ISSN 2077–9879

Eurasian Mathematical Journal

2016, Volume 7, Number 3

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia
the University of Padua

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by
the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

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EMJ: from Scopus Q4 to Scopus Q3 in two years?!

Recently the list was published of all mathematical journals included in 2015 Scopus quartiles Q1 (334 journals), Q2 (318 journals), Q3 (315 journals), and Q4 (285 journals). Altogether 1252 journals.

With great pleasure we inform our readers that the Eurasian Mathematical Journal was included in this list, currently the only mathematical journal in the Republic of Kazakhstan and Central Asia.

It was included in Q4 with the SCImago Journal & Country Rank (SJR) indicator equal to 0,101, and is somewhere at the bottom of the Q4 list. With this indicator the journal shares places from 1240 to 1248 in the list of all 2015 Scopus mathematical journals. Nevertheless, this may be considered to be a good achievement, because Scopus uses information about journals for the three previous years, i. e. for years 2013-2015, and the EMJ is in Scopus only from the first quarter of year 2015.

The SJR indicator is calculated by using a sophisticated formula, taking into account various characteristics of journals and journals publications, in particular the average number of weighted citations received in the selected year by the documents published in the selected journal in the three previous years. This formula and related comments can be viewed on the web-page

$$http://www.scimagojr.com/journalrank.php?category = 2601\&area = 2600\&page = 1\&total_size = 373$$

(Help/Journals/Understand tables and charts/Detailed description of SJR.)

In order to enter Q3 the SJR indicator should be greater than 0,250. It looks like the ambitious aim of entering Q3 in year 2017 is nevertheless realistic due to recognized high level of the EMJ.

We hope that all respected members of the international Editorial Board, reviewers, authors of our journal, representing more than 35 countries, and future authors will provide high quality publications in the EMJ which will allow to achieve this aim.

On behalf of the Editorial Board of the EMJ
V.I. Burenkov, E.D. Nursultanov, T.Sh. Kalmenov,
R. Oinarov, M. Otelbaev, T.V. Tararykova, A.M. Temirkhanova

VICTOR IVANOVICH BURENKOV

(to the 75th birthday)

On July 15, 2016 was the 75th birthday of Victor Ivanovich Burenkov, editor-in-chief of the Eurasian Mathematical Journal (together with V.A. Sadovnichy and M. Otelbaev), director of the S.M. Nikol'skii Institute of Mathematics, head of the Department of Mathematical Analysis and Theory of Functions, chairman of Dissertation Council at the RUDN University (Moscow), research fellow (part-time) at the Steklov Institute of Mathematics (Moscow), scientific supervisor of the Laboratory of Mathematical Analysis at the Russian-Armenian (Slavonic) University (Yerevan, Armenia), doctor of physical and mathematical sciences (1983), professor (1986), honorary professor of the L.N. Gumilyov Eurasian National University (Astana, Kazakhstan, 2006), honorary doctor of the Russian-Armenian (Slavonic) University (Yerevan, Armenia, 2007), honorary member of staff of the University of Padua (Italy, 2011), honorary distinguished professor of the Cardiff School of Mathematics (UK, 2014), honorary professor of the Aktobe Regional State University (Kazakhstan, 2015).

V.I. Burenkov graduated from the Moscow Institute of Physics and Technology (1963) and completed his postgraduate studies there in 1966 under supervision of the famous Russian mathematician academician S.M. Nikol'skii.

He worked at several universities, in particular for more than 10 years at the Moscow Institute of Electronics, Radio-engineering, and Automation, the RUDN University, and the Cardiff University. He also worked at the Moscow Institute of Physics and Technology, the University of Padua, and the L.N. Gumilyov Eurasian National University.

He obtained seminal scientific results in several areas of functional analysis and the theory of partial differential and integral equations. Some of his results and methods are named after him: Burenkov's theorem of composition of absolutely continuous functions, Burenkov's theorem on conditional hypoellipticity, Burenkov's method of mollifiers with variable step, Burenkov's method of extending functions, the Burenkov-Lamberti method of transition operators in the problem of spectral stability of differential operators, the Burenkov-Guliyevs conditions for boundedness of operators in Morrey-type spaces. On the whole, the results obtained by V.I. Burenkov have laid the groundwork for new perspective scientific directions in the theory of functions spaces and its applications to partial differential equations, the spectral theory in particular.

More than 30 postgraduate students from more than 10 countries gained candidate of sciences or PhD degrees under his supervision. He has published more than 170 scientific papers. The lists of his publications can be viewed on the portals MathSciNet and Math-Net.Ru. His monograph "Sobolev spaces on domains" became a popular text for both experts in the theory of function spaces and a wide range of mathematicians interested in applying the theory of Sobolev spaces.

In 2011 the conference "Operators in Morrey-type Spaces and Applications", dedicated to his 70th birthday was held at the Ahi Evran University (Kirsehir, Turkey). Proceedings of that conference were published in the EMJ 3-3 and EMJ 4-1.

The Editorial Board of the Eurasian Mathematical Journal congratulates Victor Ivanovich Burenkov on the occasion of his 75th birthday and wishes him good health and new achievements in science and teaching!

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 7, Number 3 (2016), 89 – 99

AN ANALOGUE OF THE HAHN-BANACH THEOREM FOR FUNCTIONALS ON ABSTRACT CONVEX CONES

F.S. Stonyakin

Communicated by E. Kissin

Key words: abstract convex cone, cancellation law, convex functional, Hahn-Banach theorem, convex normed come, Lemma on a support functional, strict convex normed cone, sublinear injective isometric embedding.

AMS Mathematics Subject Classification: 46A22, 46A20, 46B10.

Abstract. We prove an analogue of the Hahn-Banach theorem on the extension of a linear functional with a convex estimate for each abstract convex cone with the cancellation law. Also we consider the special class of the so-called strict convex normed cones ($SCNC$). For such structures we obtain an appropriate analogue of the Hahn-Banach separation theorem. On the base of this result we prove that each ($SCNC$) is sublinearly, injectively and isometrically embedded in some Banach space.

