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YESMUKHANBET SAIDAKHMETOVICH SMAILOV

(to the 70th birthday)

On October 18, 2016 was the 70th birthday of Yesmukhabet Saidakhmetovich Smailov, member of the Editorial Board of the Eurasian Mathematical Journal, director of the Institute of Applied Mathematics (Karaganda), doctor of physical and mathematical sciences (1997), professor (1993), honoured worker of the E.A. Buketov Karaganda State University, honorary professor of the Sh. Valikanov Kokshetau State University, honorary citizen of the Tarbagatai district of the East-Kazakhstan region. In 2011 he was awarded the Order "Kurmet" (= "Honour").

Y.S. Smailov was born in the Kyzyl-Kesek village (the Aksuat district of the Semipalatinsk region of the Kazakh SSR). He graduated from the S.M. Kirov Kazakh State University (Almaty) in 1968 and in 1971 he completed his postgraduate studies at the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR (Almaty). Starting with 1972 he worked at the E.A. Buketov Karaganda State University (senior lecturer, associate professor, professor, head of the Department of Mathematical Analysis, dean of the Mathematical Faculty; from 2004 director of the Institute of Applied Mathematics).

In 1999 the American Biographical Institute declared professor Smailov "Man of the Year" and published his biography in the "Biographical encyclopedia of professional leaders of the Millennium".

Professor Smailov is one of the leading experts in the theory of functions and functional analysis and a major organizer of science in the Republic of Kazakhstan. He had a great influence on the formation of the Mathematical Faculty of the E.A. Buketov Karaganda State University and he made a significant contribution to the development of mathematics in Central Kazakhstan. Due to the efforts of Y.S. Smailov, in Karaganda an actively operating Mathematical School on the function theory was established, which is well known in Kazakhstan and abroad.

He has published more than 140 scientific papers, two textbooks for students and one monograph. 10 candidate of sciences and 4 doctor of sciences dissertations have been defended under his supervision.

Research interests of Professor Smailov are quite broad: the embedding theory of function spaces; approximation of functions of real variables; interpolation of function spaces and linear operators; Fourier series for general orthogonal systems; Fourier multipliers; difference embedding theorems.

The Editorial Board of the Eurasian Mathematical Journal congratulates Yesmukhanbet Saidakhmetovich Smailov on the occasion of his 70th birthday and wishes him good health and new achievements in mathematics and mathematical education.

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HARMONIC ANALYSIS OF FUNCTIONS PERIODIC AT INFINITY

A. Baskakov, I. Strukova

Communicated by T.V. Tararykova

Key words: Banach space, functions slowly varying at infinity, functions periodic at infinity, Wiener's theorem, absolutely convergent Fourier series, invertibility, difference equations.

AMS Mathematics Subject Classification: 34A55, 34B05, 58C40

Abstract. In this paper we introduce the notion of vector-valued functions periodic at infinity. We characterize the sums of the usual periodic functions and functions vanishing at infinity as a subclass of these functions. Our main focus is the development of the basic harmonic analysis for functions periodic at infinity and an analogue of the celebrated Wiener's Lemma that deals with absolutely convergent Fourier series. We also derive criteria of periodicity at infinity for solutions of difference and differential equations. Some of the results are derived by means of the spectral theory of isometric group representations.

1 Introduction

Functions periodic at infinity appear naturally as bounded solutions of certain classes of differential and difference equations. In this paper we develop basic harmonic analysis for such functions. We introduce the notion of a generalized Fourier series of a function periodic at infinity; the Fourier coefficients in this case may not be constants, they are functions that are slowly varying at infinity. We prove analogues of the classical results on Ćesaro summability (Theorems 2.1 and 2.2) and convergence (Theorem 2.3). One of our main results is an extension of the celebrated Wiener's Lemma for functions periodic at infinity that have an absolutely convergent Fourier series. The proof of the theorem is based on more abstract results in [2, 9, 10]. In Section 5, we also describe the solutions of certain differential and difference equations in terms of functions periodic at infinity. We use methods of the spectral theory of locally compact Abelian group representations (Banach modules over group algebras) [3, 4, 12, 14].

2 Basic notation and statements of the main results

We begin by introducing the basic notation used in this paper for the standard function spaces. We let X be a complex Banach space and $EndX$ be the Banach algebra of all bounded linear operators (endomorphisms) on X. By $\mathbb{J}$ we denote one of the intervals $\mathbb{R}_+ = [0, \infty)$ or $\mathbb{R} = (-\infty, \infty)$. We write $C_b = C_b(\mathbb{J}, X)$ for the Banach space of X-valued bounded continuous functions on $\mathbb{J}$ with the norm $\|x\|_\infty = \sup\limits_{t\in\mathbb{J}} \|x(t)\|_X$. The closed subspace of bounded uniformly continuous functions is denoted by $C_{b,u} = C_{b,u}(\mathbb{J}, X)$. Another closed subspace, $C_0 = C_0(\mathbb{J}, X)$, consists af all functions in C_b that vanish at infinity, i.e., $\lim\limits_{|t|\to\infty} x(t) = 0$. Other subspaces of C_b that are of interest to us in this paper are introduced via the operator semigroup $S : \mathbb{J} \to End(C_b(\mathbb{J}, X))$ defined by

$$(S(t)x)(\tau) = x(t + \tau), \quad t, \tau \in \mathbb{J}. \tag{2.1}$$

It should be noted that S is a group in the case $\mathbb{J} = \mathbb{R}$.

Definition 1. A function $x \in C_{b,u}(\mathbb{J}, X)$ is called *slowly varying at infinity* if $(S(t)x - x) \in C_0(\mathbb{J}, X)$ for all $t \in \mathbb{J}$.

It should be mentioned that Definition 1 differs from the classical one given by J. Karamata in 1930 (see [26]). In [26] a positive continuous function $L : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be *weakly oscillating* if, for any $\lambda > 0$, $\frac{L(\lambda x)}{L(x)} \to 1$ as $x \to +\infty$. For example, such are the functions $\ln^\nu(x)$, $\ln\ln^\nu(x)$, ..., where $\nu \in \mathbb{R}$.

In view of Definition 1, we will call two functions $x, y \in C_b(\mathbb{J}, X)$ *C_0-equivalent* if $x - y \in C_0(\mathbb{J}, X)$. The functions listed below are slowly varying at infinity:

1) $x_1(t) = \sin(\ln(1 + t^2))$, $t \in \mathbb{R}$;

2) $x_2(t) = \operatorname{arctg} t$, $t \in \mathbb{R}$;

3) $x_3 : \mathbb{R}_+ \to X$, $x_3(t) = c + x_0(t), t \geq 0$, where c is a vector from X and x_0 is an arbitrary function from $C_0(\mathbb{R}_+, X)$;

4) any continuously differentiable function x from $C_b(\mathbb{R}, X)$ with the property $x' \in C_0(\mathbb{R}, X)$.

An equivalent definition for functions in $C_{b,u}(\mathbb{R}, X)$ is used in the theory of differential equations ([18, Section 3.6.3]), where such functions are called *stationary at infinity*.

Definition 2. A function $x \in C_{b,u}(\mathbb{J}, X)$ is called *periodic at infinity with period $\omega > 0$ (or $\omega-$periodic at infinity)* if $(S(\omega)x - x) \in C_0(\mathbb{J}, X)$, or, equivalently, $\lim\limits_{|t|\to\infty} \|x(t + \omega) - x(t)\|_X = 0$.

Every function ω-periodic at infinity is a solution of the difference equation $x(t + \omega) - x(t) = y(t)$, $t \in \mathbb{J}$, for some $y \in C_0(\mathbb{J}, X)$, that is $S(\omega)x$ and x are C_0-equivalent. We also point out that each function slowly varying at infinity is periodic at infinity with any period. The notion of a function almost periodic at infinity is given in [13].

The set of all functions slowly varying at infinity is denoted by $C_{sl,\infty} = C_{sl,\infty}(\mathbb{J}, X)$ and the set of all functions ω-periodic at infinity – by $C_{\omega,\infty} = C_{\omega,\infty}(\mathbb{J}, X)$. They form closed linear subspaces of $C_{b,u}(\mathbb{J}, X)$. The Banach space $C_\omega = C_\omega(\mathbb{J}, X)$ of all continuous $\omega-$periodic functions $f : \mathbb{J} \to X$ is a closed subspace of $C_{\omega,\infty}(\mathbb{J}, X)$. Thus, the inclusions $C_{sl,\infty}(\mathbb{J}, X) \subset C_{\omega,\infty}(\mathbb{J}, X) \subset C_{b,u}(\mathbb{J}, X)$ hold. Each of these subspaces is invariant for the operators $S(t)$, $t \in \mathbb{J}$.

