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EMJ: from Scopus Q4 to Scopus Q3 in two years?!

Recently the list was published of all mathematical journals included in 2015 Scopus quartiles Q1 (334 journals), Q2 (318 journals), Q3 (315 journals), and Q4 (285 journals). Altogether 1252 journals.

With great pleasure we inform our readers that the Eurasian Mathematical Journal was included in this list, currently the only mathematical journal in the Republic of Kazakhstan and Central Asia.

It was included in Q4 with the SCImago Journal & Country Rank (SJR) indicator equal to 0,101, and is somewhere at the bottom of the Q4 list. With this indicator the journal shares places from 1240 to 1248 in the list of all 2015 Scopus mathematical journals. Nevertheless, this may be considered to be a good achievement, because Scopus uses information about journals for the three previous years, i. e. for years 2013-2015, and the EMJ is in Scopus only from the first quarter of year 2015.

The SJR indicator is calculated by using a sophisticated formula, taking into account various characteristics of journals and journals publications, in particular the average number of weighted citations received in the selected year by the documents published in the selected journal in the three previous years. This formula and related comments can be viewed on the web-page

$$http://www.scimagojr.com/journalrank.php?category = 2601\&area = 2600\&page = 1\&total_size = 373$$

(Help/Journals/Understand tables and charts/Detailed description of SJR.)

In order to enter Q3 the SJR indicator should be greater than 0,250. It looks like the ambitious aim of entering Q3 in year 2017 is nevertheless realistic due to recognized high level of the EMJ.

We hope that all respected members of the international Editorial Board, reviewers, authors of our journal, representing more than 35 countries, and future authors will provide high quality publications in the EMJ which will allow to achieve this aim.

On behalf of the Editorial Board of the EMJ
V.I. Burenkov, E.D. Nursultanov, T.Sh. Kalmenov,
R. Oinarov, M. Otelbaev, T.V. Tararykova, A.M. Temirkhanova

VICTOR IVANOVICH BURENKOV

(to the 75th birthday)

On July 15, 2016 was the 75th birthday of Victor Ivanovich Burenkov, editor-in-chief of the Eurasian Mathematical Journal (together with V.A. Sadovnichy and M. Otelbaev), director of the S.M. Nikol'skii Institute of Mathematics, head of the Department of Mathematical Analysis and Theory of Functions, chairman of Dissertation Council at the RUDN University (Moscow), research fellow (part-time) at the Steklov Institute of Mathematics (Moscow), scientific supervisor of the Laboratory of Mathematical Analysis at the Russian-Armenian (Slavonic) University (Yerevan, Armenia), doctor of physical and mathematical sciences (1983), professor (1986), honorary professor of the L.N. Gumilyov Eurasian National University (Astana, Kazakhstan, 2006), honorary doctor of the Russian-Armenian (Slavonic) University (Yerevan, Armenia, 2007), honorary member of staff of the University of Padua (Italy, 2011), honorary distinguished professor of the Cardiff School of Mathematics (UK, 2014), honorary professor of the Aktobe Regional State University (Kazakhstan, 2015).

V.I. Burenkov graduated from the Moscow Institute of Physics and Technology (1963) and completed his postgraduate studies there in 1966 under supervision of the famous Russian mathematician academician S.M. Nikol'skii.

He worked at several universities, in particular for more than 10 years at the Moscow Institute of Electronics, Radio-engineering, and Automation, the RUDN University, and the Cardiff University. He also worked at the Moscow Institute of Physics and Technology, the University of Padua, and the L.N. Gumilyov Eurasian National University.

He obtained seminal scientific results in several areas of functional analysis and the theory of partial differential and integral equations. Some of his results and methods are named after him: Burenkov's theorem of composition of absolutely continuous functions, Burenkov's theorem on conditional hypoellipticity, Burenkov's method of mollifiers with variable step, Burenkov's method of extending functions, the Burenkov-Lamberti method of transition operators in the problem of spectral stability of differential operators, the Burenkov-Guliyevs conditions for boundedness of operators in Morrey-type spaces. On the whole, the results obtained by V.I. Burenkov have laid the groundwork for new perspective scientific directions in the theory of functions spaces and its applications to partial differential equations, the spectral theory in particular.

More than 30 postgraduate students from more than 10 countries gained candidate of sciences or PhD degrees under his supervision. He has published more than 170 scientific papers. The lists of his publications can be viewed on the portals MathSciNet and MathNet.Ru. His monograph "Sobolev spaces on domains" became a popular text for both experts in the theory of function spaces and a wide range of mathematicians interested in applying the theory of Sobolev spaces.

In 2011 the conference "Operators in Morrey-type Spaces and Applications", dedicated to his 70th birthday was held at the Ahi Evran University (Kirsehir, Turkey). Proceedings of that conference were published in the EMJ 3-3 and EMJ 4-1.

The Editorial Board of the Eurasian Mathematical Journal congratulates Victor Ivanovich Burenkov on the occasion of his 75th birthday and wishes him good health and new achievements in science and teaching!

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USE OF BUNDLES OF LOCALLY CONVEX SPACES IN PROBLEMS OF CONVERGENCE OF SEMIGROUPS OF OPERATORS. I

B. Silvestri

Communicated by V.I. Burenkov

Key words: bundles of locally convex spaces, one-parameter semigroups, spectrum and resolvent.

AMS Mathematics Subject Classification: 55R25, 46E40, 46E10; 47A10, 47D06.

Abstract. In this work we construct certain general bundles $\langle \mathfrak{M}, \rho, X\rangle$ and $\langle \mathfrak{B}, \eta, X\rangle$ of Hausdorff locally convex spaces associated with a given Banach bundle $\langle \mathfrak{E}, \pi, X\rangle$. Then we present conditions ensuring the existence of bounded sections $\mathcal{U} \in \Gamma^{x_\infty}(\rho)$ and $\mathcal{P} \in \Gamma^{x_\infty}(\eta)$ both continuous at a point $x_\infty \in X$, such that $\mathcal{U}(x)$ is a C_0-semigroup of contractions on $\mathfrak{E}_x$ and $\mathcal{P}(x)$ is a spectral projector of the infinitesimal generator of the semigroup $\mathcal{U}(x)$, for every $x \in X$.