1 Introduction

The theory of so-called abstract convex cones (or, simply, convex cones) is actively developed in recent decades (see, e.g., [4, 8, 9, 10]). In particular, such known mathematicians as J. Rädström, K. Keimel, W. Roth and others were engaged in this theory. The special types of (convex) cones in some functional spaces were studied by M. L. Goldman, P. P. Zabreiko and E. G. Bakhtigareeva (see, e.g., [1, 2, 3]). Recently, in connection with some problems of analysis, the so-called *subnormed cones* were considered by I. V. Orlov (see, e.g., [5, 6, 7]). For the (convex) cones of convex compact subsets of normed space E with separable conjugate space E^* a new analogue of the Shauder fixed-point theorem was proved by us in [11].

Our paper is devoted to an analogue of the Hahn-Banach theorem for functionals on abstract convex cones and some applications. Various analogues of the Hahn-Banach theorem on functional separation [4, 9, 10] are known in the theory of convex cones. But these results are proved only in special classes of *ordered* convex cones using *non-negative monotonic* linear functionals. We do not use such essential restrictions and consider all linear functionals.

The paper consists of three sections.

In the first section we prove an analogue of the Hahn-Banach theorem on the extension of a linear functional with a convex estimate for each abstract convex cone with the cancellation law (Theorem 2.1).

In the second section we select a special class of *convex normed cones* (CNC) X and introduce a concept of the *conjugate cone* of (CNC) (see Definition 5). On the base of

Theorem 2.1 we obtain an analogue of the well-known Lemma on a support functional in convex normed cones (see Corollary 3.1).

Finally, in the last section we consider a special type of abstract convex cones — *strict convex normed cones* ($SCNC$). On the base of Theorem 2.1 and Corollary 3.1 a theorem on separation of points in ($SCNC$) is obtained for semi-bounded linear functionals (see Theorem 4.1) and for bounded convex functionals (see Theorem 4.2). In fact, we construct some elements of the duality theory in ($SCNC$). Also for each $X \in (SCNC)$ we prove the existence of a sublinear isometric embedding $\varphi : X \to E$ in a Banach space E (see Theorem 4.3). Generally, one cannot replace the condition of sublinearity of such embedding by the condition of usual linearity (see Example 8).

2 An analogue of the Hahn-Banach theorem for abstract convex cones

We start with some auxiliary concepts. Recall that *abstract convex cone* or *convex cone* is a collection X of elements with operations of addition and non-negative scalar multiplication, where X is a commutative semigroup under addition, such that for arbitrary numbers $\lambda, \mu \geqslant 0$ and elements $x, y \in X$ the following relations hold:

$$1 \cdot x = x; \quad (\lambda\mu)x = \lambda(\mu x); \quad 0 \cdot x = 0; \quad \lambda(x+y) = \lambda x + \lambda y; \quad (\lambda+\mu)x = \lambda x + \mu x.$$

Note, that for many results in the theory of convex cones the following *cancellation law* is essential:

$$x + y = y + z \Longleftrightarrow x = z \text{ for all } x, y, z \in X. \tag{2.1}$$

Definition 1. A mapping $p : X \to \mathbb{R}$ is called a *convex functional*, if for each $x, y \in X$ and $\lambda \geqslant 0$ the following conditions hold:

$$p(x) \geqslant 0, \; p(\lambda x) = \lambda p(x) \text{ and } p(x+y) \leqslant p(x) + p(y).$$

We also introduce an analogue of subspaces in the class of convex cones.

Definition 2. We say that Y is a *subcone* of X, if $Y \subset X$, Y is a convex cone and for all $x \in X$ and $y, z \in Y \subset X$ the condition $z = x + y$ means that $x \in Y$.

For example, for each fixed elements $x_1, x_2, ..., x_n \in X$ the set

$$Y = \left\{ x \in X \,\middle|\, x + \sum_{k=1}^{n} \mu_k x_k = \sum_{k=1}^{n} \lambda_k x_k \text{ for all } \lambda_k, \mu_k \geqslant 0, \, k = \overline{1,n} \right\}$$

is a subcone of X. One more example is considered in the proof of Theorem 4.1.

Now we obtain an analogue of the Hahn-Banach theorem on the extension of a linear functional from a subcone $Y \subset X$ on the whole cone X for the class of convex cones X with the cancellation law, briefly $X \in (CL)$.

Theorem 2.1. *Let X be a convex cone with the cancellation law, $p : X \to \mathbb{R}$ be a convex functional on X, Y be a subcone of X. Assume that $\ell : Y \to \mathbb{R}$ is a linear functional with the estimate $\ell(y) \leqslant p(y)$ for all $y \in Y$.*

Then there exists a linear functional $L : X \to \mathbb{R}$ such that $L(x) \leqslant p(x)$ for all $x \in X$ and $L(y) = \ell(y)$ for all $y \in Y$.