In the case $X = \mathbb{C}$ the symbol X will be omitted in the notation of the spaces in question. For example, the space $C_{\omega,\infty}(\mathbb{J}, \mathbb{C})$ will be denoted by $C_{\omega,\infty}(\mathbb{J})$.

The space $C_{sl,\infty}(\mathbb{R})$ has the following properties:

1. $C_{sl,\infty}(\mathbb{R})$ is not separable, since the family of functions $\{x_\alpha, \alpha \geq 0\}$ in $C_b(\mathbb{R}_+)$ with $x_\alpha(t) = \exp(i\alpha \ln(1 + t))$, $\alpha, t \geq 0$, has the property $\|x_\alpha - x_\beta\|_\infty \geq \sqrt{2}$ for each $\alpha, \beta \in \mathbb{R}_+$, $\alpha \neq \beta$.

2. The space $C_{sl,\infty}(\mathbb{R})$ is an algebra under pointwise multiplication.

In the case X is a Banach algebra the function spaces under consideration are Banach algebras under pointwise multiplication $(xy)(t) = x(t)y(t),\ t \in \mathbb{J}$, if the functions x, y belong to a corresponding subspace. Moreover, each of these algebras is commutative if X is commutative and is a C^*-algebra if X is a C^*-algebra. In particular, the algebras $C_{sl,\infty}(\mathbb{J})$ and $C_{\omega,\infty}(\mathbb{J})$ are C^*-algebras.

Let $\mathcal{B}$ be a Banach algebra. By $L^1(\mathbb{R}, \mathcal{B})$ we denote the Banach algebra of all absolutely integrable functions $f : \mathbb{R} \to \mathcal{B}$ with the multiplication defined by convolution:

$$(f * g)(t) = \int_{\mathbb{R}} f(t-s)g(s)ds,\ t \in \mathbb{R},\ f, g \in L^1(\mathbb{R}, \mathcal{B}).$$

In the case $\mathcal{B} = \mathbb{C}$ we have $End\,\mathcal{B} \approx \mathbb{C}$ and we write $L^1(\mathbb{R}) = L^1(\mathbb{R}, \mathbb{C})$.

The class of functions periodic at infinity contains bounded solutions to many difference and differential equations (see Section 5). This is a consequence of the fact that the Laplace convolution

$$(f \overset{\mathcal{L}}{*} x)(t) = \int_0^t f(t-\tau)x(\tau)d\tau = \int_0^t f(\tau)x(t-\tau)d\tau$$

of functions $x \in C_\omega(\mathbb{J}, X)$ and $f \in L^1(\mathbb{J}, End\,X)$, while not usually a periodic function, is always periodic at infinity.

Definition 3. Given a function $x \in C_{\omega,\infty}(\mathbb{J}, X)$, we define its *canonical Fourier coefficients* $x_n : \mathbb{J} \to X,\ n \in \mathbb{Z}$, via

$$x_n(t) = \frac{1}{\omega}\int_0^\omega x(t+\tau)e^{-i\frac{2\pi n}{\omega}(t+\tau)}d\tau,\ t, \tau \in \mathbb{J},\ n \in \mathbb{Z}. \tag{2.2}$$

The series

$$\sum_{n\in\mathbb{Z}} x_n(t)e^{i\frac{2\pi n}{\omega}t},\ t \in \mathbb{J}, \tag{2.3}$$

is then called the *canonical Fourier series* of the function x.

Obviously, if $x \in C_\omega(\mathbb{R}, X)$ then $x_k(t) \equiv x_k = \frac{1}{\omega}\int_0^\omega x(\tau)e^{-i\frac{2\pi n}{\omega}\tau}d\tau$, $t \in \mathbb{R},\ k \in \mathbb{Z}$, are the standard Fourier coefficients of a function x. Since C_0-equivalence plays a major role in the theory of $C_{\omega,\infty}$ functions, we introduce the following definition.

Definition 4. An arbitrary series of the form

$$\sum_{n\in\mathbb{Z}} y_n(t)e^{i\frac{2\pi n}{\omega}t},\ t \in \mathbb{J}, \tag{2.4}$$

is called a *generalized Fourier series* of a function $x \in C_{\omega,\infty}(\mathbb{J}, X)$ provided that the functions $y_n \in C_{b,u}(\mathbb{J}, X),\ n \in \mathbb{Z}$, are C_0-equivalent to the canonical Fourier coefficients $x_n,\ n \in \mathbb{Z}$, defined by (2.2).

If a function $\overline{x} \in C_{b,u}(\mathbb{R}, X)$ coincides with $x \in C_\omega(\mathbb{R}, X)$ on $\mathbb{R}_+$ and $\lim\limits_{t\to-\infty} \|\overline{x}(t)\| = 0$ then $\overline{x} \in C_{\omega,\infty}(\mathbb{R}, X)$ and it has a generalized Fourier series of the form $\sum\limits_{n\in\mathbb{Z}} y_n(t)e^{i\frac{2\pi n}{\omega}t},\ t \in \mathbb{R}$, where $y_n(t) \equiv x_n(t),\ n \in \mathbb{Z},\ t \geq 0,\ y_n(t) = 0$ for all $t \leq -1$, and y_n is continuous.

Lemma 2.1. *The canonical Fourier coefficients $x_n,\ n \in \mathbb{Z}$, defined by (2.2) are slowly varying at infinity, i.e. $x_n \in C_{sl,\infty}(\mathbb{J}, X),\ n \in \mathbb{Z}$.*

Proof. The statement follows from the equality

$$x_n(t+\omega) - x_n(t) =$$
$$= \frac{1}{\omega}\int\limits_0^\omega (S(\omega)x - x)(t+\tau)e^{-i\frac{2\pi n}{\omega}(t+\tau)}d\tau,\ t \in \mathbb{J},\ n \in \mathbb{Z}.$$

□

Definition 4 and Lemma 2.1 imply that the Fourier coefficients of any generalized Fourier series satisfy $y_n \in C_{sl,\infty}(\mathbb{J}, X),\ n \in \mathbb{Z}$. We also get the following two statements about Ćesaro sums.

Theorem 2.1. *For each function $x \in C_{\omega,\infty}(\mathbb{J}, X)$ there exists a sequence of functions (x_n^0) in $C_0(\mathbb{J}, X)$ such that*

$$\lim_{n\to\infty} \sup_{t\in\mathbb{J}} \|x(t) - \sum_{k=-n}^{n}\left(1 - \frac{|k|}{n+1}\right)x_k(t)e^{i\frac{2\pi k}{\omega}t} - x_n^0(t)\| = 0,$$

where $x_k,\ k \in \mathbb{Z}$, are the canonical Fourier coefficients of a function x.

Theorem 2.2. *For any function $x \in C_{\omega,\infty}(\mathbb{J}, X)$ and any $\varepsilon > 0$ there exists a sequence of functions (x_n^0) in $C_0(\mathbb{J}, X)$ and a sequence of functions (y_n) in $C_{sl,\infty}(\mathbb{J}, X)$ such that*

$$\lim_{n\to\infty} \sup_{t\in\mathbb{J}} \|x(t) - \sum_{k=-n}^{n}\left(1 - \frac{|k|}{n+1}\right)y_k(t)e^{i\frac{2\pi k}{\omega}t} - x_n^0(t)\| = 0;$$

and for each $k \in \mathbb{Z}$ the function y_k is C_0-equivalent to the canonical Fourier coefficient x_k defined by (2.2) and admits a holomorphic extension to an entire function of exponential type at most ε.

In view of the above results we introduce the following notion of convergence of generalized Fourier series.

Definition 5. A generalized Fourier series

$$x(t) \sim \sum_{n\in\mathbb{Z}} y_n(t)e^{i\frac{2\pi n}{\omega}t},\ t \in \mathbb{J},$$

of a function $x \in C_{\omega,\infty}(\mathbb{J}, X)$ is called *convergent to x with respect to the subspace $C_0(\mathbb{J}, X)$*, or *C_0-convergent*, if there exists a sequence (x_n^0) of functions in $C_0(\mathbb{J}, X)$ such that

$$\lim_{n\to\infty} \sup_{t\in\mathbb{J}} \|x(t) - \sum_{k=-n}^{n} y_k(t)e^{i\frac{2\pi k}{\omega}t} + x_n^0(t)\| = 0.$$

The notion of C_0-convergence is well defined because it does not depend on the choice of a generalized Fourier series of x. This follows from the C_0-equivalence of the respective Fourier coefficients of different generalized Fourier series to each other (and to the respective canonical Fourier coefficients).

