Here, i cannot be of the form $1 + kp$, since for such i, the automorphism $x \to x^{1+kp}$ is of order a power of p, whereas y has order q. Hence, $yxy^{-1}x^{-1} = x^{i-1}$, where $i - 1$ is not divisible by p. This implies that x^{i-1} is also a generator of the cyclic group $\langle x \rangle = C_{p^2}$, i.e. we have $G' = \langle x \rangle$.

Then, an IA-automorphism of G is of the form

$$x \to x^a,\ 1 \leq a \leq p^2, (a, p) = 1, \quad y \to x^b y,\ 0 \leq b \leq p^2. \tag{3.4}$$

Let $x_1 = x^a$ and $y_1 = x^b y$ with a, b chosen subject to the above conditions. Clearly, $x_1^{p^2} = 1$, $y_1^q = 1$, and also $y_1 x_1 y_1^{-1} = x_1^i$.

Thus, for the specified choices of a, b, the elments $x_1 = x^a$ and $y_1 = x^b y$ satisfy the same relations as x, y. Hence, (3.4) defines an automorphism of G for all possible values of a and b. So, $|IA(G)| = (p^2 - p)p^2$.

2. $G = (C_p \times C_p) \rtimes C_q$.

The group $C_p \times C_p$ contains $p + 1$ subgroups of order p. Let $C_p \times C_p = \langle x, y \rangle$ and $C_q = \langle z \rangle$. Then z permutes the $p + 1$ subgroups of order p under conjugation. We should consider two cases.

Case 1: C_q normalizes some C_p. In this case, z must fix one subgroup of order p under conjugation, say, without loss of generality, $\langle x \rangle$. Thus, $z\langle x \rangle z^{-1} = \langle x \rangle$.

Then z permutes remaining p subgroups of order p under conjugation, and again since q does not divide p, there must be another subgroup of order p which is invariant under conjugation by z, without loss of generality, we can say that it is $\langle y \rangle$. Hence,

$$z\langle y \rangle z^{-1} = \langle y \rangle.$$

Thus, to determine the structure of G, it is sufficient to know the value of zxz^{-1} and zyz^{-1} (one can easily note that z can not fix both x and y under conjugation, otherwise G will be abelian). For getting these value, we have to consider following two cases.

z **fixes only one of** x **and** y **by conjugation.** Without loss of generality, consider $zxz^{-1} = x$. Then $zyz^{-1} = y^i$ for some i with condition that $i \neq 1$ and $i^q \equiv 1 \ (mod\ p)$ (hence y, z generate $C_p \rtimes C_q$). Then G has the representation

$$G = \langle x, y, z : \ x^p = y^p = z^q = 1, xy = yx, zxz^{-1} = x, zyz^{-1} = y^i \rangle, \ i^q \equiv 1 (mod\ p).$$

In fact, G has the form

$$\langle x \rangle \times (\langle y \rangle \rtimes \langle z \rangle) = C_p \times (C_p \rtimes C_q).$$

Clearly, $IA(G) = IA(C_p \rtimes C_q)$, and hence the number of IA-automorphisms of this group is $p(p-1)$.

z **does not fix any of** x **and** y **by conjugation.** Without loss of generality, consider $zxz^{-1} = x^i$ with condition that $i \neq 1$ and $i^q \equiv 1 (mod\ p)$. Also, the action of z on $\langle y \rangle$ is given by

$$zyz^{-1} = y^j \text{ with } j \neq 1 \text{ and } j^q \equiv 1 (mod\ p).$$

Thus, G has the representation

$$G = \langle x, y, z : x^p = y^p = z^q = 1, xy = yx, zxz^{-1} = x^i, zyz^{-1} = y^j \rangle, \text{ where, } i^q \equiv j^q \equiv 1 (mod\ p).$$

Here $G' = \langle x, y \rangle$, and so any IA-automorphism of G is of the form

$$x \to x^a y^b, \ y \to x^c y^d, \ z \to x^e y^f z,$$

where e, f are arbitrary in $\mathbb{Z}_p$, whereas a, b, c, d are so chosen that $\langle x, y \rangle = \langle x^a y^b, x^c y^d \rangle$, i.e. the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible.

With a, b, c, d satisfying the said conditions and e, f arbitrary in Z_p, it is easy to see that the above map is an automorphism of G. Thus, $|IA(G)| = |GL(2, p)|.p.p$.