Proof. 1) Suppose that, for a fixed element $e \in X$

$$X = Lin(e, Y) = \{x \in X \mid x + \lambda_1 e + y_1 = \lambda_2 e + y_2 \text{ for some } y_1, y_2 \in Y \text{ and } \lambda_1, \lambda_2 \geqslant 0\}.$$

By virtue of (2.1) from $x + \lambda_1 e + y_1 = \lambda_2 e + y_2$ we have $x + (\lambda_1 - \lambda_2)e + y_1 = y_2$ for the case of $\lambda_1 > \lambda_2$, or $x + y_1 = (\lambda_2 - \lambda_1)e + y_2$ otherwise. Consequently, for each $x \in Lin(e, Y)$ we can consider $x + \lambda e + y_1 = y_2$ or $x + y_1 = \lambda e + y_2$ for some $\lambda \geqslant 0$ and $y_1, y_2 \in Y$. Let us define L on $X = Lin(e, Y)$ in the following way:

1) If $x_1 \in X$: $x_1 + y_{12} = \lambda_1 e + y_{11}$ for some y_{11}, $y_{12} \in Y$ and $\lambda_1 \geqslant 0$ then

$$L(x_1) := \lambda_1 L(e) + \ell(y_{11}) - \ell(y_{12}).$$

2) If there exists $x_2 \in X$: $x_2 + \lambda_2 e + y_{22} = y_{21}$ for some y_{21}, $y_{22} \in Y$ and $\lambda_2 \geqslant 0$ then

$$L(x_2) := \ell(y_{21}) - \ell(y_{22}) - \lambda_2 L(e).$$

It is necessary to prove the existence of a number $L(e)$, such that $L(x_i) \leqslant p(x_i)$ for each $i = 1, 2$, or equivalently

$$\lambda_1 L(e) + \ell(y_{11}) - \ell(y_{12}) \leqslant p(x_1), \quad -\lambda_2 L(e) + \ell(y_{21}) - \ell(y_{22}) \leqslant p(x_2). \tag{2.2}$$

The case $\lambda_i = 0$ for some $i = 1, 2$ is obvious: $x_i \in Y$. So, without loss of generality, we assume that $\lambda_i > 0$ for $i = 1, 2$ and we can rewrite conditions (2.2) as follows:

$$L(e) \leqslant p\left(\frac{x_1}{\lambda_1}\right) + \ell\left(\frac{y_{12}}{\lambda_1}\right) - \ell\left(\frac{y_{11}}{\lambda_1}\right), \quad L(e) \geqslant \ell\left(\frac{y_{21}}{\lambda_2}\right) - \ell\left(\frac{y_{22}}{\lambda_2}\right) - p\left(\frac{x_2}{\lambda_2}\right).$$

for all possible $y_{11}, y_{12}, y_{21}, y_{22} \in Y$ and $\lambda_i > 0$ $(i = 1, 2)$.

Let us check the useful inequality:

$$\ell\left(\frac{y_{21}}{\lambda_2}\right) - \ell\left(\frac{y_{22}}{\lambda_2}\right) - p\left(\frac{x_2}{\lambda_2}\right) \leqslant p\left(\frac{x_1}{\lambda_1}\right) + \ell\left(\frac{y_{12}}{\lambda_1}\right) - \ell\left(\frac{y_{11}}{\lambda_1}\right). \tag{2.3}$$

for all possible $y_{11}, y_{12}, y_{21}, y_{22} \in Y$ and $\lambda_i > 0$ $(i = 1, 2)$.

Indeed, by virtue of the cancellation law of X from the correlations

$$\frac{x_1}{\lambda_1} + \frac{y_{12}}{\lambda_1} = \frac{y_{11}}{\lambda_1} + e \ \text{ and } \ \frac{x_2}{\lambda_2} + \frac{y_{22}}{\lambda_2} + e = \frac{y_{21}}{\lambda_2}$$

we have

$$\frac{x_1}{\lambda_1} + \frac{x_2}{\lambda_2} + \frac{y_{12}}{\lambda_1} + \frac{y_{22}}{\lambda_2} = \frac{y_{11}}{\lambda_1} + \frac{y_{21}}{\lambda_2},$$

i.e. $\frac{x_1}{\lambda_1} + \frac{x_2}{\lambda_2} \in Y$, so Y is a subcone of X (see Definition 2). Further,

$$p\left(\frac{x_1}{\lambda_1}\right) + \ell\left(\frac{y_{12}}{\lambda_1}\right) - \ell\left(\frac{y_{11}}{\lambda_1}\right) - \left(\ell\left(\frac{y_{21}}{\lambda_2}\right) - \ell\left(\frac{y_{22}}{\lambda_2}\right) - p\left(\frac{x_2}{\lambda_2}\right)\right) =$$

$$= p\left(\frac{x_1}{\lambda_1}\right) + p\left(\frac{x_2}{\lambda_2}\right) + \ell\left(\frac{y_{12}}{\lambda_1} + \frac{y_{22}}{\lambda_2}\right) - \ell\left(\frac{y_{11}}{\lambda_1} + \frac{y_{21}}{\lambda_2}\right) =$$

$$= p\left(\frac{x_1}{\lambda_1}\right) + p\left(\frac{x_2}{\lambda_2}\right) + \ell\left(\frac{y_{12}}{\lambda_1} + \frac{y_{22}}{\lambda_2}\right) - \ell\left(\frac{x_1}{\lambda_1} + \frac{x_2}{\lambda_2} + \frac{y_{12}}{\lambda_1} + \frac{y_{22}}{\lambda_2}\right) =$$

$$= p\left(\frac{x_1}{\lambda_1}\right) + p\left(\frac{x_2}{\lambda_2}\right) - \ell\left(\frac{x_1}{\lambda_1} + \frac{x_2}{\lambda_2}\right) \geqslant p\left(\frac{x_1}{\lambda_1} + \frac{x_2}{\lambda_2}\right) - \ell\left(\frac{x_1}{\lambda_1} + \frac{x_2}{\lambda_2}\right) \geqslant 0$$

in view of $\ell(y) \leqslant p(y)$ for all $y \in Y$.