1 Introduction

This work consists of three parts of which the present represents the first one. We construct certain general bundles $\langle \mathfrak{M}, \rho, X\rangle$ and $\langle \mathfrak{B}, \eta, X\rangle$ of Hausdorff locally convex spaces associated with a given Banach bundle $\langle \mathfrak{E}, \pi, X\rangle$. Then we present conditions ensuring the existence of bounded sections $\mathcal{U} \in \Gamma^{x_\infty}(\rho)$ and $\mathcal{P} \in \Gamma^{x_\infty}(\eta)$ both continuous at a point $x_\infty \in X$, such that $\mathcal{U}(x)$ is a C_0-semigroup of contractions on $\mathfrak{E}_x$ and $\mathcal{P}(x)$ is a spectral projector of the infinitesimal generator of the semigroup $\mathcal{U}(x)$, for every $x \in X$.

Here $\mathfrak{W} \doteqdot \langle \mathfrak{M}, \rho, X\rangle$ and $\langle \mathfrak{B}, \eta, X\rangle$ are special kind of bundles of Hausdorff locally convex spaces (bundle of $\Omega-$spaces [10]) while $\mathfrak{V} \doteqdot \langle \mathfrak{E}, \pi, X\rangle$ is a suitable Banach bundle such that the common base space X is a completely regular topological space and the filter of neighbourhoods of x_∞ admits a countable basis[1]. Moreover for all $x \in X$ the stalk $\mathfrak{M}_x \doteqdot \overset{-1}{\rho}(x)$ is a topological subspace of the space $\mathcal{C}_c(\mathbb{R}^+, \mathcal{L}_{S_x}(\mathfrak{E}_x))$ with the topology of compact convergence, of all continuous maps defined on $\mathbb{R}^+$ and with values in $\mathcal{L}_{S_x}(\mathfrak{E}_x)$, and the stalk $\mathfrak{B}_x \doteqdot \overset{-1}{\eta}(x)$ is a topological subspace of $\mathcal{L}_{S_x}(\mathfrak{E}_x)$. Here $\mathfrak{E}_x \doteqdot \overset{-1}{\pi}(x)$, while $\mathcal{L}_{S_x}(\mathfrak{E}_x)$, is the space, of all linear bounded maps on $\mathfrak{E}_x$ with the topology of uniform convergence over the subsets of $S_x \subset Bounded(\mathfrak{E}_x)$ which depends, for all $x \in X$, on the same subspace $\mathcal{E} \subseteq \Gamma(\pi)$. Here $\rho : \mathfrak{M} \to X$, $\eta : \mathfrak{B} \to X$, and $\pi : \mathfrak{E} \to X$ are the projection maps of the respecive bundles, $\Gamma^{x_\infty}(\rho)$ is the set of all bounded sections of $\mathfrak{W}$ continuous at x_∞ with respect to the topology on the bundle space $\mathfrak{M}$ and $\Gamma(\pi)$ is the set of all bounded continuous sections of $\mathfrak{V}$.

[1] in particular X a metric space and x_∞ any point of X.

An essential factor is that the continuity at x_∞ of $\mathcal{U}$ and $\mathcal{P}$ derives by a sort of continuity at the same point of the section $\mathcal{T}$ of the graphs of the infinitesimal generators of the semigroups in the range of $\mathcal{U}$, where the sort of continuity has to be understood in the following sense. For every $x \in X$ let $\mathcal{T}(x)$ be the graph of the infinitesimal generator T_x of the semigroup $\mathcal{U}(x)$, then

$$\begin{cases} \mathcal{T}(x_\infty) = \{\phi(x_\infty) \mid \phi \in \Phi\} \\ \Phi \subseteq \Gamma^{x_\infty}(\pi_{\mathsf{E}\oplus}) \\ (\forall x \in X)(\forall \phi \in \Phi)(\phi(x) \in \mathcal{T}(x)), \end{cases} \tag{1.1}$$

where $\Gamma^{x_\infty}(\pi_{\mathsf{E}\oplus})$ is the set of all bounded sections of the direct sum of bundles $\mathfrak{V} \oplus \mathfrak{V}$ which are continuous at x_∞.

Hence for any $v \in Dom(T_{x_\infty})$ there exists a bounded section ϕ of $\mathfrak{V} \oplus \mathfrak{V}$ such that

$$\begin{cases} (v, T_{x_\infty} v) = \lim_{x \to x_\infty} (\phi_1(x), \phi_2(x)) \\ (\phi_1(x), \phi_2(x)) \in Graph(T_x), \forall x \in X - \{x_\infty\}, \end{cases} \tag{1.2}$$

where the limit is with respect to the topology on the bundle space of $\mathfrak{V} \oplus \mathfrak{V}$[2].

The main strategy for obtaining the continuity at x_∞ of $\mathcal{U}$ and $\mathcal{P}$, it is to correlate the topologies on $\mathfrak{M}$ and $\mathfrak{B}$, with the topology on $\mathfrak{E}$. Thus it is clear that the construction of the right structures has a prominent role.

It is well-known the relative freedom of choice of the topology on the bundle space of any bundle of $\Omega-$spaces. More exactly fixed a suitable linear space say G of bounded sections there exists always a topology on the bundle space such that all the maps in G are continuous. Moreover if X is compact one can find a topology such that G is the whole space of bounded continuous sections [10, Theorem 5.9]. This freedom of choice allows the construction of examples of the above-mentioned correlations of topologies.

From the following simple result Corollary 3.1 and without entering in the definition of the topology of a bundle of $\Omega-$space, we can recognize the power of determining the right set $\Gamma(\zeta)$ of continuous sections of a general bundle $\langle \mathfrak{Q}, \zeta, X\rangle$ of $\Omega-$space. Let $f \in \prod_{x\in X}^b \mathfrak{Q}_x$ be any bounded section and $x_\infty \in X$ such that there exists a section $\sigma \in \Gamma(\zeta)$ such that $\sigma(x_\infty) = f(x_\infty)$. Then

$$f \in \Gamma^{x_\infty}(\zeta) \Leftrightarrow (\forall j \in J)(\lim_{z \to x_\infty} \nu_j^z (f(z) - \sigma(z)) = 0), \tag{1.3}$$

where J is a set such that $\{\nu_j^z \mid j \in J\}$ is a directed fundamental set of seminorms of the locally convex space $\mathfrak{Q}_z \doteqdot \overset{-1}{\zeta}(z)$ for all $z \in X$. About the problem of establishing if there are continuous bounded sections intersecting f in x_∞, we can use an important result of the theory of Banach bundles, stating that any Banach bundle over a locally compact base space is full, namely for any point of the bundle space there exists a section passing on it. While for more general bundles of $\Omega-$spaces we can use the above described freedom.