Here $GL(2, p)$ is the group of automorphisms of $C_P \times C_p = \langle x, y \rangle$, and the last contribution of $p.p$ number of automorphisms is by the automorphisms of the type

$$x \to x, \ y \to y, \ z \to (x^e y^f)z.$$

These automorphisms form a group isomorphic to $C_p \times C_p = \langle x, y \rangle$; they are just multiplications of z by elements of $\langle x, y \rangle$. Further, these automorphisms are the identities on G' (and also on G/G', as these are IA-automorphisms). Hence, these automorphisms form a normal subgroup of $IA(G)$ of order p^2, and the automorphisms

$$x \to x^a y^b, \ y \to x^c y^d, \ z \to z,$$

form a subgroup of order $|GL(2,p)|$ in $IA(G)$. Thus, $IA(G) \cong (C_P \times C_p) \rtimes GL(2,p)$.

Case 2. C_q **does not normalize any** C_p**.** In this case, $\langle z \rangle$ does not normalize any subgroup of order p in $\langle x, y \rangle$, i.e. z acts faithfully by conjugation on collection of subgroups of order p in $\langle x, y \rangle$.

Let us consider $C_p \times C_p$ as a vector space, and conjugation of z induces an automorphism or invertible linear map from this vector space to itself. Since z is not normalizing any subgroup of order p, we have that the transformation has no invariant subspace of dim 1.

Thus, if $T_z : \langle x, y \rangle \to \langle x, y \rangle$ denotes the transformation induced by z, then T_z has no eigenvalue in $\mathbb{Z}_p$. If v is any vector in the vector space $\langle x, y \rangle$, then $T_z(v)$ and v should be linearly independent, so they should span the whole space, i.e. they form a basis. With respect to this basis, the matrix of T_z will be of the form

$$\begin{bmatrix} 0 & a \\ 1 & b \end{bmatrix}.$$

Group theoretically, under conjugation by z this is same as

$$x \to y \text{ and } y \to x^a y^b, \ \textit{i.e. } zxz^{-1} = y \text{ and } zyz^{-1} = x^a y^b.$$

But there is a unique group of order $p^2 q$ under such conditions (see [4]), and therefore the group G has the form

$$G = \langle\ x, y, z : x^p = y^p = z^q = 1, xy = yx, zxz^{-1} = y, zyz^{-1} = x^a y^b \rangle,$$

where $a, b \in \mathbb{Z}_p$ are so chosen that the above matrix has no eigenvalues in $\mathbb{Z}_p$ and it is order of q. Here $G/\langle x, y \rangle$ is cyclic, hence $G' \subseteq \langle x, y \rangle$, and since no subgroup of order p in $\langle x, y \rangle$ is normal in G, so G' must be $\langle x, y \rangle$.

Therefore, an IA-automorphism of G is of the form

$$x \to x^k y^l, y \to x^r y^s, z \to (x^t y^u)z, \text{ where the matrix } \begin{bmatrix} k & r \\ l & s \end{bmatrix} \in GL(2,p), \text{ and } t, u \in \mathbb{Z}_p.$$

Denoting the members in the right-hand sides of the above map by x_1, y_1, z_1 respectively, the above map is an automorphism of G if x_1, y_1, z_1 satisfy the same relations as x, y, z, i.e. $z_1 x_1 z_1^{-1} = y_1,\ z_1 y_1 z_1^{-1} = x_1^a y_1^b$.

If $z_1 x_1 z_1^{-1} = y_1$ then $z_1(x^k y^l)z_1^{-1} = x^r y^s$ i.e. $y^k (x^a y^b)^l = x^r y^s$. We have, $r = al$ and $s = k + bl$. Thus, from the known values of k, l the values of r, s are automatically determined. Moreover, with these values of r, s, it is easy to see that the relation $z_1 y_1 z_1^{-1} = x_1^a y_1^b$ is satisfied automatically. The order of $x^t y^u z$ must be q, since if it is divisible by p, then $x^t y^u z$ will commute with some element of order p. Since elements of order p are in the unique Sylow-p subgroup $\langle x, y \rangle$, z will also commute, this is a contradiction. Hence, the order of $x^t y^u z$ is divisible by q only, and, as it should divide the order of the group also, it must be q.

Thus, for $(k, l) \neq (0, 0)$ we find r, s by the above formula, and taking t, u arbitrarily in $\mathbb{Z}_p$, we get an IA-automorphism given by

$$x \to x^k y^l,\ y \to x^{al} y^{k+bl},\ z \to (x^t y^u)z$$

(k, l) which should be non zero, has $p^2 - 1$ choices, and (t, u) has p^2 choices. Hence $|IA(G)| = (p^2 - 1)p^2$.