For the sake of convenience, we make the following agreements. If $x_1 \in X$ and for any $\lambda_1 > 0$ and $y_{11}, y_{12} \in Y$ the equality $x_1 + y_{12} = \lambda_1 e + y_{11}$ does not hold, then we assume that:

$$p\left(\frac{x_1}{\lambda_1}\right) + \ell\left(\frac{y_{12}}{\lambda_1}\right) - \ell\left(\frac{y_{11}}{\lambda_1}\right) := +\infty. \tag{2.4}$$

If $x_2 \in X$ and for any $\lambda_2 > 0$ and $y_{21}, y_{22} \in Y$ there equality $x_2 + y_{22} + \lambda_2 e = y_{21}$ does not hold, then we assume that:

$$\ell\left(\frac{y_{21}}{\lambda_2}\right) - \ell\left(\frac{y_{22}}{\lambda_2}\right) - p\left(\frac{x_2}{\lambda_2}\right) := -\infty. \tag{2.5}$$

By (2.3) $\alpha_e \leqslant \beta_e$, where

$$\alpha_e := \sup_{\lambda_2 > 0,\, y_{21}, y_{22} \in Y} \left\{\ell\left(\frac{y_{21}}{\lambda_2}\right) - \ell\left(\frac{y_{22}}{\lambda_2}\right) - p\left(\frac{x_2}{\lambda_2}\right)\right\}$$

and

$$\beta_e := \inf_{\lambda_1, > 0,\, y_{11}, y_{12} \in Y} \left\{p\left(\frac{x_1}{\lambda_1}\right) + \ell\left(\frac{y_{12}}{\lambda_1}\right) - \ell\left(\frac{y_{11}}{\lambda_1}\right)\right\}.$$

We can put $L(e) \in [\alpha_e; \beta_e]$ and conditions (2.2) hold.

2) It is clear, that ℓ can be similarly extended to the chain of subcones

$$Lin(e_1, Y) \subset Lin(e_2, Y_1) \subset \ldots \subset Lin(e_n, Y_{n-1}) \subset \ldots,$$

where $Y_k = Lin(e_k, Y_{k-1})$, $Y = Y_0$, $Y_1 \subset Y_2 \subset \ldots$

Let us introduce the following order: $(Z, L_Z) \leqslant (Z_1, L_{Z_1})$ if $Z \subset Z_1$ and a linear functional L_{Z_1} extends L_Z. By Zorn's lemma each totally ordered set has a majorant: the union. Hence there is a maximal element (Y_m, L_m). By the arguments in 1), $Y_m = X$. □

3 Convex normed cones and an analogue of the Lemma on a support functional

It is interesting to obtain conditions under which some abstract convex cone with a norm may be isometrically embedded in a linear normed space with enough convenient properties of the corresponding embedding. It is clear, that such a norm in X should satisfy the following property:

$$x + y = 0 \Longrightarrow \|x\| = \|y\| \quad \text{for all} \quad x, y \in X. \tag{3.1}$$

Generally, the last property (3.1) does not follow from the standard axioms of a norm in the class of convex cones. Below we give such an example (see Example 1).

In this section we select a special class of *convex normed cones* (CNC) X with the norm satisfying (3.1). We consider some examples of (CNC) and obtain an analogue of the well-known Lemma on a support functional for such cones (see Corollary 3.1). Let us start with the definition of a norm in convex cones.

Definition 3. We say that a norm is a function $\|\cdot\| : X \to \mathbb{R}$ satisfying condition (3.1) and the following standard axioms: for all $x, y \in X$ and $\lambda \geqslant 0$

$$\|x\| \geqslant 0; \quad \|x\| = 0 \Leftrightarrow x = 0; \quad \|\lambda x\| = \lambda\|x\|; \quad \|x + y\| \leqslant \|x\| + \|y\|. \tag{3.2}$$

We say, that X is a convex normed cone (CNC), if there is a norm on X.

Let us show that condition (3.1) does not follow from the other three standard axioms of a norm in abstract convex cones.

Example 1. Let $X = (-\infty; +\infty)$. The non-negative scalar multiplication is standard. The addition $x_1 \oplus x_2$ we introduce in the following way: $x_1 \oplus x_2 := \min\{x_1, x_2\}$. Axioms (3.2) hold for the function $\|x\| := |x|$. However, $\|0\| \neq \|1\|$, although $0 \oplus 1 = 0$. Thus X is not a convex normed cone.

Note that for the addition $x_1 \oplus x_2 := \max\{x_1, x_2\}$ the set $X = [0; +\infty)$ is a convex normed cone with the standard norm. Also the set $X = (-\infty; +\infty)$ with the standard norm is a convex normed cone for the usual addition and scalar multiplication. Let us give some other simple examples of convex normed cones.

Example 2. Let X be the collection of all bounded real-valued functions $f : [0; 1] \to \mathbb{R}$ with the usual addition and scalar multiplication. In this case we can consider the usual sup-norm $\|f\| := \sup\limits_{0 \leqslant t \leqslant 1} |f(t)|$.

Example 3. Let X be the collection of all convex compact subsets A of a normed linear space E with the Minkowsky addition and the usual scalar multiplication. The norm in K is introduced in a natural way: $\|A\| := \max\limits_{a \in A} \|a\|_E$.

Let us pass to an analogue of the well-known Lemma on a support functional in (CNC). We start with an auxiliary concepts of a semi-bounded linear functional and a conjugate cone.

Definition 4. We say that a linear functional $\ell : X \to \mathbb{R}$ is *semi-bounded* on X, if for some $C > 0$ the following inequality holds: $\ell(x) \leqslant C\|x\|$ for all $x \in X$.

Clearly, the set X^* of all semi-bounded linear functionals on X is a convex cone if we introduce the addition of functionals and scalar multiplication in the usual way.

Definition 5. The collection of linear semi-bounded functionals on X is called the *conjugate cone* of X and we denote it by X^*.

By Theorem 2.1 immediately follows an analogue of the well-known Lemma on a support functional in the class of convex normed cones with the cancellation law.