Definition 6. By the *modulus of continuity at infinity* of a function $x \in C_{b,u}(\mathbb{J}, X)$ we call the function $\omega_\infty(\cdot, x) : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\omega_\infty(\delta, x) = \lim_{\mu\to\infty} \sup_{|t|\le\delta, |\tau|\ge\mu} \|x(t+\tau) - x(\tau)\|_X,\ \delta \in \mathbb{R}_+.$$

Theorem 2.3. *Every generalized Fourier series of a function $x \in C_{\omega,\infty}(\mathbb{J}, X)$ converges to x with respect to the subspace $C_0(\mathbb{J}, X)$ provided that $\lim\limits_{n\to\infty} \omega_\infty(n^{-1}, x) \ln n = 0$.*

Definition 7. We say that a function $x \in C_{\omega,\infty}(\mathbb{J}, X)$ has *an absolutely convergent Fourier series* if it has a generalized Fourier series (2.4) such that $\sum\limits_{n\in\mathbb{Z}} \|y_n\| < \infty$.

We note that if a twice continuously differentiable function $x \in C_{\omega,\infty}$ satisfies $x', x'' \in C_{b,u}$ then $x', x'' \in C_{\omega,\infty}$. After directly computing the canonical Fourier coefficients of x'' one sees that, in this case, the canonical Fourier series of x converges absolutely.

We also observe that if a function x has an absolutely convergent Fourier series then its canonical Fourier series C_0-converges to x, but it does not have to converge absolutely.

In the case X is a Banach algebra, the functions in $C_{\omega,\infty}(\mathbb{J}, X)$ with absolutely convergent Fourier series form a closed subalgebra of $C_{\omega,\infty}(\mathbb{J}, X)$. This subalgebra is denoted by $A_{\omega,\infty}(\mathbb{J}, X)$ or $A_{\omega,\infty}(\mathbb{J})$ in the case $X = \mathbb{C}$.

Theorem 2.5, which is the main result of this paper, is devoted to functions in $A_{\omega,\infty}(\mathbb{J}, X)$. It is an extension of the following well-known Wiener's Tauberian Lemma (see, for example, [1, 2, 8, 10, 11, 21, 25] for many other extensions and applications of the result).

Theorem 2.4. *[28]. If $f \in A_\omega(\mathbb{R})$ and $f(t) \neq 0$ for all $t \in \mathbb{R}$, then also $1/f \in A_\omega(\mathbb{R})$.*

From this point on we let X to be a Banach algebra with the identity $\mathbf{e}$. We also let $e \in C_b(\mathbb{J}, X)$ to be the identity function, i.e., $e(t) \equiv \mathbf{e}$.

Definition 8. A function $x \in C_b(\mathbb{J}, X)$ is called *invertible with respect to the subspace* $C_0(\mathbb{J}, X)$ or C_0-*invertible*, if there is a function $y \in C_b(\mathbb{J}, X)$ such that $xy - e, yx - e \in C_0(\mathbb{J}, X)$, i.e. the functions xy and yx are both C_0-equivalent to x. In this case the function y is called an *inverse function* of x *with respect to the subspace* $C_0(\mathbb{J}, X)$ or a C_0-*inverse function* of x.

We note that all C_0-inverses of a function $x \in C_b(\mathbb{J}, X)$ are C_0-equivalent to each other.

Theorem 2.5. *Let X be a unital Banach algebra. If a C_0-invertible function $a \in C_{\omega,\infty}(\mathbb{J}, X)$ has an absolutely convergent Fourier series then each of its C_0-inverse functions also has an absolutely convergent Fourier series.*

The proofs of this and the following theorems are in Section 4. To formulate the next result, let us consider a sequence of operators (A_N) in $End C_{b,u}(\mathbb{J}, X)$ defined by $A_N = \frac{1}{N}\sum\limits_{k=0}^{N-1} S(k\omega),\ N \ge 1$. It is clear that $\|A_N\| = 1,\ N \ge 1$.

The following theorem answers the question of when a function periodic at infinity can be written as a sum of a periodic function and a function vanishing at infinity.

Theorem 2.6. *A function* $x \in C_{\omega,\infty}(\mathbb{J}, X)$ *can be written in the form* $x = x_1 + x_0$ *with* $x_1 \in C_\omega(\mathbb{J}, X)$ *and* $x_0 \in C_0(\mathbb{J}, X)$ *if and only if the limit* $\lim\limits_{N\to\infty} A_N x$ *exists in* $C_{b,u}(\mathbb{J}, X)$.

3 Harmonic analysis of periodic vectors and operators

The main result of this paper is proved by using a more general extension of Wiener's lemma from [9]. In this section we review the general theory of periodic vectors and operators which allows us to explain the result. This is a part of the spectral theory of Banach modules and group representations which appears in various sources such as [3, 4, 12, 14, 22].

In this section $\mathscr{X}$ is a complex Banach space and $T : \mathbb{R} \to End\mathscr{X}$ is a strongly continuous isometric representation. The Banach space $\mathscr{X}$ is given a structure of a Banach $L^1(\mathbb{R})$-module via the formula

$$fx = \int\limits_{\mathbb{R}} f(t)T(-t)xdt,\ x \in \mathscr{X},\ f \in L^1(\mathbb{R}). \tag{3.1}$$

By $\widehat{f} : \mathbb{R} \to \mathbb{C}$ we denote the Fourier transform

$$\widehat{f}(\lambda) = \int\limits_{\mathbb{R}} f(t)e^{-i\lambda t}dt,\ \lambda \in \mathbb{R}, \tag{3.2}$$

of a function $f \in L^1(\mathbb{R})$.

Definition 9. By the *Beurling spectrum* of a vector $x \in \mathscr{X}$ we mean the set $\Lambda(x) \subset \mathbb{R}$ defined by
$\Lambda(x) = \{\lambda_0 \in \mathbb{R} : fx \neq 0$ for any $f \in L^1(\mathbb{R})$ with $\widehat{f}(\lambda_0) \neq 0\}$
or, equivalently,
$\Lambda(x) = \mathbb{R}\backslash\{\mu_0 \in \mathbb{R} :$ there is a function $f \in L^1(\mathbb{R})$ such that $\widehat{f}(\mu_0) \neq 0$ and $fx = 0\}$.

In the following lemma [3, 12] we collect the basic properties of the Beurling spectrum of an element in a Banach module.

Lemma 3.1. *The following properties hold for all* $f \in L^1(\mathbb{R})$ *and* $x \in \mathscr{X}$*:*

1. *for any* $f \in L^1(\mathbb{R})$ *the equality* $fx = 0$ *implies that* $x = 0$ *(the* $L^1(\mathbb{R})$*-module* $\mathscr{X}$ *is non-degenerate);*

2. $\Lambda(x)$ *is a closed subspace of* $\mathbb{R}$ *and* $\Lambda(x) = \emptyset$ *if and only if* $x = 0$;

3. $\Lambda(fx) \subset (supp\widehat{f}) \cap \Lambda(x)$;

4. $fx = 0$ *in case* $(supp\widehat{f}) \cap \Lambda(x) = \emptyset$ *and* $fx = x$ *in case* $\Lambda(x)$ *is a compact and* $\widehat{f} = 1$ *in a neighborhood of* $\Lambda(x)$*;*

5. $\Lambda(x) = \{\lambda_0\}$ *is a singleton if and only if the vector* x *satisfies* $T(t)x = \exp(i\lambda_0 t)x,\ t \in \mathbb{R}$, *and* $x \neq 0$.

The Banach space $C_b(\mathbb{R}, X)$, even though the translation representation (2.1) is not strongly continuous on it, still has the structure of a Banach $L^1(\mathbb{R})$-module (see [3], [14]) defined via convolution

$$(f * x)(t) = \int_{\mathbb{R}} f(\tau)\,(S(-\tau)x)\,(t)d\tau = \int_{\mathbb{R}} f(\tau)x(t-\tau)d\tau = \int_{\mathbb{R}} f(t-\tau)x(\tau)d\tau, \qquad (3.3)$$

$t \in \mathbb{R},\ f \in L^1(\mathbb{R}),\ x \in C_b(\mathbb{R}, X)$. The module structure is still non-degenerate and the other properties of Lemma 3.1 hold s well. The subspace $C_{b,u}(\mathbb{R}, X)$ is then a closed submodule of $C_b(\mathbb{R}, X)$ with the structure given by (3.1) with $\mathscr{X} = C_{b,u}(\mathbb{R}, X)$ and $T(t) = S(t),\ t \in \mathbb{R}$.

Definition 10. The *nonessential spectrum* $\Lambda_0(x)$ of a function $x \in C_b(\mathbb{R}, X)$ is the set of all $\lambda_0 \in \Lambda(x)$ such that there is a function $f \in L^1(\mathbb{R})$ satisfying $\widehat{f}(\lambda_0) \neq 0$ and $f * x \in C_0(\mathbb{R}, X)$. The set $\Lambda_{ess}(x) = \Lambda(x)\backslash\Lambda_0(x)$ is called *the essential spectrum* of x.