[2]Later we shall see that the topology on the bundle space of $\mathfrak{V} \oplus \mathfrak{V}$ will be constructed in order to ensure that the limit in (1.2) is equivalent to say that $v = \lim_{x\to x_\infty} \phi_1(x)$ and $T_{x_\infty} v = \lim_{x\to x_\infty} \phi_2(x)$, both limits with respect to the topology on the bundle space $\mathfrak{E}$.

The criterium we used for determining the correlations between $\mathfrak{M}$ (resp.$\mathfrak{B}$) and $\mathfrak{E}$ is that of extending to a general bundle of $\Omega-$spaces two properties of the topology of the space $\mathcal{C}_c(Y, \mathcal{L}_s(Z))$.

Here Z is a normed space, S is a set of bounded subsets of Z, $\mathcal{L}_s(Z)$ is the space of all linear continuous maps on Z with the pointwise topology, finally $\mathcal{C}_c(Y, \mathcal{L}_s(Z))$ is the space of all continuous maps on a topological space Y with values in $\mathcal{L}_s(Z)$ with the topology of uniform convergence over the compact subsets of Y.

In order to simplify the notation we here shall consider Z as a Banach space and take $\mathcal{L}_S(Z) = B_s(Z)$, i.e. the space of all bounded linear operators on Z with the strong operator topology.

Let X be a compact space

$$\begin{aligned}\mathcal{M} \doteqdot &\{F \in \mathcal{C}_b(X, \mathcal{C}_c(Y, B_s(Z))) \mid (\forall K \in Comp(Y))\\ &(C(F, K) \doteqdot \sup_{(x,s)\in X\times K} \|F(x)(s)\|_{B(Z)} < \infty)\}\\ \mathsf{M}_x \doteqdot &\overline{\{F(x) \mid F \in \mathcal{M}\}}\end{aligned}$$

Let $\mathfrak{V} \doteqdot \langle \mathfrak{E}, \pi, X\rangle$ denote the trivial bundle with constant stalk Z so $\Gamma(\pi) \simeq \mathcal{C}_b(X, Z)$, set

$$\begin{cases}\mathcal{A}_x \doteqdot \{\mu^K_{(v,x)} \mid K \in Comp(Y), v \in \Gamma(\pi)\},\\ \mu^K_{(v,x)} : \mathsf{M}_x \ni G \mapsto \sup_{s\in K} \|G(s)v(x)\|,\\ \mathsf{M} \doteqdot \{\langle \mathsf{M}_x, \mathcal{A}_x\rangle\}_{x\in X}.\end{cases} \tag{1.4}$$

Then by using Lemma 5.2 and [10, Theorem 5.9] we can construct a bundle of $\Omega-$spaces say $\mathfrak{V}(\mathsf{M}, \mathcal{M})$ whose stalk at x is the locally convex space $\langle \mathsf{M}_x, \mathcal{A}_x\rangle$ and whose space of bounded continuous sections $\Gamma(\pi_{\mathsf{M}})$ is such that $\Gamma(\pi_{\mathsf{M}}) \simeq \mathcal{M}$.

Let $f \in \prod_{x\in X} \mathsf{M}_x$ be such that $(\forall K \in Comp(Y))(\sup_{(x,s)\in X\times K} \|f(x)(s)\|_{B(Z)} < \infty)$ then according to Theorem 5.1 we obtain that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) with

1. $(\forall K \in Comp(Y))(\forall v \in \Gamma(\pi))$
$$(\lim_{x\to x_\infty} \sup_{s\in K} \|f(x)(s)v(x) - f(x_\infty)(s)v(x)\| = 0);$$
2. $f \in \Gamma^{x_\infty}(\pi_{\mathsf{M}})$;
3. $f : X \to \mathcal{C}_c(Y, B_s(Z))$ continuous at x_∞.

Moreover if Y is locally compact for all $t \in Y$

$$\Gamma(\pi_{\mathsf{M}})_t \bullet \Gamma(\pi) \subseteq \Gamma(\pi). \tag{1.5}$$

Therefore we constructed two bundles $\mathfrak{V}$ and $\mathfrak{V}(\mathsf{M}, \mathcal{M})$ whose topologies are (I) stalkwise related by $\{\mathcal{A}_x\}_{x\in X}$ in (1.4) *and for which hold* (1) $\Leftrightarrow$ (2) *and* (II) *globally related by* (1.5). *Finally $\Gamma^{x_\infty}(\pi_{\mathsf{M}})$ coincides with the subset of all maps $f : X \to \mathcal{C}_c(Y, B_s(Z))$ continuous at x_∞ such that* $(\forall K \in Comp(Y))(\sup_{(x,s)\in X\times K} \|f(x)(s)\|_{B(Z)} < \infty)$. The extension at general bundles of the property (I) leads to the concept of $(\Theta, \mathcal{E})$ **−structure**, provided in Definition 6 see Lemma 5.1, while the generalization of the property (II) leads to the concept of compatible $(\Theta, \mathcal{E})$ −structure, given in Definition 6.

A similar and more important global correlation between $\mathfrak{M}$ and $\mathfrak{E}$, this time for the case in which the topology on each stalk $\mathfrak{M}_x$ is that of the pointwise convergence instead of the compact convergence, is that encoded in [19, eq (4.12)] in the definition of invariant $(\Theta, \mathcal{E}, \mu)$ −structures provided in [19, Definition 10]. This closes the discussion about the relationship between the topologies on $\mathfrak{M}$ and $\mathfrak{E}$, in particular between those on $\mathfrak{B}$ and $\mathfrak{E}$ [3]

Briefly we recall what here has to be understood as a classical stability problem in order to understand how to generalize it through the language of bundles. The classical stability problem could be so described. Fixed a Banach space Z find a sequence $\{S_n : D_n \subseteq Z \to Z\}$ of possibly unbounded linear operators in Z and a sequence $\{P_n\} \subset B(Z)$ where P_n is a spectral projector of S_n for $n \in \mathbb{N}$, such that

(A) whenever there exists an operator $S : D \subset Z \to Z$ such that $S = \lim_{n\to\infty} S_n$ with respect to a suitable topology or in any other generalized sense,

(B) then there exists a spectral projector $P \in B(Z)$ of S such that $P = \lim_{n\to\infty} P_n$ with respect to the strong operator topology.