Corollary 3.1. *Let X be a (CNC) with the cancellation law. Then for each fixed $x_0 \in X\backslash\{0\}$ there exists $\ell \in X^* \setminus \{0\}$, such that $\ell(x_0) = \|x_0\|$ and $\ell(x) \leqslant \|x\|$ for all $x \in X$.*

Proof. It is easy to see that $Y = \{x \in X \,|\, x + \mu x_0 = \lambda x_0, \lambda \text{ and } \mu \geqslant 0\}$ is a subcone of X. Since $X \in (CL)$, then $x + \mu x_0 = \lambda x_0$ means that $x = (\lambda - \mu)x_0$ (for $\lambda \geqslant \mu$) or $x + (\mu - \lambda)x_0 = 0$ (for $\mu > \lambda$). Note that for $\mu > \lambda$ from $x + (\mu - \lambda)x_0 = 0$ and (3.1) we have $\|x\| = (\mu - \lambda)\|x_0\|$. We put $\ell(x) := (\lambda - \mu)\|x_0\|$ and apply Theorem 2.1. □

Remark 1. An analogous result is known in the special class of strict ordered normed cones X for non-negative linear functionals $f : X \to \mathbb{R}$ (see [10], Theorem 2.14). However, in general, the equality $f(x_0) = \|x_0\|$ is not possible (see Remark after Theorem 2.14 in [10]). Note, that the condition $\ell(x_0) = \|x_0\|$ in Corollary 3.1 is essential for some results of our paper (see Theorems 4.1, 4.2 and 4.3).

4 On the isometric embedding of convex normed cone in a Banach space

In the work [8] it was proved that if a convex cone X has a metric $d : X \times X \longrightarrow \mathbb{R}$ with special properties, then X is linearly injectively and isometrically (with respect to the metric) embedded in some linear normed space.

Naturally it is interesting to get conditions of the existence of an isometrical embedding of an abstract convex normed cone in a normed space without using of the metric d, but only using the norm. In this section on the base of Theorem 2.1 and Corollary 3.1 we study this problem for a special class of the so-called *strict convex normed cones* $(SCNC)$ X. We prove that semi-bounded linear functionals separate points in such cones (see Theorem 4.1). Further we give analogues of Corollary 3.1 and Theorem 4.1 for bounded convex functionals (see Theorem 4.2) and prove the existence of a sublinear isometrical embedding $\varphi : X \to E$ for each $(SCNC)$ X in some Banach space E (see Theorem 4.3). We start with auxiliary definitions, notations and examples.

Definition 6. We say that X is a strict cone, if the following property holds:

$$x + y = 0 \Longrightarrow x = y = 0 \ \text{ for all } \ x, y \in X. \tag{4.1}$$

Note that strict convex cones were considered earlier, for example, in [10]. In each strict convex cone X we can consider the following partial order [10]:

$$x \preceq y \ \text{ if } \ y = x + z \ \text{ for some } z \in X. \tag{4.2}$$

Now we introduce the following property of *order separability* for strict convex cones.

Definition 7. We say that an abstract convex cone X is a strict convex normed cone $(SCNC)$, if X is a strict convex normed cone with the cancellation law and for all $x, y \in X$:

$$\text{for each non-negative numbers } \ \alpha < 1 < \beta \quad \alpha x \preceq y \preceq \beta x \Longrightarrow y = x, \tag{4.3}$$

and

$$x \preceq y \Longrightarrow \|x\| \leqslant \|y\|. \tag{4.4}$$

Note that in strict cones property (3.1) follows from the other axioms of norm (since $x + y = 0$ means $x = y = 0$ and $\|x\| = \|y\| = 0$). Now we consider some examples of strict convex normed cones.

Example 4. Let $X = [0; +\infty) =: \mathbb{R}_+$ be the collection of all non-negative numbers with the usual addition and scalar multiplication. For each $a, b \in \mathbb{R}_+$ $a + b = 0$ means $a = b = 0$ and

$$a \leqslant b \Longleftrightarrow b = a + c \text{ for some } c \geqslant 0.$$

Clearly, $\alpha b \preceq a \preceq \beta b$ for all $\alpha < 1 < \beta$ means $b \leqslant a \leqslant b$, i.e. $a = b$.

Analogously with the previous example we can give the following examples.

Example 5. Let X be the collection of all sequences $a = \{a_n\}_{n=1}^{\infty}$, where $a_n \geqslant 0$ for each $n \in \mathbb{N}$. 0 is a zero sequence. The addition and scalar multiplication in X we introduce standardly: $a + b := \{a_n + b_n\}_{n=1}^{\infty}$ and $\lambda \cdot a := \{\lambda \cdot a_n\}_{n=1}^{\infty}$ for each $\lambda \geqslant 0$ and $a = \{a_n\}_{n=1}^{\infty}$, $b = \{b_n\}_{n=1}^{\infty}$ from X. Note, that

$$a \preceq b \Longleftrightarrow a_n \leqslant b_n \ \text{ for each } \ n \in \mathbb{N}.$$

Example 6. Let X be the collection of all non-negative bounded real-valued functions $f : [0; 1] \to \mathbb{R}_+$ with the usual addition and scalar multiplication. In this case for each $f, g \in X$

$$f \preceq g \Longleftrightarrow f(t) \leqslant g(t) \ \text{ for all } \ t \in [0; 1].$$

Example 7. Let X be the collection of all segments $A = [a; b] \subset \mathbb{R}^+$ with the Minkowsky addition and usual scalar multiplication. In this case we can consider the following order: for each $A, B \in X$

$$A \preceq B \Longleftrightarrow \quad \text{there exists } C \in X \ \text{ such that } \ B = A + C.$$

Now we pass to an analogue of the Hanh-Banach separation theorem for $(SCNC)$.

Theorem 4.1. *Let X be a $(SCNC)$. Then for all $e_1 \neq e_2$ $(e_1, e_2 \in X)$ there exists a functional $\ell \in X^*\backslash\{0\}$, such that $\ell(e_1) \neq \ell(e_2)$ and $\ell(e_1) > 0$ or $\ell(e_2) > 0$.*

Proof. 1) If $\|e_1\| \neq \|e_2\|$ (for example, $e_1 = 0$ or $e_2 = 0$) then for each $i = 1, 2$ by Corollary 3.1 there exists $\ell_i \in X^*\backslash\{0\}$, such that $\ell_i(e_i) = \|e_i\|$. If $\|e_1\| < \|e_2\|$ then by Corollary 3.1 we have $\ell_2(e_1) \leqslant \|e_1\| < \|e_2\| = \ell_2(e_2)$, so that $\ell_2(e_1) \neq \ell_2(e_2) > 0$. The case of $\|e_2\| < \|e_1\|$ is considered analogously.