Definition 11. Let $x \in C_b(\mathbb{R}_+, X)$. By $\overline{x} \in C_b(\mathbb{R}, X)$ we denote a function which coincides with x on $\mathbb{R}_+$ and possesses a property $\lim\limits_{t\to-\infty} \overline{x}(t) = 0$. By the *essential spectrum* $\Lambda_{ess}(x)$ of a function x we mean the set $\Lambda_{ess}(\overline{x})$.

The above definition is well posed because $\Lambda_{ess}(x)$ does not depend on the chosen extension $\overline{x} \in C_b(\mathbb{R}, X)$ of the function x on $\mathbb{R}$. Indeed, all such extensions are C_0-equivalent.

We note that the essential spectrum of a function $x \in C_b(\mathbb{R}, X)$ defined by $x(t) = \exp(it^2)$, $t \in \mathbb{R}$, is empty. We also observe that $\Lambda_{ess}(x) \subset \frac{2\pi}{\omega}\mathbb{Z}$ for $x(t) = y(t) + x_0(t),\ t \geq 0$, for $y \in C_\omega(\mathbb{R}_+, X)$ and $x_0 \in C_0(\mathbb{R}_+, X)$.

In another example, let $z \in C_\omega(\mathbb{R}, X)$ be an odd periodic function. It is obvious that the function $z_1 : t \mapsto z(|t|)$ is not periodic. But it is periodic at infinity with period ω and $\Lambda_{ess}(z_1) \subseteq \frac{2\pi}{\omega}\mathbb{Z}$.

Definition 12. A vector $x_0 \in \mathscr{X}$ is called T-*periodic* with a period $\omega > 0$, or (ω, T)-*periodic,* if the equation $T(\omega)x_0 = x_0$ holds.

The set of all (ω, T)-periodic vectors in $\mathscr{X}$ we denote by $\mathscr{X}_\omega = \mathscr{X}_\omega(T)$. It is a closed subspace of $\mathscr{X}$ invariant under operators $T(t),\ t \in \mathbb{R}$.

Theorem 3.1. *A vector $x_0 \in \mathscr{X}$ is (ω, T)-periodic (in other words, $x_0 \in \mathscr{X}_\omega$) if and only if the following condition is satisfied:*

$$\Lambda(x_0) \subset \frac{2\pi}{\omega}\mathbb{Z}. \qquad (3.4)$$

Proof. Necessity. Let x_0 be a vector from $\mathscr{X}_\omega$. By the definition we get $T(\omega)x_0 - x_0 = 0$. Then for any $f \in L^1(\mathbb{R})$ from (3.1) we have

$$f(T(\omega)x_0 - x_0) = \int_{\mathbb{R}} f(\tau)T(-\tau)(T(\omega)x_0 - x_0)d\tau$$

$$= \int_{\mathbb{R}} f(\tau)T(-\tau + \omega)x_0 d\tau - fx_0$$

$$= \int_{\mathbb{R}} (T(\omega)f)(u)T(-u)x_0 du - fx_0 = (T(\omega)f - f)x_0 = 0.$$

If $\lambda_0 \notin \frac{2\pi}{\omega}\mathbb{Z}$ one can consider a function $f \in L^1(\mathbb{R})$ with the property $\widehat{f}(\lambda_0) \neq 0$. In this case one has $\widehat{g}(\lambda_0) = (e^{i\lambda_0\omega} - 1)\widehat{f}(\lambda_0) \neq 0$ for the function $g = S(\omega)f - f \in L^1(\mathbb{R})$. Thus, there is a function $g \in L^1(\mathbb{R})$ with the properties $gx = (S(\omega)f - f)x = 0$ and $\widehat{g}(\lambda_0) \neq 0$. By Definition 10 one has $\lambda_0 \notin \Lambda(x_0)$ and inclusion (3.4) is proved.

Sufficiency. Suppose that for $\Lambda(x_0)$ condition (3.4) is fulfilled. Let us consider a vector $y_0 = T(\omega)x_0 - x_0$ and a function $f \in L^1(\mathbb{R})$ such that $\operatorname{supp}\widehat{f}$ is compact. By Lemma 3.1 we have that

$$\Lambda(fy_0) \subset supp\widehat{f} \cap \Lambda(y_0) \subset supp\widehat{f} \cap \Lambda(x_0) \subset supp\widehat{f} \cap \frac{2\pi}{\omega}\mathbb{Z}$$

is a finite set of the from $\left\{\frac{2\pi}{\omega}k_1, ..., \frac{2\pi}{\omega}k_n\right\}$, $k_1, ..., k_n \in \mathbb{Z}$.

It follows by the proof of Theorem 1 in [3] and Theorem 3.2.7 in [12] that the vector fx_0 can be written as

$$fx_0 = x_1 + ... + x_n,$$

where $\Lambda(x_j) = \left\{\frac{2\pi}{\omega}k_j\right\}$, $T(t)x_j = e^{i\frac{2\pi}{\omega}k_j t}x_j$, $1 \leq j \leq n$. So we get $fy_0 = f(T(\omega)x_0 - x_0) = (T(\omega) - I)fx_0 = \sum\limits_{j=1}^{n} \left(e^{i2\pi k_j} - 1\right) x_j = 0$.

Since the set of functions in $L^1(\mathbb{R})$, whose Fourier transforms have compact support, is dense in $L^1(\mathbb{R})$, and the $L^1(\mathbb{R})$-module $\mathscr{X}$ is non-degenerate by Lemma 3.1, the equation $T(\omega)x_0 - x_0 = 0$ holds. Hence, $x_0 \in \mathscr{X}_\omega$. □

Since the equations $T(t+\omega)x - T(t)x = T(t)(T(\omega)x - x) = 0$, $t \in \mathbb{R}$, hold for all $x \in \mathscr{X}_\omega$, it follows that the function $\varphi_x : \mathbb{R} \to \mathscr{X}$, $\varphi_x(t) = T(t)x$, is continuous and (ω, T)-periodic. Its Fourier series is

$$\varphi_x(t) \sim \sum_{n\in\mathbb{Z}} x_n e^{i\frac{2\pi n}{\omega}t},$$

where

$$x_n = \frac{1}{\omega}\int_0^\omega T(\tau)xe^{-i\frac{2\pi n}{\omega}\tau}d\tau, \ n \in \mathbb{Z}. \tag{3.5}$$

Definition 13. Given $x \in \mathscr{X}_\omega$, the series $\sum\limits_{n\in\mathbb{Z}} x_n$ is called its *Fourier series* if the vectors x_n, $n \in \mathbb{Z}$, called the *Fourier coefficients* of x, are given by (3.5).

If the Fourier series of a vector $x \in \mathscr{X}$ is absolutely convergent, i.e. $\sum\limits_{n\in\mathbb{Z}} \|x_n\| < \infty$, then we write $x = \sum\limits_{n\in\mathbb{Z}} x_n$.

Lemma 3.2. *For each $f \in L^1(\mathbb{R})$ and $x \in \mathscr{X}_\omega$ we have $fx \in \mathscr{X}_\omega$, and the Fourier series of fx is given by $fx \sim \sum\limits_{k=-\infty}^{\infty} \widehat{f}\left(\frac{2\pi k}{\omega}\right) x_k$.*

Proof. First, formula (3.1) directly implies that $fx \in \mathscr{X}_\omega$. Next, letting $y = fx$, formula (3.5) yields $y_k = \frac{1}{\omega}\int\limits_0^\omega T(\tau)(fx)e^{-i\frac{2\pi k}{\omega}\tau}d\tau$. Finally, by (3.1) and (3.2) we get

$$
\begin{aligned}
y_k &= \frac{1}{\omega}\int\limits_0^\omega T(\tau)\left(\int\limits_{\mathbb{R}} f(s)T(-s)xds\right)e^{-i\frac{2\pi k}{\omega}\tau}d\tau \\
&= \frac{1}{\omega}\int\limits_0^\omega \left(\int\limits_{\mathbb{R}} f(s)T(\tau - s)xds\right)e^{-i\frac{2\pi k}{\omega}\tau}d\tau \\
&= \int\limits_{\mathbb{R}} f(s)\left(\frac{1}{\omega}\int\limits_0^\omega T(\tau - s)xe^{-i\frac{2\pi k}{\omega}\tau}d\tau\right)ds \\
&= \int\limits_{\mathbb{R}} f(s)\left(\frac{e^{-i\frac{2\pi k}{\omega}s}}{\omega}\int\limits_0^\omega T(t)xe^{-i\frac{2\pi k}{\omega}t}dt\right)ds \\
&= \int\limits_{\mathbb{R}} f(s)x_k e^{-i\frac{2\pi k}{\omega}s}ds = \widehat{f}\left(\frac{2\pi k}{\omega}\right)x_k,
\end{aligned}
$$

and the lemma is proved. □

Below we list a few of the large number of the results of the classical Fourier theory that were derived for (ω, T)-periodic vectors in Banach modules, for example, in [3, 4]. The subspace $\mathscr{X}_\omega$ and the following lemma were also considered in [23, Theorem 16.7.2].