Here a spectral projector of an operator S in a Banach space is a continuous projector associated with a closed S−invariant subspace Z_0 such that $\sigma(S \upharpoonright Z_0) \subset \sigma(S)$, where $\sigma(T)$ is the spectrum of the operator T.

In [12, Ch IV] one finds many stability theorems in which the limit in (A) has to be understood with respect to the metric induced by the so called gap between the corresponding closed graphs.

Additional stability theorems, even for operators defined in different spaces, are available. They have been obtained by using the concept of *Transition Operators* introduced by Victor I. Burenkov, see for expample [4], [5] and [6]. Instead to their stability theorems Massimo Lanza de Cristoforis and Pier Domenico Lamberti employed functional analytic approaches, see for examples [15], [16], [14].

If we try to generalize the classical stability problem to the case in which Z is replaced by any sequence $\{Z_n\}$ of Banach spaces and S_n is defined in Z_n for all n, then we would face the following difficulty. How can we adapt the definition of the gap given by Kato to the case of a sequence of different spaces? More in general in which sense has to be understood the convergence of operators defined in different spaces.

A first step toward the generalization to the case of different spaces of the classical stability problem is the following result of Thomas G. Kurtz [13].

Theorem 1.1 (2.1. **of [13]).** *For each n, let $U_n(t)$ be a strongly continuous contraction semigroup defined on L_n with the infinitesimal operator A_n. Let $A = ex - \lim_{n\to\infty} A_n$. Then there exists a strongly continuous semigroup $U(t)$ on L such that $\lim_{n\to\infty} U_n(t)Q_n f = U(t)f$ for all $f \in L$ and $t \in \mathbb{R}^+$ if and only if the domain $D(A)$ is dense and the range $R(\lambda_0 - A)$ of $\lambda_0 - A$ is dense in L for some $\lambda_0 > 0$. If the above conditions hold A is the infinitesimal generator of U and we have*

$$\lim_{n\to\infty} \sup_{0\leq s\leq t} \|U_n(s)Q_n f - Q_n U(s) f\|_n = 0, \tag{1.6}$$

[3]Indeed it is sufficient to take $Y = \{pt\}$ i.e. one point space.

for every $f \in L$ *and* $t \in \mathbb{R}^+$.

Here $\langle L, \|\cdot\|\rangle$ is a Banach space, $\{\langle L_n, \|\cdot\|_n\rangle\}_{n\in\mathbb{N}}$ is a sequence of Banach spaces, $\{Q_n \in B(L, L_n)\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \|Q_n f\|_n = \|f\|$ for all $f \in L$. Let $f \in L$ and $\{f_n\}_{n\in\mathbb{N}}$ such that $f_n \in L_n$ for every $n \in \mathbb{N}$, thus he set [4]

$$f = \lim_{n\to\infty} f_n \Leftrightarrow \lim_{n\to\infty} \|f_n - Q_n f\|_n = 0. \tag{1.7}$$

Moreover if $A_n : Dom(A_n) \subseteq L_n \to L_n$ he defined

$$\begin{cases} Graph(ex - lim_{n\to\infty} A_n) \doteqdot \{\lim_{n\in\mathbb{N}} s_0(n) \mid s_0 \in \Phi_0\} \\ \Phi_0 \doteqdot \{(f_n, A_n f_n)_{n\in\mathbb{N}} \in (Z \times Z)^{\mathbb{N}} \mid \\ (\forall n \in \mathbb{N})(f_n \in Dom(A_n)) \wedge (\exists \lim_{n\in\mathbb{N}} (f_n, A_n f_n))\}, \end{cases} \tag{Gr}$$

where $(f, g) = \lim_{n\in\mathbb{N}} (f_n, A_n f_n))$ if and only if $f = \lim_{n\in\mathbb{N}} f_n$ and $g = \lim_{n\in\mathbb{N}} A_n f_n$ and all these limits are those defined in (1.7). Whenever $Graph(ex - lim_{n\to\infty} A_n)$ is a graph in L Kurtz denoted by $ex - lim_{n\to\infty} A_n$ the corresponding operator in L.

The Kurtz's approach did not make use of the bundle theory, and, except when imposing stronger assumptions, it cannot be implemented in terms of bundles of $\Omega-$spaces.

What follows results fundamental for understanding the strategy behind this work. (1.3) essentially generalizes (1.7). More importantly *if the topology on* $\mathfrak{M}$ *and that on* $\mathfrak{E}$ *are related by a* $(\Theta, \mathcal{E})$ *−structure (for a very simple model see* (1.9)*) then the convergence* (1.3) *essentially generalizes the convergence* (1.6) *of the sequence of semigroups* $\{U_n\}_{n\in\mathbb{N}}$ *to the semigroup* U.[5]

[4] Notice the strong similarity of (1.7) with (1.3).