2) Now we assume $\|e_1\| = \|e_2\|$, $e_1 \neq 0$ and $e_2 \neq 0$. Set $Y := \{\lambda e_1 | \lambda \geqslant 0\}$.

Note that Y is a subcone of X. Indeed, in view of the cancellation law of X (2.1) for each $\lambda \geqslant \mu \geqslant 0$ from $x + \mu e_1 = \lambda e_1$ we have $x = (\lambda - \mu)e_1$. The case of $\lambda < \mu$ is impossible in view of (4.1) X $(x + (\mu - \lambda)e_1 = 0$ means $x = e_1 = 0)$.

For each $x = \lambda e_1 \in Y$ we can define $\ell \in X^*$, such that $\ell(x) := \lambda\|e_1\|$. Clearly, for $e_2 \in Y\backslash\{e_1\}$ we have $\ell(e_2) \neq \|e_1\| = \ell(e_1) > 0$.

If $e_2 \notin Y$ then analogously to the proof of Theorem 2.1 we consider the set $Y_1 = Lin(e_2, Y)$. By virtue of the cancellation law of X for each $y_1 = \mu_1 e_1$ and $y_2 = \mu_2 e_1 \in Y$ from $x + \lambda_2 e_2 + \mu_2 e_1 = \lambda_1 e_2 + \mu_1 e_1$ we have

$$x = (\lambda_1 - \lambda_2)e_2 + (\mu_1 - \mu_2)e_1, \ \text{ for } \lambda_1 \geqslant \lambda_2 \text{ and } \mu_1 \geqslant \mu_2,$$

$$x + (\lambda_2 - \lambda_1)e_2 = (\mu_1 - \mu_2)e_1, \text{ for } \lambda_1 < \lambda_2 \text{ and } \mu_1 \geqslant \mu_2,$$
$$x + (\mu_2 - \mu_1)e_1 = (\lambda_1 - \lambda_2)e_2, \text{ for } \lambda_1 \geqslant \lambda_2 \text{ and } \mu_1 < \mu_2.$$

The equality $x + (\mu_2 - \mu_1)e_1 + (\lambda_2 - \lambda_1)e_2 = 0$ means $x = (\mu_2 - \mu_1)e_1 = (\lambda_2 - \lambda_1)e_2 = 0$ in view of (4.1). Hence, we can consider only three types of elements $x_1, x_2, x_3 \in Y_1$:

a) $x_1 = \lambda_1 e_2 + \alpha_1\lambda_1 e_1$ for some $\alpha_1, \lambda_1 \geqslant 0$;

b) $x_2 + \lambda_2 e_2 = \alpha_2\lambda_2 e_1$ in case of existing $x_2 \in X$ for some $\alpha_2, \lambda_2 \geqslant 0$;

c) $x_3 + \alpha_3\lambda_3 e_1 = \lambda_3 e_2$ in case of existing $x_3 \in X$ for some $\alpha_3, \lambda_3 \geqslant 0$.

For $x_1, x_2, x_3 \in X$ conditions (2.2) on $\ell(e_2)$ can be written in the following way:

$$\lambda_1\ell(e_2) + \alpha_1\lambda_1\ell(e_1) \leqslant p(x_1), \;\; -\lambda_2\ell(e_2) + \alpha_2\lambda_2\ell(e_1) \leqslant p(x_2), \;\; \lambda_3\ell(e_2) - \alpha_3\lambda_3\ell(e_1) \leqslant p(x_3),$$

or equivalently, for all possible $\alpha_i > 0$ and $\lambda_i > 0$ $(i = 1, 2, 3)$

$$\ell(e_2) \leqslant p\left(\frac{x_1}{\lambda_1}\right) - \alpha_1\ell(e_1), \;\; \ell(e_2) \geqslant \alpha_2\ell(e_1) - p\left(\frac{x_2}{\lambda_2}\right), \;\; \ell(e_2) \leqslant p\left(\frac{x_3}{\lambda_3}\right) + \alpha_3\ell(e_1).$$

According to the proof of Theorem 2.1 we can choose $\ell(e_2) \in [\alpha_{e_2}; \beta_{e_2}]$ (we use agreements (2.4) and (2.5)), where for all possible $\alpha_i > 0$ and $\lambda_i > 0$ $(i = 1, 2, 3)$

$$\alpha_{e_2} = \sup_{\alpha_2, \lambda_2 > 0} \left\{\alpha_2\ell(e_1) - p\left(\frac{x_2}{\lambda_2}\right)\right\}$$

and

$$\beta_{e_2} = \inf_{\alpha_1, \alpha_3, \lambda_1, \lambda_3 > 0} \left\{p\left(\frac{x_1}{\lambda_1}\right) - \alpha_1\ell(e_1); \; p\left(\frac{x_3}{\lambda_3}\right) + \alpha_3\ell(e_1)\right\}.$$

Without loss of generality we assume, that there exist x_2 and x_3 for some $\alpha_i > 0$ and $\lambda_i > 0$ $(i = 2, 3)$. Since

$$\alpha_2 e_1 = \frac{x_2}{\lambda_2} + e_2 \;\text{ and }\; \alpha_3 e_1 + \frac{x_3}{\lambda_3} = e_2,$$

then $\alpha_3 e_1 \preceq e_2 \preceq \alpha_2 e_1$ for some $\alpha_2 \geqslant \alpha_3 \geqslant 0$.