Lemma 3.3. *The operators $P_n \in End\, \mathscr{X}_\omega$, $n \in \mathbb{Z}$, defined by*

$$
P_n x = \frac{1}{\omega}\int\limits_0^\omega e^{-i\frac{2\pi n}{\omega}\tau}T(\tau)xd\tau, \quad x \in \mathscr{X}_\omega,
$$

are projectors and $x_n = P_n x$, $n \in \mathbb{Z}$, are the Fourier coefficients of x. Moreover, $T(t)P_n = e^{i\frac{2\pi n}{\omega}t}P_n$, $t \in \mathbb{R}$, $n \in \mathbb{Z}$, and $\|P_n\| = 1$ if $P_n \neq 0$.

Lemma 3.4. *For each $x \in \mathscr{X}_\omega$ the following equation holds:*

$$
\lim_{n\to\infty}\|x_n\| = 0,
$$

where x_n, $n \in \mathbb{Z}$, are Fourier coefficients of x.

Proof. Let us consider an arbitrary vector x in the domain $D(A)$ of the infinitesimal generator [20] of the operator semigroup T. We get the following estimate:

$$
\|x_n\| = \left\|\frac{1}{\omega}\int\limits_0^\omega e^{-i\frac{2\pi n}{\omega}\tau}T(\tau)xd\tau\right\| = \left\|\frac{1}{\omega}\int\limits_0^\omega \frac{\omega}{-2\pi i n}e^{-i\frac{2\pi n}{\omega}\tau}T(\tau)Axd\tau\right\|
$$

$$\leq \frac{\|Ax\|}{|n|},\ n \in \mathbb{Z}\backslash\{0\},$$

i.e. $\lim_{n\to\infty} \|x_n\| = 0$. Since $D(A)$ is dense in $\mathscr{X}_\omega$ the property $\lim_{n\to\infty} \|x_n\| = 0$ is fulfilled for all $x \in \mathscr{X}_\omega$. □

The following two theorems are contained in the more general results in [16], see also [27]. For the sake of completeness we give the proofs.

Definition 14. The function $\omega(\cdot, x) : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\omega(\delta, x) = \sup_{|t|\leq\delta} \|T(t)x - x\|,$$

is called the *modulus of continuity* of a vector x.

Theorem 3.2. *Let $x \in \mathscr{X}_\omega$. Then*

$$\lim_{n\to\infty} \left\| x - \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) x_k \right\| = 0,$$

where x_k, $k \in \mathbb{Z}$, are the Fourier coefficients of x.

Proof. Let us consider an arbitrary periodic vector $x \in \mathscr{X}_\omega$ and the functions $f_n \in L^1(\mathbb{R})$ defined by

$$f_n(t) = \frac{\omega}{4\pi^4 t^2 (n+1)} \sin^2 \frac{(n+1)\pi t}{\omega}, \quad t \in \mathbb{R},\ n \in \mathbb{N}.$$

Note that these functions have the following Fourier coefficients:

$$\widehat{f}_n(\lambda) = \begin{cases} 1 - \frac{\omega|\lambda|}{2\pi(n+1)} & , \quad |\lambda| \leq \frac{2\pi(n+1)}{\omega}, \\ 0 & , \quad |\lambda| > \frac{2\pi(n+1)}{\omega}; \end{cases} \qquad \lambda \in \mathbb{R},\ n \in \mathbb{N}.$$

Hence,

$$\widehat{f}_n(\frac{2\pi k}{\omega}) = \begin{cases} 1 - \frac{|k|}{n+1} & , \quad |k| \leq n+1, \\ 0 & , \quad |k| > n+1; \end{cases} \qquad k \in \mathbb{Z},\ n \in \mathbb{N}.$$

Lemma 3.2 implies that the convolution of f_n and x is given by

$$f_n x = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) x_k,\ n \in \mathbb{N}.$$

This leads to the following estimate:

$$\|x - \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) x_k\| = \|x - f_n x\|$$

$$= \| \int_{\mathbb{R}} f_n(\tau) x d\tau - \int_{\mathbb{R}} f_n(\tau) T(-\tau) x d\tau \| = \| \int_{\mathbb{R}} f_n(\tau) \left(x - T(-\tau)x\right) d\tau \|$$

$$\leq \| \int_{-\delta_n}^{\delta_n} f_n(\tau)\omega(\delta_n, x)d\tau\| + 2\|x\| \| \int_{\mathbb{R}\setminus(-\delta_n,\delta_n)} f_n(\tau)d\tau\|$$

$$\leq \omega(\delta_n, x) \int_{-\delta_n}^{\delta_n} f_n(\tau)d\tau + 2\|x\| \int_{\mathbb{R}\setminus(-\delta_n,\delta_n)} f_n(\tau)d\tau$$

$$\leq \omega(\delta_n, x) \int_{-\infty}^{\infty} f_n(\tau)d\tau + \frac{\omega\|x\|}{2\pi^4(n+1)} \int_{\mathbb{R}\setminus(-\delta_n,\delta_n)} \frac{\sin^2 \frac{(n+1)\pi\tau}{\omega}}{\tau^2} d\tau$$

$$\leq \omega(\delta_n, x) + \frac{\omega\|x\|}{\pi^4(n+1)} \int_{\delta_n}^{\infty} \frac{d\tau}{\tau^2} \leq \omega(\delta_n, x) + \frac{\omega\|x\|}{\pi^4(n+1)\delta_n} \to 0,\ n \to \infty,$$

for any sequence $\{\delta_n\}_{n=0}^{\infty}$ with the properties $\lim\limits_{n\to\infty} \delta_n = 0$ and $\lim\limits_{n\to\infty} (n+1)\delta_n = \infty$. □

Theorem 3.3. *If $x \in \mathscr{X}_\omega$ then*

$$\left\| x - \sum_{k=-n}^{n} x_k \right\| \leq Const \cdot \omega\left(\frac{1}{n}, x\right) \ln n, n \in \mathbb{N}. \tag{3.6}$$

In particular, the Fourier series of x converges to x if

$$\lim_{n\to\infty} \omega\left(\frac{1}{n}, x\right) \ln n = 0.$$

Proof. Let $E_n[x]$, $n \geq 1$, be the best trigonometric polynomial approximation of x of order n. From [29, p.123] we have

$$\left\| x - \sum_{k=-n}^{n} x_k \right\| \leq (L_n + 1)E_n[x], \tag{3.7}$$

where L_n, $n \geq 1$, is the Lebesgue constant, which satisfies the estimate (see [29, p.115])

$$L_n = 4\pi^{-2} \ln n + O(1) \simeq 4\pi^{-2} \ln n \text{ as } n \to \infty. \tag{3.8}$$

To prove our result we need the estimate for $E_n(x)$, $n \geq 1$. We consider a function $f \in L^1(\mathbb{R})$ with the following properties:

1. $\widehat{f}(0) = 1,\ \widehat{f}(\lambda) = \widehat{f}(-\lambda),\ \lambda \in \mathbb{R}$;
2. $\mathrm{supp}\widehat{f} \subset [-1, 1]$;
3. $f \geq 0$;
4. $\int\limits_{-\infty}^{\infty} |t|f(t)dt = M_f < \infty$.

Note that we need the third property only to simplify the proof. For an arbitrary $\alpha > 0$ we let $f_\alpha(t) = \alpha f(\alpha t)$, $t \in \mathbb{R}$. Then we have

$$E_\alpha[x] \le \|x - f_\alpha x\| = \| \int_{\mathbb{R}} (T(-\tau)x - x) f_\alpha(\tau) d\tau \|$$

$$\le \int_{\mathbb{R}} \|T(-\tau)x - x\| f_\alpha(\tau) d\tau$$

$$= \int_{-\alpha}^{\alpha} \|T(-\tau)x - x\| f_\alpha(\tau) d\tau + \int_{|\tau|\ge\alpha} \|T(-\tau)x - x\| f_\alpha(\tau) d\tau$$

$$\le \omega(\frac{1}{\alpha}, x) \int_{-\alpha}^{\alpha} f_\alpha(\tau) d\tau + \omega(\frac{1}{\alpha}, x) \int_{|\tau|\ge\alpha} (|\tau|\alpha + 1) f_\alpha(\tau) d\tau$$

$$\le \omega(\frac{1}{\alpha}, x) \left(2 + \int_{|\tau|\ge\alpha} |\tau|\alpha f_\alpha(\tau) d\tau \right) \le \omega(\frac{1}{\alpha}, x) \left(2 + \int_{|t|\ge\alpha^2} |t| f(t) dt \right)$$

$$\le \omega(\frac{1}{\alpha}, x) \left(2 + \int_{\mathbb{R}} |t| f(t) dt \right) \le (2 + M_f)\, \omega(\frac{1}{\alpha}, x),$$

and the desired estimate (3.6) follows from (3.7) and (3.8). □

Remark 1. For scalar periodic functions, the best approximation estimate in terms of the modulus of continuity was derived by Jackson [24]. In [15] an analogous estimate was obtained for bounded, uniformly continuous functions on $\mathbb{R}$ using the function $f(t) = \frac{96}{\pi t^4} \sin^4 \frac{t}{4}$, $t \in \mathbb{R}$.