3. $G = C_q \rtimes C_{p^2}$. Here C_{p^2} acts on C_q by conjugation. The kernel of this action is a proper subgroup of C_{p^2} (if the kernel is whole C_{p^2}, then $G = C_q \times C_{p^2}$). Obviously here $G' = \langle x \rangle$, and G has the representation

$$G = \langle x, y : x^q = y^{p^2} = 1, yxy^{-1} = x^i \rangle, \text{ where } i^{p^2} \equiv 1 \ (mod\ q).$$

Consider the case $i^p \equiv 1 \ (mod\ q)$. In this case, the subgroup $\langle y^p \rangle$ acts trivially on $\langle x \rangle$, since $y^p x y^{-p} = x^{i^p} = x^{1+kq} = x$. It follows that $\langle y^p \rangle \subseteq Z(G)$, and in fact we have equality (otherwise, the order of the center will be p^2 or pq, and $G/Z(G)$ will be then cyclic, a contradiction).

Now, an IA-automorphism has the form

$$x \to x^a, \quad y \to x^b y, \text{ where } 1 \leq a \leq q \text{ and } 0 \leq b < q.$$

Clearly, each choice of a, b gives an automorphism, and hence $|IA(G)| = q(q-1)$. Also, note that $|Z(G)| = |\langle y^p \rangle| = p \Rightarrow Inn(G) = pq$.

Now, consider the case $i^p \not\equiv 1 \ (mod\ q)$ but $i^{p^2} \equiv 1 \ (mod\ q)$. This case is similar to the previous one, and hence $|IA(G)| = q(q-1)$.

4. $G = C_q \rtimes (C_p \times C_p)$. Here $C_p \times C_p$ acts on C_q by conjugation, hence we have a homomorphism

$$C_p \times C_p \to Aut(C_q).$$

Since $Aut(C_q)$ is cyclic, and $C_p \times C_p$ is non-cyclic, the above homomorphism has a non-trivial kernel, and also it should be a proper subgroup of $C_p \times C_p$, otherwise G will be abelian.

Let $C_q = \langle x \rangle$ and $C_p \times C_p = \langle y, z \rangle$. Without loss of generality, we can assume that z is in the kernel of action of $C_p \times C_p$ on C_q, i.e $zxz^{-1} = x$. This implies that $z \in Z(G)$, hence $\langle z \rangle$ is a direct abelian factor of G, and hence

$$G = (\langle x \rangle \rtimes \langle y \rangle) \times \langle z \rangle = (C_q \rtimes C_p) \times C_p.$$

Here $G' = \langle x \rangle = C_q$ and $Z(G) \cap G' = 1$. Therefore $IA(G) = IA(C_q \rtimes C_p) = q(q-1)$.

References

[1] J.L. Alperin, R.B. Bell, *Groups and representations.* (Graduate texts in mathematics) (162) Springer; (1995).

[2] A. Ahmad, A. Magidin, R.F. Morse, *Two generator p-groups of nilpotency class 2 and their conjugacy classes.* Pub. Math. Debrecen, (81) (2012), 145-166.

[3] S. Bachmuth, *Automorphisms of free metabelian groups.* Trans. Amer. Math. Soc. (118) (1965), 93-104.

[4] W. Burnside, *The theory of groups of finite order.* 2^{nd} Edition Dover Publication Inc., (1955).

[5] S. Fouladi, *On the order of the automorphisms group of a p-groups of maximal class.* 20^{th} seminar on Algebra, 2-3 Ordibehesht, (1388) (2009), 71-72.

[6] A. Gustavo, *An introduction to finite p-groups: regular p-groups and groups of maximal class.* Notes in XVI Escola de Algebra, Bransilla 2000.

[7] M. Kumar, L.R. Vermani, *"Hasse Principle" for extraspecial p-groups.* Proc. Japan Acad., (76) A (2000), 123-125.

[8] D.J.S. Robinson, *A course in the theory of groups*, Newyork Inc.: Springer-Verlag, 1996.

[9] M. Suzuki, *Group theory II*, Springer-Verlag, 1986.

Ranjit G. Ghumde
Department of Mathematics
Ramdeobaba College of Engineering & Management
440013, Nagpur, India
Email: ranjitghumde@gmail.com

Suresh H. Ghate
Department of Mathematics
R.T.M. Nagpur University
440033, Nagpur, India.
Email: sureshghate@gmail.com

Received: 28.03.2016