If $\inf\{\alpha_2 - \alpha_3\} = 0$ then from $\|e_1\| = \|e_2\|$ and $\alpha_3\|e_1\| \leqslant \|e_2\| \leqslant \alpha_2\|e_1\|$ we have $\sup\alpha_3 = \inf\alpha_2 = 1$. By (4.3) $e_1 = e_2$, that is impossible.

We can put $\inf\{\alpha_2 - \alpha_3\} = \delta > 0$. By virtue of the cancellation law of X we have

$$\frac{x_2}{\lambda_2} + \frac{x_3}{\lambda_3} + \alpha_3 e_1 = \alpha_2 e_1 \;\text{ and }\; \frac{x_2}{\lambda_2} + \frac{x_3}{\lambda_3} = (\alpha_2 - \alpha_3)e_1 \in Y.$$

Hence,

$$\left\|\frac{x_3}{\lambda_3}\right\| + \alpha_3\ell(e_1) - \left(\alpha_2\ell(e_1) - \left\|\frac{x_2}{\lambda_2}\right\|\right) = \left\|\frac{x_2}{\lambda_2}\right\| + \left\|\frac{x_3}{\lambda_3}\right\| + \alpha_3\ell(e_1) - \ell\left((\alpha_2 - \alpha_3)e_1 + \alpha_3 e_1\right) =$$

$$= \left\|\frac{x_2}{\lambda_2}\right\| + \left\|\frac{x_3}{\lambda_3}\right\| - \ell\left((\alpha_2 - \alpha_3)e_1\right) \geqslant \left\|\frac{x_2}{\lambda_2} + \frac{x_3}{\lambda_3}\right\| - \frac{1}{2}\|(\alpha_2 - \alpha_3)e_1\| =$$

$$= \|(\alpha_2 - \alpha_3)e_1\| - \frac{1}{2}\|(\alpha_2 - \alpha_3)e_1\| = \frac{1}{2}\|(\alpha_2 - \alpha_3)e_1\| \geqslant \frac{1}{2}\delta\|e_1\| > 0 \text{ and}$$

$$\left\|\frac{x_1}{\lambda_1}\right\| - \alpha_1\ell(e_1) - \left(\alpha_2\ell(e_1) - \left\|\frac{x_2}{\lambda_2}\right\|\right) = \left\|\frac{x_1}{\lambda_1}\right\| + \left\|\frac{x_2}{\lambda_2}\right\| - \frac{1}{2}\ell\left(\alpha_1 e_1 + \alpha_2 e_1\right) =$$

$$= \|\alpha_1 e_1 + e_2\| + \left\|\frac{x_2}{\lambda_2}\right\| - \frac{1}{2}\left\|\alpha_1 e_1 + \alpha_2 e_1\right\| \geqslant \frac{1}{2}\left\|\alpha_1 e_1 + e_2\right\| + \frac{1}{2}\left\|\frac{x_2}{\lambda_2}\right\| + \frac{1}{2}\left\|\alpha_1 e_1 + e_2 + \frac{x_2}{\lambda_2}\right\| -$$

$$-\frac{1}{2}\left\|\alpha_1 e_1 + \alpha_2 e_1\right\| = \frac{1}{2}\left\|\alpha_1 e_1 + e_2\right\| + \frac{1}{2}\left\|\frac{x_2}{\lambda_2}\right\| \geqslant \frac{1}{2}\|e_2\| > 0$$

in view of $\|x\| \leqslant \|y\|$ for $x = e_2 \preceq y = \alpha_1 e_1 + e_2$ (see (4.4)).

Consequently, $\beta_{e_2} \geqslant \alpha_{e_2} + \min\left\{\frac{1}{2}\delta\|e_1\|;\ \frac{1}{2}\|e_2\|\right\}$, i.e. $\beta_{e_2} > \alpha_{e_2}$ and we can choose $\ell \in X^*$ such that $\ell(e_2) \neq \ell(e_1) = \frac{1}{2}\|e_1\| > 0$. □

Now we consider analogues of Corollary 3.1 and Theorem 4.1 for convex *bounded* functionals on X.

Definition 8. We say that a convex functional p is *bounded* on X, if for some $C > 0$ the following inequality holds: $p(x) \leqslant C\|x\|$ for all $x \in X$.

Denote by X^*_{sub} the collection of all bounded convex functionals on X. Obviously, for each $\ell \in X^*$ the functional $p(x) := \max\{0, \ell(x)\} \in X^*_{sub}$. We can introduce the norm on the convex cone X^*_{sub} in the following way:

$$\|p\|_* := \sup_{x \neq 0}\left\{\frac{p(x)}{\|x\|}\right\}. \tag{4.5}$$

By Corollary 3.1 and Theorem 4.1 the next fact for each $(SCNC)$ immediately follows.

Theorem 4.2. *Let X be a $(SCNC)$. Then for all $x_0, x_1, x_2 \in X$:*

(i) if $x_0 \neq 0$ then there exists $p \in X^*_{sub} \setminus \{0\}$ such that $\|p\|_* = 1$ and $p(x_0) = \|x_0\|$;

(ii) if $x_1 \neq x_2$ then there exists a functional $p \in X^*_{sub} \setminus \{0\}$ such that $p(x_1) \neq p(x_2)$.

Proof. Note, that the condition of $\ell(e_1) > 0$ or $\ell(e_2) > 0$ in Theorem 4.1 is essential for this result. □

Clearly, the set $\widehat{X}^*_{sub} = \{\alpha p_1 + \beta p_2 \mid p_1, p_2 \in X^*_{sub},\ \alpha, \beta \in \mathbb{R}\}$ is a linear space if we introduce the addition of functionals and scalar multiplication in the usual way ($X^*_{sub} \subset \widehat{X}^*_{sub}$). The norm on $\widehat{X}^*_{sub}$ is introduced analogously to (4.5).

Definition 9. We say that $\widehat{X}^*_{sub}$ is a *subconjugate space* of X.