[5] Indeed if we set assume that there exists for every $n \in \mathbb{N}$ $S_n \in B(L_n, L)$ such that $S_n Q_n = Id$ then (1.6) would become

$$(\forall t \in \mathbb{R}^+)(\forall f \in L)(\lim_{n\to\infty} \sup_{0\leq s\leq t} \|(U_n(s) - Q_n U(s) S_n) Q_n f\|_n = 0). \tag{1.8}$$

Moreover let $\langle \mathfrak{M}, \rho, X\rangle$ and $\langle \mathfrak{E}, \pi, X\rangle$ be set as in the beginning and assume that $\{\nu^z_{(K,v)} \mid (K, v) \in Comp(Y), v \in \mathcal{E}\}$ is a fundamental set of seminorms on $\mathfrak{M}_z$ for every $z \in X$, where $\mathcal{E} \subseteq \Gamma(\pi)$. Finally assume that for all $K \in Comp(Y)$, $v \in \mathcal{E}$ and for all $z \in X$ and $f^z \in \mathfrak{M}_z$

$$\nu^z_{(K,v)}(f^z) \doteqdot \sup_{s\in K} \|f^z(s) v(z)\|_z. \tag{1.9}$$

Thus (1.3) would read: if there exists $\sigma \in \Gamma(\rho)$ such that $\sigma(x_\infty) = F(x_\infty)$ then

$$F \in \Gamma^{x_\infty}(\rho) \Leftrightarrow (\forall K \in Comp(Y))(\forall v \in \mathcal{E})(\lim_{z\to x_\infty} \sup_{s\in K} \|(F(z) - \sigma(z)) v(z)\|_z = 0). \tag{1.10}$$

Therefore by setting X the Alexandroff compactification of $\mathbb{N}$, $x_\infty = \infty$ and for all $n \in \mathbb{N}$

$$\begin{cases} \mathfrak{E}_n \doteqdot L_n,\ \mathfrak{E}_\infty \doteqdot L \\ \mathfrak{M}_n \doteqdot \mathcal{C}_c(\mathbb{R}^+, B_s(L_n)) \\ \mathfrak{M}_\infty \doteqdot \mathcal{C}_c(\mathbb{R}^+, B_s(L)) \\ \mathcal{E} \doteqdot \{Qf \mid f \in L\}, \end{cases} \tag{1.11}$$

if there exist conditions under which we can obtain that

$$\begin{cases} \{Qf \mid f \in L\} \subseteq \Gamma(\pi) \\ \{QVS \mid V \in \mathsf{U}(L)\} \subseteq \Gamma(\rho), \end{cases} \tag{1.12}$$

We used the word "essentially" due to the difficulty to build a couple of Kurtz' bundles, namely two bundles of $\Omega-$spaces $\langle \mathfrak{E}, \pi, X\rangle$ and $\langle \mathfrak{M}, \rho, X\rangle$ such that X is the Alexandroff compactification of $\mathbb{N}$ and (1.11), (1.12) hold. In any case it is possible under strong assumptions, see [19, Section 5]. Despite the difficulty of constructing Kurtz's bundles, since the above remark we opted to investigate to which extent the Kurtz's Theorem 1.1 can be extended in the framework of bundles of $\Omega-$spaces, by using the concept of $(\Theta, \mathcal{E})-$structure.

It is now clear that, in the way of extending the Kurtz's Theorem, we replace the sequence of Banach spaces $\{L_n\}_{n\in\mathbb{N}\cup\{\infty\}}$ where $L_\infty \doteqdot L$, with a Banach bundle $\mathfrak{E}$, while we replace the sequence $\{\mathcal{C}_c\left(\mathbb{R}^+, B_s(L_n)\right)\}_{n\in\mathbb{N}\cup\{\infty\}}$ by the bundle of $\Omega-$spaces $\mathfrak{M}$. Hence the Kurtz' convergences (1.6) and (1.7) will be replaced by the *convergences of sections on the bundles spaces* $\mathfrak{M}$ *and* $\mathfrak{E}$ respectively. In this view definition (Gr) has to be replaced by that of Pre-Graph section Definition 9 (essentially (1.1)), while the case in which $Graph(ex-lim_{n\to\infty} A_n)$ is a graph in L with that of Graph section Definition 8. Hence it arises as a natural question which topology has to be selected for the bundle space of $\mathfrak{V}\oplus\mathfrak{V}$.

An essential tool used in the definition of $Graph(ex-lim_{n\to\infty} A_n)$ in (Gr) is that of convergence of a sequence $(f_n, A_n f_n)$ in the direct sum of the spaces $L_n \oplus L_n$, given by construction as the convergence of both the sequences in L_n in the meaning of (1.7).

It is exactly this factorization the property which we want to preserve when selecting the right topology on the bundle space of $\mathfrak{V}\oplus\mathfrak{V}$.

It is a well-known result the solution of this problem in the special case of Banach bundles. We generalize this result for a finite direct sum of general bundles of $\Omega-$spaces, by constructing in Theorem 4.1 a directed family of seminorms on the direct sum of Hausdorff locally convex spaces that generates the product topology.

This result along with **Lemma 4.1** allow to define the direct sum of (full) bundles of $\Omega-$spaces as given in Definition 2

The result that the topology on each stalk is the product topology, encoded in (4.6)*, the choice provided in* (4.7) *of a set that will become a subset of bounded continuous sections of the direct sum of bundles and the general convergence criterium in* (1.3)*, allow to show the claimed factorization property in Corollary 4.1. Namely any continuous map from X at values in the direct sum $\bigoplus_{i=1}^n \mathfrak{E}_i$ of bundles is continuous at a point if and only if all its n components are continuous at the same point.*

In [18, Theorem 2.1] we resolve the claim of extending the Kurtz's result to the setting of bundles of $\Omega-$spaces. More exacly we construct an element of the set $\Delta_\Theta\langle\mathfrak{V}, \mathfrak{W}, \mathcal{E}, X, \mathbb{R}^+\rangle$ Definition 11. Roughly and limited to singletons we have that the singleton $\{\langle\mathcal{T}, x_\infty, \Phi\rangle\}$ belongs to $\Delta_\Theta\langle\mathfrak{V}, \mathfrak{W}, \mathcal{E}, X, \mathbb{R}^+\rangle$ if and only if $\mathcal{T}(x)$ is the graph of the infinitesimal generator T_x of a C_0-semigroup $\mathcal{U}(x)$ on $\mathfrak{E}_x$, for all $x\in X$, (1.1) holds true and

$$\mathcal{U} \in \Gamma^{x_\infty}(\rho). \tag{1.13}$$

Thus, according to the discussed way of extending the Kurtz' theorem, to find an element in the set $\Delta_\Theta\langle\mathfrak{V}, \mathfrak{W}, \mathcal{E}, X, \mathbb{R}^+\rangle$ means to find an extension of Theorem 1.1.