Along with the isometric (not necessary periodic) representation $T : \mathbb{R} \to End\mathscr{X}$ we consider the representation $\widetilde{T} : \mathbb{R} \to End(End\mathscr{X})$ defined by $\widetilde{T}(t)A = T(t)AT(-t)$, $t \in \mathbb{R}$, $A \in End\mathscr{X}$.

Definition 15. An operator $A \in End\mathscr{X}$ is called *periodic* with respect to the representation $\widetilde{T}$ *with period* $\omega > 0$, or $(\omega, \widetilde{T})$-periodic, if

$$\widetilde{T}(\omega)A = T(-\omega)AT(\omega) = A,$$

i.e. A commutes with $T(\omega)$, and the function $t \mapsto T(t)AT(-t) : \mathbb{R} \to End\mathscr{X}$ is continuous in the uniform operator topology.

The set of $(\omega, \widetilde{T})$-periodic operators is a closed subalgebra of $End\mathscr{X}$. We denote it by $End_\omega\mathscr{X} = (End\mathscr{X})_\omega$. The above definition is consistent with the notion of (ω, T)-periodicity introduced in Definition 12. There we did not require the continuity of the function because it automatically follows since the representation T is assumed to be strongly continuous.

As in Definition 8 let us consider the Fourier series

$$A \sim \sum_{n\in\mathbb{Z}} A_n \tag{3.9}$$

of an operator A with respect to the presentation $\widetilde{T}$, i.e.

$$A_n = \frac{1}{\omega}\int_0^\omega e^{i\frac{2\pi n}{\omega}\tau} T(\tau)AT(-\tau)d\tau.$$

The notion of Fourier series for operators in the algebra $C_{2\pi}(\mathbb{R})$ was introduced in [19]. More generally, the concept was considered, for example, in [4, 9, 11, 12].

The following result appears in [2, 9].

Theorem 3.4. *If an ω-periodic continuously invertible operator $A \in End_\omega \mathscr{X}$ has an absolutely convergent Fourier series (3.9) then the inverse operator $B = A^{-1}$ is also ω-periodic and has an absolutely convergent Fourier series $A^{-1} \sim \sum_{n\in\mathbb{Z}} B_n$.*

Remark 2. In [2] the result is formulated for elements of Banach algebras that are almost periodic with respect to a group of algebra automorphisms. In the periodic case the two formulations are equivalent as we essentially show in the proof of Theorem 2.5. We chose to present the result for operators rather than general Banach algebras because we feel it to be more instructive.

4 Proofs of the main results

In this section X is a unital Banach algebra and $\mathscr{X} = \mathscr{X}(\mathbb{J}, X)$ is the quotient space $C_{b,u}(\mathbb{J}, X)/C_0(\mathbb{J}, X)$. Then $\mathscr{X}$ is a Banach space with the norm $\|\widetilde{x}\| = \inf_{y\in x+C_0(\mathbb{J},X)} \|y\|$, where $\widetilde{x} = x + C_0(\mathbb{J}, X)$ is the equivalence class of the function x. Moreover, $\mathscr{X}$ is a Banach algebra with the multiplication defined by

$$\widetilde{x}\widetilde{y} = \widetilde{xy}, \;\; \widetilde{x}, \widetilde{y} \in \mathscr{X}. \tag{4.1}$$

By $\mathscr{X}^\omega = \mathscr{X}^\omega(\mathbb{J}, X)$ we denote the quotient space $C_{\omega,\infty}(\mathbb{J}, X)/C_0(\mathbb{J}, X)$, which we view as a closed Banach subalgebra of $\mathscr{X}$.

Clearly the subspaces $C_{\omega,\infty}(\mathbb{R}, X)$ and $C_0(\mathbb{R}, X)$ are closed submodules of the $L^1(\mathbb{R})$-module $C_{b,u}(\mathbb{R}, X)$, and the algebras $\mathscr{X}(\mathbb{R}, X)$ and $\mathscr{X}^\omega(\mathbb{R}, X)$ are both quotient modules. Formula (4.1), however, does not allow us to define the structure of an $L^1(\mathbb{R})$-module in $C_b(\mathbb{R}_+, X)$. Nevertheless, the quotient spaces $\mathscr{X}(\mathbb{J}, X)$ and $\mathscr{X}^\omega(\mathbb{J}, X)$ have such a structure when $\mathbb{J} \in \{\mathbb{R}_+, \mathbb{R}\}$.

Indeed, the case $\mathbb{J} = \mathbb{R}$ is obvious, and in the case $\mathbb{J} = \mathbb{R}_+$ the strongly continuous isometric group $\widetilde{S} : \mathbb{R} \to End(\mathscr{X}(\mathbb{R}_+, X))$,

$$\widetilde{S}(t)\widetilde{x} = \widetilde{S(t)x}, \; t \in \mathbb{R}, \; \widetilde{x} \in \mathscr{X}(\mathbb{R}_+, X), \tag{4.2}$$

is defined in the following way. In (4.2) by $\widetilde{S(t)x}$, $t \geq 0$, we mean the equivalence class of the translation $S(t)x$ of x as in (2.1), and by $\widetilde{S(t)x}$, $t < 0$, we mean the equivalence class containing the continuous function $x_t \in C_b(\mathbb{R}_+, X)$ defined by

$$x_t(s) = \begin{cases} x(s+t) & , \quad s+t > 0, \\ -t^{-1}x(0)s & , \quad s+t \leq 0, \ s \geq 0. \end{cases}$$

With the above notation, the formula

$$f\widetilde{x} = \int\limits_{\mathbb{R}} f(\tau)\widetilde{S}(-\tau)\widetilde{x}d\tau, \ f \in L^1(\mathbb{R}), \ \widetilde{x} \in \mathscr{X}(\mathbb{J}, X), \ \mathbb{J} \in \{\mathbb{R}_+, \mathbb{R}\}, \tag{4.3}$$

defines the structure of a Banach $L^1(\mathbb{R})$-module on $\mathscr{X}$ and on $\mathscr{X}^\omega$.

Directly by the definition of representation $\widetilde{S}$ we get the equality $\widetilde{S}(\omega)\widetilde{x} = \widetilde{x}$, $\widetilde{x} \in \mathscr{X}^\omega$. Therefore, the function $t \mapsto \widetilde{S}(t)\widetilde{x} : \mathbb{R} \to \mathscr{X}^\omega$ is continuous and ω-periodic, i.e. it belongs to the Banach space $C_\omega(\mathbb{R}, \mathscr{X}^\omega)$. Thus, we have proved the following result.

Lemma 4.1. *A function $x \in C_{b,u}(\mathbb{J}, X)$ is ω-periodic at infinity if and only if the equivalence class $\widetilde{x} = x + C_0(\mathbb{J}, X)$ is $(\omega, \widetilde{S})$-periodic.*

As a corollary of Lemma 4.1 it follows that $\mathscr{X}^\omega = \mathscr{X}_\omega$, i.e. every class $\widetilde{x} \in \mathscr{X}^\omega$ is an $(\omega, \widetilde{S})$-periodic vector in $\mathscr{X}$ according to Definition 12.

Proofs of Theorems 2.1, 2.2, 2.3. follows directly by Lemma 4.1 and Theorems 3.2 and 3.3, where $\mathscr{X}_\omega = C_{\omega,\infty}(\mathbb{J}, X)/C_0(\mathbb{J}, X)$. □

We are now ready to present the proof of the characterization result.