By Theorem 4.2 (ii) for each $X \in (SCNC)$ functionals from $X^*_{sub} \subset \widehat{X}^*_{sub}$ separate points in X. It means that we can introduce *the second conjugate Banach space* for each $X \in (SCNC)$ in a natural way:

$$X^{**} := \left(\widehat{X}^*_{sub}\right)^* = \left\{\psi : \widehat{X}^*_{sub} \to \mathbb{R} \mid \psi \text{ is linear, } |\psi(p)| \leqslant C\|p\|_* \text{ for all } p \in \widehat{X}^*_{sub}\right\},$$

where E^* is the conjugate space of a linear normed space E with the standard norm.

For all $x \in X$ there exists a functional $\psi_x : \widehat{X}^*_{sub} \to \mathbb{R}$ such that $\psi_x(p) = p(x)$ for each $\widehat{p} \in \widehat{X}^*_{sub}$. We consider the following natural partial order in X^{**}:

$$\psi_1 \preceq \psi_2, \text{ if } \psi_1(p) \leqslant \psi_2(p) \text{ for all } p \in X^*_{sub}.$$

Clearly, for all $x_1, x_2 \in X$, $\lambda_1, \lambda_2 \geqslant 0$ and $p \in X^*_{sub}$

$$\psi_{\lambda_1 x_1 + \lambda_2 x_2}(p) = p(\lambda_1 x_1 + \lambda_2 x_2) \leqslant \lambda_1 p(x_1) + \lambda_2 p(x_2) = [\lambda_1 \psi_{x_1} + \lambda_2 \psi_{x_2}](p), \text{ i.e.}$$

$$\psi_{\lambda_1 x_1 + \lambda_2 x_2} \preceq \lambda_1 \psi_{x_1} + \lambda_2 \psi_{x_2}.$$

Hence, for each $X \in (SCNC)$ there is a sublinear (subadditive and homogeneous) embedding $\varphi(x) = \psi_x(\cdot)$ in the space X^{**}. By Theorem 4.2 φ is an injective and isometrical embedding, i.e. the following theorem holds.

Theorem 4.3. *Let X be a (SCNC). Then X is sublinearly, injectively and isometrically embedded in the second conjugate Banach space X^{**}.*

Note, that injective isometrical embedding in Theorem 4.3 may be non-linear.

Example 8. Let X be the collection of all pairs of non-negative numbers (a, b), for which $(a, b) = 0 \iff a = 0$. We can define the norm $\|(a, b)\|_X := a$. X is linearly injectively embedded in the linear space E of all pairs of real numbers (a, b). Moreover, each corresponding linear injective embedding

$$\varphi((a, b)) = (\alpha_1 a + \beta_1 b, \alpha_2 a + \beta_2 b)$$

is not isometric for all fixed numbers $\alpha_{1,2}$ and $\beta_{1,2}$ such that $(\alpha_1, \alpha_2) \neq (0, 0)$ and $(\beta_1, \beta_2) \neq (0, 0)$. Indeed, for each possible norm $q(\cdot)$ on E

$$q(\varphi(a, b)) = q((\alpha_1, \alpha_2)a + (\beta_1, \beta_2)b) \geqslant q((\beta_1, \beta_2))b - q((\alpha_1, \alpha_2))a$$

and $\lim\limits_{b \to +\infty} q(\varphi(a, b)) = +\infty$ for each fixed number a. Consequently, there exists a pair $(a, b) \in X$ such that $q((a, b)) \neq \|(a, b)\|_X$, i.e. φ is not isometric.

The previous example leads us to the following conclusion: generally, abstract convex cones can be described only on the base of the non-linear analysis.

Acknowledgments

The author would like to thank the unknown reviewer for useful comments and suggestions. This work was supported by grant of the President of Russian Federation for young candidates of sciences, project no. MK-2915.2015.1.

References

[1] E.G. Bakhtigareeva, M.L. Goldman, *Associate norms and optimal embeddings for a class of two-weighted integral quasi-norms*. Fund. Appl. Math. 19 (2014), no. 5, 3 – 33 (in Russian).

[2] E.G. Bakhtigareeva, *Optimal Banach function space for a given cone of decreasing functions in a weighted L_p-space*. Eurasian Math. J. 6 (2015), no. 1, 6 – 25.

[3] M.L. Goldman, P.P. Zapreiko, *Optimal Banach function spaces generalized with the cone of nonnegative increasing functions*. Trans. Inst. Math. Nat. Akad. Sci. Belarus. 22 (2014), no. 1, 24 – 34 (in Russian).

[4] K. Keimel, *Topological Cones: Functional Analysis in a T_0-Setting*. Semigroup Forum. 77 (2008), 109 – 142.

[5] I.V. Orlov, *Inverse and implicat function theorems in the class of subsmooth maps*. Math. Notes. 99 (2016), no. 4, 619 – 622.

[6] I.V. Orlov, *Introduction to sublinear analysis*. Contemp. Math. Fundam. Direct. 53 (2014), 64 – 132 (in Russian).

[7] I.V. Orlov., S.I. Smirnova, *Invertibility of multivalued sublinear operators*. Eurasian Math. J. 6 (2015), no. 4, 44 – 58.

[8] J.H. Rädström, *An embedding theorem for space of convex sets*. Proc. Amer. Math. Soc. 3 (1952), 165 – 169.

[9] W. Roth, *Hahn-Banach type theorems for locally convex cones*. Journal of the Australian Math. Soc. (Ser. A). 68 (2000), no. 1, 104 – 125.

[10] P. Selinger, *Towards a semantics for higher-order quantum computation*. Proceedings of the 2nd International Workshop on Quantum Programming Languages. Turku Centre for Computer Science General Publication. 33 (2004), 127 – 143.

[11] F.S. Stonyakin, *Analogs of the Shauder theorem that use anticompacta*. Math. Notes. 99 (2016), no. 6, 954 – 958.

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Received: 27.04.2016