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where $(Qf)(n) \doteqdot Q_n f$, $(Qf)(\infty) \doteqdot f$, while $(QVS)(n) \doteqdot Q_n V S_n$, $(QVS)(\infty) \doteqdot V$, for all $n\in\mathbb{N}$ and $\mathsf{U}(L)$, is the set of all C_0-semigroup on L, then by (1.10) and (1.8) follows that

$$\mathcal{U} \in \Gamma^{\infty}(\rho),$$

where $\mathcal{U}(n) \doteqdot U_n$ and $\mathcal{U}(\infty) \doteqdot U$.

Finally let us outline how the main result of the entire work [19, Theorem 4.2] extends the classical stability problem at operators defined in different spaces. It provides the existence of an element $\langle \mathcal{T}, \Phi, x_\infty \rangle$ whose singleton belongs to the intersection of the set $\Delta_\Theta \langle \mathfrak{V}, \mathfrak{W}, \mathcal{E}, X, \mathbb{R}^+ \rangle$ with the set $\Delta \langle \mathfrak{V}, \mathfrak{D}, \Theta, \mathcal{E} \rangle$ which ammounts to what follows. There exists $\mathcal{U}$ satisfying (1.13) and there exists a section

$$\mathcal{P} \in \Gamma^{x_\infty}(\eta), \tag{1.14}$$

satisfying (1.15) with T_x the infinitesimal generator of the C_0-semigroup $\mathcal{U}(x)$ for all $x \in X$. Actually the result is stronger since it establishes that $\mathcal{P}(x)$ is a *spectral projector* of T_x for all $x \in X$.

Roughly speaking and limited to singletons we have what follows, see Definition 10 for the precise and general definition. Given a $(\Theta, \mathcal{E})-$structure $\langle \mathfrak{V}, \mathfrak{D}, X, \{pt\} \rangle$ and denoted $\mathfrak{D} \doteqdot \langle \mathfrak{B}, \eta, X \rangle$, we have that the singleton of $\langle \mathcal{T}, \Phi, x_\infty \rangle$ belongs to $\Delta \langle \mathfrak{V}, \mathfrak{D}, \Theta, \mathcal{E} \rangle$ if and only if for all $x \in X$ the set $\mathcal{T}(x)$ is a graph in $\mathfrak{E}_x$, (1.1) holds true and there exists $\mathcal{P} \in \Gamma^{x_\infty}(\eta)$ such that $\mathcal{P}(x)$ is a projector on $\mathfrak{E}_x$ for all $x \in X$ and

$$\mathcal{P}(x) T_x \subseteq T_x \mathcal{P}(x), \tag{1.15}$$

where T_x is the operator in $\mathfrak{E}_x$ whose graph is $\mathcal{T}(x)$.

In others words $\langle \mathcal{T}, \Phi, x_\infty \rangle \in \Delta \langle \mathfrak{V}, \mathfrak{D}, \Theta, \mathcal{E} \rangle$ if and only if $\mathcal{T}$ is a section of graphs in $\mathfrak{E}$ continuous at x_∞ in the sense of (1.2) and such that there exists a section $\mathcal{P}$ of projectors on $\mathfrak{E}$ continuous at x_∞ such that $\mathcal{P}$ *commutes with* $\mathcal{T}$ in the meaning of (1.15).

Notice that (1.15) is satisfied by any element of the resolution of the identity of a spectral operator [8, Definition 18.2.1]. Moreover whenever T_x is the infinitesimal generator of a C_0-semigroup $\mathcal{W}_T(x)$ of contractions on $\mathfrak{E}_x$, the most important case in this work, it results that (1.15) is the property satisfied by all the spectral projectors of the form

$$\mathcal{P}(x) \doteqdot -\frac{1}{2\pi i} \int_\Gamma R(-T_x; \zeta)\, d\zeta,$$

where $R(-T_x; \zeta)$ is the resolvent map of the operator $-T_x$ and Γ is a suitable closed curve on the complex plane. Hence we can consider the commutation in (1.15) as the defining property of what we here consider as the interesting bundle $\mathcal{P}$ of projectors associated with $\mathcal{T}$. Therefore *as* (1.13) *represents the extension of the Kurtz's theorem so* (1.14) *realizes our initial claim to extend in the framework of bundles of* $\Omega-$*spaces the classical stability problem. Moreover the two solutions* $\mathcal{U}$ *and* $\mathcal{P}$ *are correlated since* $\mathcal{P}(x)$ *is a spectral projector of the infinitesimal generator* T_x *of the semigroup* $\mathcal{U}(x)$ *for all* $x \in X$*, in particular* (1.15) *holds true.*

The main results of this work are the following ones

1. Construction of a suitable directed fundamental set of seminorms of the topological direct sum of a finite family of Hausdorff locally convex spaces, and construction of $\mathcal{E}^{\oplus}$ satisfying $FM(3)-FM(4)$ with respect to $\mathsf{E}^{\oplus}$ (Theorem 4.1 and Lemma 4.1);

2. Factorization property of the convergence in any direct sum of bundles of $\Omega-$spaces (Corollary 4.1);

3. Characterization of sections of $\mathfrak{W}$ continuous at a point when $\langle \mathfrak{V}, \mathfrak{W}, X, Y\rangle$ is a $(\Theta, \mathcal{E})-$structure, (Lemma 5.1);

4. Construction of a $(\Theta, \mathcal{E})-$structure $\langle \mathfrak{V}, \mathfrak{W}, X, Y\rangle$ and characterization of a subset of $\Gamma^{x_\infty}(\rho)$ when $\mathfrak{V}$ is trivial, (Theorem 5.1);

5. Construction of an element in the set $\Delta_{\Theta}\langle \mathfrak{V}, \mathfrak{W}, \mathcal{E}, X, \mathbb{R}^{+}\rangle$, ([18, Theorem 2.1, Corollary 3.1, Corollary 4.3, Theorem 4.4]);

6. Conditions yielding the bounded equicontinuity of which in hypothesis (ii) of [18, Theorem 2.1] ([18, Corollary 3.1]);

7. Conditions yielding the [18, eq. (2.14)] ([18, Proposition 4.2]);

8. [18, Lemma 4.4, Theorem 4.1, Theorem 4.2, Theorem 4.3, Corollary 4.1];

9. Laplace duality property [18, Corollary 4.2];

10. Consequence of being an $\langle \nu, \eta, E, Z, T\rangle$ invariant set V with respect to $\mathcal{F}$ ([19, Proposition 2.1]);