Proof of Theorem 2.6. Necessity. Let $x \in C_{\omega,\infty}(\mathbb{J}, X)$ be a function such that $x = x_1 + x_0$, where $x_1 \in C_\omega(\mathbb{J}, X)$ and $x_0 \in C_0(\mathbb{J}, X)$. Then $A_N(x_1 + x_0) = x_1 + A_N x_0$, $N \geq 1$. Since $x_0 \in C_0(\mathbb{J}, X)$ we get $\lim\limits_{N\to\infty} A_N x_0 = 0$ and hence $\lim\limits_{N\to\infty} A_N x = x_1$.
Sufficiency. Let us consider a function $x \in C_{\omega,\infty}(\mathbb{J}, X)$ such that the limit $\lim\limits_{N\to\infty} A_N x = y$ exists. We shall prove that there exist two functions $x_1 \in C_\omega(\mathbb{J}, X)$ and $x_0 \in C_0(\mathbb{J}, X)$ such that $x = x_1 + x_0$.

From

$$\begin{aligned} S(\omega)y - y &= \lim_{N\to\infty}\left(\frac{1}{N}\sum_{k=0}^{N-1} S((k+1)\omega)x - \frac{1}{N}\sum_{k=0}^{N-1} S(k\omega)x\right) \\ &= \lim_{N\to\infty}\left(\frac{1}{N}(S(N\omega)x - x)\right) = 0 \end{aligned}$$

we get that $y \in C_\omega(\mathbb{J}, X)$ which means that $A_N y = y$ for all $N \geq 1$.

Using the notation $x - y = x_0 \in C_{\omega,\infty}(\mathbb{J}, X)$ we get the following equalities:

$$\lim_{N\to\infty} A_N x_0 = \lim_{N\to\infty} A_N(x - y) = \lim_{N\to\infty} (A_N x - y) = y - y = 0. \tag{4.4}$$

Along with the operators A_N, $N \geq 1$, one can consider a sequence $(\widetilde{A_N})$, $N \geq 1$, of operators from $End\mathscr{X}^\omega$ defined by $\widetilde{A_N} = \frac{1}{N}\sum\limits_{k=0}^{N-1} \widetilde{S}(k\omega)$. Clearly, on one hand, $\widetilde{A_N}\widetilde{x_0} = \widetilde{x_0}$ for all $N \geq 1$. On the other hand, (4.4) implies $\lim_{N\to\infty} \widetilde{A_N \widetilde{x_0}} = \widetilde{0}$, and therefore, $\widetilde{x_0} = \widetilde{0}$. Thus, we have proved that $x_0 \in C_0(\mathbb{J}, X)$ and, hence, $x = y + x_0$ for $y \in C_\omega(\mathbb{J}, X)$, $x_0 \in C_0(\mathbb{J}, X)$. $\square$

This result can also be obtained with the help of [5, 6, 7], but our proof is easier.

Example 1. Let us consider a continuously differentiable function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\operatorname{supp} \varphi \subset [0, 1]$ and $\varphi(\frac{1}{2}) = 1$. Let us also consider an arbitrary number sequence (α_n), $n \geq 1$, with the property $\lim_{n\to\infty} \alpha_n = 0$. Then one can construct the following sequence of functions from $C_{sl,\infty}(\mathbb{R}_+)$:

$$x_1(t) = \begin{cases} \varphi\left(\frac{t-2^m}{\ln(m+2)}\right) & , \quad t \in [2^m, 2^m + \ln(m+2)], \ m \geq 0, \\ 0 & , \quad t \notin \bigcup\limits_{m\geq 0} [2^m, 2^m + \ln(m+2)]; \end{cases}$$

$$x_n(t) = \begin{cases} \alpha_n x_1\left(t - (n-1)\ln(n+2)\right) & , \quad t \geq 2^n, \\ 0 & , \quad t \in [0, 2^n), \ n \geq 2. \end{cases}$$

One should mention that the functions x_n, $n \in \mathbb{Z}$, have disjoint supports and $\|x_n\|_\infty = \alpha_n$, $n \geq 1$.

The series $\sum\limits_{n=1}^{\infty} x_n(t) e^{int}$ converges absolutely to a 2π-periodic function x. Therefore, it is a generalized (but not the canonical) Fourier series of x (see Definitions 3 and 4). Clearly $\|\widetilde{x_n}\| = \alpha_n$, $n \geq 1$. From the arbitrariness of the sequence (α_n) with the property $\lim_{n\to\infty} \alpha_n = 0$ we get that the Fourier coefficients of a function from $C_{2\pi}(\mathbb{R})$ can converge to zero as slowly as we wish.

Proof of Theorem 2.5. Let a function $a \in C_{\omega,\infty}(\mathbb{J}, X)$ be C_0-invertible at infinity and a function $b \in C_b(\mathbb{J}, X)$ be one of its C_0-inverses. In this case $\widetilde{a}\widetilde{b} = \widetilde{b}\widetilde{a} = \widetilde{e}$, which is the identity of the Banach algebra $\mathscr{X}$. Let us consider an operator $A \in End\, \mathscr{X}$ defined by

$$A\widetilde{x} = \widetilde{a}\widetilde{x}, \ \widetilde{x} \in \mathscr{X}.$$

It is $(\omega, \widetilde{S})$-periodic and its Fourier coefficients A_n, $n \in \mathbb{Z}$, are given by $A_n \widetilde{x} = \widetilde{a_n}\widetilde{x}$, $\widetilde{x} \in \mathscr{X}$, where $a_n \in C_{sl,\infty}(\mathbb{J}, X)$, $n \in \mathbb{Z}$, are the canonical Fourier coefficients of a. Since $\|\widetilde{a_n}\| = \inf\limits_{x_0 \in C_0(\mathbb{J},X)} \|a_n + x_0\|$, the series $\sum\limits_{n\in\mathbb{Z}} \|A_n\| = \sum\limits_{n\in\mathbb{Z}} \|\widetilde{a_n}\|$ is absolutely convergent. The operator A is continuously invertible and its inverse $B = A^{-1} \in End\, \mathscr{X}$ is given by $B\widetilde{x} = \widetilde{b}\widetilde{x}$. Theorem 3.4 implies that the inverse operator B is also $(\omega, \widetilde{S})$-periodic (with respect to the presentation $\widetilde{S}$) and its Fourier series $B \sim \sum\limits_{n\in\mathbb{Z}} B_n$ is absolutely convergent.

Since $B_n \widetilde{x} = \widetilde{b_n}\widetilde{x}$, $\widetilde{x} \in C_{b,u}(\mathbb{J}, X)/C_0(\mathbb{J}, X)$, where $\widetilde{b_n}$, $n \in \mathbb{Z}$, are Fourier coefficients of $\widetilde{b}$, and $\|B_n\| = \|\widetilde{b_n}\|$ we get

$$\sum_{n\in\mathbb{Z}} \|B_n\| = \sum_{n\in\mathbb{Z}} \|\widetilde{b_n}\| < \infty.$$

This implies that the function b has an absolutely convergent Fourier series. $\square$

5 Periodic at infinity solutions of difference and differential equations

In this section we illustrate the utility of the notions introduced and studied in this paper by obtaining certain results about solutions of difference and differential equations. For example, we shall use the following spectral criterion that is rather helpful while proving the periodicity at infinity of bounded solutions. Just as in the previous section, we let X be a unital Banach algebra and define $\mathscr{X} = C_{b,u}(\mathbb{J}, X)/C_0(\mathbb{J}, X)$.

Theorem 5.1. *A function $x \in C_{b,u}(\mathbb{J}, X)$ is ω-periodic at infinity if and only if*

$$\Lambda_{ess}(x) \subset \frac{2\pi}{\omega}\mathbb{Z}.$$

Proof. As we mentioned earlier, the quotient space $\mathscr{X}$ is a Banach $L^1(\mathbb{R})$-module. Directly from Definitions 9 and 10 we get the equality $\Lambda(\widetilde{x}) = \Lambda_{ess}(x)$. Without loss of generality one can assume $\mathbb{J} = \mathbb{R}$. Then the assertion of the theorem follows by Lemma 4.1 and Theorem 3.1. □

Let us consider the following difference equation:

$$x(t+1) = Bx(t) + y_0(t),\ t \in \mathbb{J},\ \mathbb{J} \in \{\mathbb{R}_+, \mathbb{R}\}, \tag{5.1}$$

where $B \in End\, X,\ y_0 \in C_0(\mathbb{J}, X)$.

Theorem 5.2. *If the spectrum $\sigma(B)$ of the operator B satisfies*

$$\sigma(B) \cap \mathbb{T} \subset \{1\}. \tag{5.2}$$

then each uniformly continuous and bounded solution $x_0 : \mathbb{J} \to X$ of difference equation (5.1) is 1-periodic at infinity.

Proof. Let us consider a function $x_0 \in C_{b,u}(\mathbb{J}, X)$ that satisfies the difference equation (5.1), i.e. $S(1)x_0 - Bx_0 = y_0$. Since $y_0 \in C_0(\mathbb{J}, X)$ we get

$$\widetilde{S}(1)\widetilde{x_0} - B\widetilde{x_0} = \widetilde{0}. \tag{5.3}$$

By Theorem 5.1 it suffices to prove the inclusion $\Lambda(\widetilde{x_0}) \subset 2\pi\mathbb{Z}$.