11. Construction of a set $\Delta_{\Theta}\langle \mathfrak{V}, \mathfrak{D}, \mathfrak{W}, \mathcal{E}, X, \mathbb{R}^{+}\rangle$ by using an $\langle \nu, \eta, \mathfrak{G}, K(\Gamma), \mathbb{R}^{+}\rangle$ invariant set V with respect to $\{\overline{F}_T\}$ ([19, Corollary 3.1]);

12. A bundle version of the Lebesgue theorem for a $\mu-$related couple $\langle \mathfrak{V}, \mathfrak{Z}\rangle$ ([19, Theorem 4.1]);

13. [19, Lemma 4.1, Lemma 4.2, Corollary 4.1]

14. Construction of a section of spectral projectors continuous at a point, given a section of semigroups continuous at the same point ([19, Corollary 4.2])

15. The Main result of the entire work namely the construction of an element in the set $\Delta\langle \mathfrak{V}, \mathfrak{D}, \Theta, \mathcal{E}\rangle$ ([19, Theorem 4.2]).

The main structures defined in this work are the following ones

1. Direct sum of full bundles of $\Omega-$spaces (Definition 2);

2. (Invariant) $(\Theta, \mathcal{E})-$structure $\langle \mathfrak{V}, \mathfrak{W}, X, Y\rangle$, (Definition 6);

3. Graph section $\langle \mathcal{T}, x_\infty, \Phi\rangle$, (Definition 8);

4. $\Delta \langle \mathfrak{V}, \mathfrak{D}, \Theta, \mathcal{E} \rangle$, (Definition 10);

5. $\Delta_{\Theta} \langle \mathfrak{V}, \mathfrak{W}, \mathcal{E}, X, \mathbb{R}^{+} \rangle$, (Definition 11);

6. $\Delta_{\Theta} \langle \mathfrak{V}, \mathfrak{D}, \mathfrak{W}, \mathcal{E}, X, \mathbb{R}^{+} \rangle$; (Definition 12);

7. $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^{+} \rangle$ with the Laplace duality property, ([18, Definition 2]);

8. $\mathsf{U}-$Spaces ([18, Definition 7]);

9. The locally convex space $\mathfrak{G}$ ([18, Definition 9]);

10. $\langle \nu, \eta, E, Z, T \rangle$ invariant set V with respect to $\mathcal{F}$ ([19, Definition 2]);

11. $\mu-$related couple $\langle \mathfrak{V}, \mathfrak{Z} \rangle$ ([19, Definition 9]);

12. (Invariant) $(\Theta, \mathcal{E}, \mu)-$ structure $\langle \mathfrak{V}, \mathfrak{Q}, X, Y \rangle$ ([19, Definition 10]);

13. $(\Theta, \mathcal{E})-$ structure $\langle \mathfrak{V}, \mathfrak{V}(\mathsf{M}, \Gamma(\xi)), X, Y \rangle$ underlying a $(\Theta, \mathcal{E}, \mu)-$ structure $\langle \mathfrak{V}, \mathfrak{Q}, X, Y \rangle$ ([19, Definition 12]).

2 Notation

For any two sets X, Y we let Y^X denote the set of maps defined on X and at values in Y. Let $Graph(X \times Y)$ denote the set of subsets of $X \times Y$ which are graphs, while for any map f let $Graph(f)$ denote its graph. If $\mathcal{B}$ is a base of a filter on X, we let $\mathfrak{F}_{\mathcal{B}}^{X}$ denote the filter on X generated by the base $\mathcal{B}$. If S is any set then $\mathcal{P}_{\omega}(S)$ denotes the set of all finite subsets of S. If τ is any topology on X and $x \in X$, then $\mathcal{I}_x^{\tau}$ denotes the filter of neighbourhoods of x of the topological space $\langle X, \tau \rangle$. Let u.s.c. mean upper semicontinuous. All vector spaces are assumed to be over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, Hlcs stands for Hausdorff locally convex spaces. We say that $\mathsf{V} \doteqdot \{\langle V_x, \mathcal{A}_x \rangle\}_{x \in X}$ is a *nice* family of Hlcs if $\{V_x\}_{x \in X}$ is a family of Hlcs and there exists a set J for which $\forall x \in X$ the set $\mathcal{A}_x \doteqdot \{\mu_j^x\}_{j \in J}$ is a directed [6] family of seminorms on V_x generating the locally convex topology on it. For any family of seminorms K on a vector space V we call the directed family of seminorms associated with K the set $\{\sup F \mid F \in \mathcal{P}_{\omega}(\Gamma)\}$ with the order relation of pointwise order on $\mathbb{R}^V$. fss stands for "fundamental set of seminorms". Given two locally convex spaces (lcs) E and F we denote by $\mathcal{L}(E, F)$ the linear space of all linear and continuous maps on E with values in F, and set $\mathcal{L}(E) \doteqdot \mathcal{L}(E, E)$, moreover let $\Pr(E) \doteqdot \{P \in \mathcal{L}(E) | P \circ P = P\}$ denote the set of all continuous projectors on E. Let S be a set of bounded subsets of a lcs E, thus $\mathcal{L}_S(E)$ denotes the lcs whose underlining linear space is $\mathcal{L}(E)$ and whose locally convex topology is that of uniform convergence over the subsets in S. When E is a normed space and S is the set of all finite parts of E, then $\mathcal{L}_S(E)$ will be denoted by $B_s(E)$, while $B(E)$ denotes $\mathcal{L}(E)$ with the usual norm topology. Let $\{E_i\}_{i \in I}$ a family of lcs. Then we denote by τ_0, τ_b, τ_l, $\tau_{\mathfrak{l}}$ the topology on $\bigoplus_{i \in I} E_i$ induced by the product topology on $\prod_{i \in I} E_i$, that induced by the box topology on $\prod_{i \in I} E_i$ (see [11]), the direct sum topology, Ch. 4, §3 of [11] and the lc-direct sum topology Ch. 6, §6 of [11] respectively.

[6]I.e. $(\forall j_1, j_2 \in J)(\exists\, j \in J)(\mu_{j_1}^x, \mu_{j_1}^x \leq \mu_j^x)$ with the order relation of pointwise order on $\mathbb{R}^{V_x}$.