Let us take an arbitrary $\lambda_0 \in \mathbb{R}\backslash 2\pi\mathbb{Z}$ and choose a function $f \in L^1(\mathbb{R})$ such that $\widehat{f}(\lambda_0) \neq 0$, $\operatorname{supp} \widehat{f}$ is compact, and $(\operatorname{supp} \widehat{f}) \cap 2\pi\mathbb{Z} = \emptyset$. We shall prove that $f\widetilde{x_0} = 0$. Formula (5.4) implies

$$f(\widetilde{S}(1)\widetilde{x_0} - B\widetilde{x_0}) = (S(1)f)\widetilde{x_0} - Bf\widetilde{x_0} = f_1\widetilde{x_0} - Bf\widetilde{x_0} = \widetilde{0}, \tag{5.4}$$

where by f_1 we denote the function $S(1)f \in L^1(\mathbb{R})$.

In the case $\mathbb{J} = \mathbb{R}_+$ we shall use the notation $\overline{x_0} \in C_{b,u}(\mathbb{R}, X)$ for an arbitrary extension of the function x_0 on $\mathbb{R}$ with the property $\lim\limits_{t\to-\infty} \overline{x_0}(t) = 0$. In the case $\mathbb{J} = \mathbb{R}$ we set $\overline{x_0} = x_0$.

Formula (5.4) implies the inclusion $(f_1 - Bf) * \overline{x_0} \in C_0(\mathbb{J}, X)$. Since $\sigma(B) \cap \mathbb{T} \subset \{1\}$, there is a neighbourhood $V \subset \mathbb{T}$ of $\gamma_0 = e^{i\lambda_0}$ such that the resolvent $\lambda \mapsto R(e^{i\lambda}, B) : V \to End\, X$ of the operator B is well defined.

Let us consider a function $\varphi \in L^1(\mathbb{R})$ such that its Fourier transform $\widehat{\varphi}$ is an infinitely differentiable function with the properties: $\widehat{\varphi}(\lambda_0) \neq 0$ and $\operatorname{supp}\widehat{\varphi} \subset [\lambda_0 - \delta, \lambda_0 + \delta]$, where $\delta > 0$ is sufficiently small to ensure $e^{i\lambda} \in \rho(B)$ for $|\lambda - \lambda_0| \leq \delta$. Since the function $\lambda \mapsto (e^{i\lambda}I - B)^{-1} : \mathbb{R} \to End\, X$ is holomorphic, we get that the function

$$\widehat{F}(\lambda) = \begin{cases} \widehat{\varphi}(\lambda)(e^{i\lambda}I - B)^{-1} & , \quad \lambda \in [\lambda_0 - \delta, \lambda_0 + \delta], \\ 0 & , \quad \lambda \notin [\lambda_0 - \delta, \lambda_0 + \delta]; \end{cases}$$

is the Fourier transform of some function $F : \mathbb{R} \to End\, X$.

By the equalities

$$\widehat{F * (f_1 - Bf)}(\lambda) = \widehat{F}(\lambda)(e^{i\lambda}I - B)\widehat{f}(\lambda) = \widehat{\varphi}(\lambda)\widehat{f}(\lambda)I,$$

$\lambda \in \mathbb{R}$, and formula (5.4) it follows that $F * (f_1 - Bf) * \overline{x_0} = (\varphi * f) * \overline{x_0} \in C_0(\mathbb{R}, X)$. Let us introduce the notation $\varphi * f = g$. Then $\widehat{g}(\lambda_0) = \widehat{\varphi}(\lambda_0)\widehat{f}(\lambda_0) \neq 0$, and, therefore, λ_0 does not belong to the essential spectrum of $\overline{x_0}$.

Theorem 5.1 now implies that the function $\overline{x_0}$ is 1-periodic at infinity. Clearly, the function x_0 is also 1-periodic at infinity. □

Corollary 5.1. *Assume that an operator $B \in End\, X$ satisfies condition (5.2). Let us consider the nonlinear equation*

$$x(t+1) = Bx(t) + f(t, x(t)),\ t \geq 0, \tag{5.5}$$

where the function $t \mapsto f(t,x)$ is uniformly continuous with respect to x in any bounded subset of X and the equation $\lim\limits_{t\to\infty} \sup\limits_{\|x\|\leq R} \|f(t,x)\| = 0$ holds for any $R > 0$. Then each uniformly continuous and bounded solution of equation (5.5) is 1-periodic at infinity.

Corollary 5.2. *If an operator $B \in End\, X$ satisfies condition (5.2), $F_0 \in C_0(\mathbb{R}_+, End\, X)$, and a function $g : X \to \mathbb{C}$ is cotinuous then each uniformly continuous and bounded solution of equation*

$$x(t+1) = Bx(t) + F_0(t)g(x),\ t \geq 0,$$

is 1-periodic at infinity.

Corollary 5.3. *If an operator $B \in End\, X$ satisfies condition (5.2) and $F_0 \in C_0(\mathbb{R}_+, EndX)$ then each uniformly continuous and bounded solution of the equation*

$$x(t+1) = (B + F_0(t))x(t),\ t \geq 0,$$

is 1-periodic at infinity.

Let us now consider a linear differential equation

$$\dot{x}(t) - Ax(t) = y(t),\ t \in \mathbb{J}, \tag{5.6}$$

where $y \in L^1(\mathbb{R}, X)$ and $A : D(A) \subset X \to X$ is a generator of a C_0-semigroup $U : \mathbb{R}_+ \to End\, X$.

Definition 16. A function $x : \mathbb{J} \to X$ is called *a mild solution* of (5.6) (see [17]) if the equality

$$x(t) = U(t-s)x(s) + \int_s^t U(t-\tau)y(\tau)d\tau, \tag{5.7}$$

holds for all $s \le t,\ s,t \in \mathbb{J}$.

We note that in the case $\mathbb{J} = \mathbb{R}_+$ the last equality has to hold only for $s = 0$ and $t \ge 0$. It is obvious that x is uniformly continuous.

Theorem 5.3. *If the inclusion*

$$\sigma(U(1)) \cap \mathbb{T} \subset \{1\}. \tag{5.8}$$

holds then each mild solution of (5.6) that is bounded on $\mathbb{J}$ *is* 1*-periodic at infinity, i.e.* $x \in C_{1,\infty}(\mathbb{J}, X)$.

Proof. Assume condition (5.8). Let $x : \mathbb{J} \to X$ be a mild solution of (5.6) that is bounded on $\mathbb{J}$. By setting $s = t$ in (5.7) and considering the function x at the point $t+1$ we get the following equality:

$$x(t+1) = U(1)x(t) + \int_t^{t+1} U(t+1-\tau)f(\tau)d\tau,\ t \in \mathbb{J}.$$

We shall write $U(1) = B,\ \int_t^{t+1} U(t+1-\tau)f(\tau)d\tau = y_0(t),\ t \in \mathbb{J}$. Next, we will show that $y_0 \in C_0(\mathbb{J}, X)$. We have

$$\|y_0(t)\| = \| \int_t^{t+1} U(t+1-\tau)y(\tau)d\tau\| \le M \int_t^{t+1} \|y(\tau)\|d\tau \to 0 \quad \text{as} \quad |t| \to \infty,$$

which follows by the boundedness of U and the estimate

$$\int_t^{t+1} \|y(\tau)\|d\tau \le \int_n^{n+1} \|y(\tau)\|d\tau + \int_{n+1}^{n+2} \|y(\tau)\|d\tau \to 0 \quad \text{as} \quad |n| \to \infty$$

where $y \in L^1(\mathbb{R}, X)$ and $n = \lfloor t \rfloor$ is the floor of t.

Hence, x satisfies difference equation (5.1). Equation (5.8) yields condition (5.2) of Theorem 5.2 and, therefore, $x \in C_{1,\infty}(\mathbb{J}, X)$. □

Theorem 5.4. *Let us consider a bounded semigroup* $U : \mathbb{R}_+ \to End\, X$*. If the spectrum of its infinitesimal generator* A *satisfies*

$$\sigma(A) \cap i\mathbb{R} \subset i\frac{2\pi}{\omega}\mathbb{Z}, \tag{5.9}$$

then each function $\varphi_x : \mathbb{R}_+ \to X$ *defined by* $\varphi_x(t) = U(t)x,\ t \ge 0,$ *is* ω*-periodic at infinity.*

Definition 16 directly implies that each function $\varphi_x,\ x \in X$, is a bounded mild solution of (5.6) and, therefore, the conditions of Theorem 5.3 are satisfied.

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