Let X, Y be two topological spaces then $Comp(X)$ is the set of all compact subsets of X, while $\mathcal{C}(X,Y)$ is the set of all continuous maps on X valued in Y, while $\mathcal{C}_c(X,Y)$ is the topological space of all continuous maps on X valued in Y with the topology of uniform convergence over the compact subsets of Y. If Y is a uniform space then $\mathcal{C}^b(X,Y)$ is the space of all bounded maps in $\mathcal{C}(X,Y)$, while $\mathcal{C}_c^b(X,Y) = \mathcal{C}_c(X,Y) \cap \mathcal{C}^b(X,Y)$. If E is a lcs then $\mathcal{C}_c(X,E)$ is a lcs, while if E is a Hlcs and $Comp(X)$ is a covering of X, for example if X is a locally compact space, then $\mathcal{C}_c(X,E)$ is a Hlcs. Let Y be a locally compact space, $\mu \in Radon(Y)$ and $E \in Hlcs$, then $\mathfrak{L}_1(Y,E,\mu)$ denotes the linear space of all scalarly essentially $\mu-$integrable maps $f : Y \to E$ such that its integral belongs to E, see [3, Ch. 6], while $Meas(Y,E,\mu)$ denotes the linear space of all $\mu-$measurable maps $f : Y \to E$. Let E be a topological vector space, and $\langle \mathcal{L}(E), \tau \rangle$ the topological vector space whose underlying linear space is $\mathcal{L}(E)$ provided by the topology τ. Thus $\mathsf{U}(\langle \mathcal{L}(E), \tau \rangle)$ is the set of all continuous semigroup morphisms defined on $\mathbb{R}^+$ and with values in $\langle \mathcal{L}(E), \tau \rangle$. Moreover if $\|\cdot\|$ is any seminorm on $\mathcal{L}(E)$ (not necessarly continuous with respect to τ) we set $\mathsf{U}_{\|\cdot\|}(\langle \mathcal{L}(E), \tau \rangle)$ as the subset of all $U \in \mathsf{U}(\langle \mathcal{L}(E), \tau \rangle)$ such that $\|U(s)\| \leq 1$, for all $s \in \mathbb{R}^+$. Let $\mathsf{U}_{is}(\langle \mathcal{L}(E), \tau \rangle)$ be the subset of all $U \in \mathsf{U}(\langle \mathcal{L}(E), \tau \rangle)$ such that there exists a fundamental set of seminorms K on E such that $U(s)$ is an isometry with respect to any element in K, for all $s \in \mathbb{R}^+$. We use throughout this work the notation of [10] and often when referring to Banach bundles those of [9]. In particular $\langle \langle \mathfrak{E}, \tau \rangle, p, X, \mathfrak{N} \rangle$ or simply $\langle \mathfrak{E}, p, X \rangle$, whenever τ and $\mathfrak{N}$ are known, is a bundle of $\Omega-$spaces (1.5. of [10]), where we denote by τ the topology on $\mathfrak{E}$ while with $\mathfrak{N} \doteqdot \{\nu_j \mid j \in J\}$ the directed set of seminorms on $\mathfrak{E}$ (1.3. of [10]). Thus we set $\mathfrak{N}_x \doteqdot \{\nu_j^x \mid j \in J\}$ with $\nu_j^x \doteqdot \nu_j \upharpoonright \mathfrak{E}_x$ and $\mathfrak{E}_x \doteqdot \overset{-1}{p}(x)$, for all $x \in X$ and $j \in J$. Moreover for any $U \subseteq X$ we call $\Gamma_U(p)$ the space of bounded continuous sections of $\langle \langle \mathfrak{E}, \tau \rangle, p, X, \mathfrak{N} \rangle$ on U defined by

$$\Gamma_U(p) \doteqdot \mathcal{C}(U, \mathfrak{E}) \bigcap \prod_{x \in U}^{b} \langle \mathfrak{E}_x, \mathfrak{N}_x \rangle$$

where

$$\prod_{x \in U}^{b} \langle \mathfrak{E}_x, \mathfrak{N}_x \rangle \doteqdot \big\{\sigma \in \prod_{x \in U} \mathfrak{E}_x \mid (\forall j \in J)(\sup_{x \in U} \nu_j^x(\sigma(x)) < \infty)\big\}.$$

Let $U \subseteq X$ and $x \in U$ set

$$\Gamma_U^x(p) \doteqdot \big\{f \in \prod_{x \in U}^{b} \langle \mathfrak{E}_x, \mathfrak{N}_x \rangle \mid f \text{ is continuous at } x\big\}.$$

So $\Gamma_U(p) = \bigcap_{x \in U} \Gamma_U^x(p)$. We set $\Gamma(p) \doteqdot \Gamma_X(p)$ and $\Gamma^x(p) \doteqdot \Gamma_X^x(p)$ for any $x \in X$. The definition of trivial bundle of $\Omega-$spaces is given in 1.8. of [10]. Whenever we mention the properties $FM(3), FM(4)$ we always mean those provided in [10, §5] and recalled in Definition 13. If $\mathfrak{A} \doteqdot \langle \langle \mathfrak{B}, \tau \rangle, \xi, X, \mathfrak{N} \rangle$ is a bundle of $\Omega-$spaces, $x \in X$ and Q, S are subsets of $\prod_{z \in X} \mathfrak{B}_z$, we set

$$\begin{aligned} Q_S^x &\doteqdot \{H \in Q \mid (\exists\, F \in S)(H(x) = F(x))\}, \\ Q_\diamond^x &\doteqdot Q_{\Gamma(\xi)}^x, \\ \Gamma_S^x(\xi) &\doteqdot (\Gamma^x(\xi))_S^x, \\ \Gamma_\diamond^x(\xi) &\doteqdot (\Gamma^x(\xi))_\diamond^x. \end{aligned} \tag{2.1}$$

3 Continuous sections of bundles of $\Omega-$spaces

In this section we provide simple but helpful results concerning convergence in bundles of $\Omega-$spaces and more specifically characterizations of the continuity of sections at a certain point.

Proposition 3.1. *Let $\mathfrak{V} = \langle\langle \mathfrak{E}, \tau\rangle, \pi, X, \mathfrak{N}\rangle$ be a bundle of $\Omega-$spaces where $\mathfrak{N} \doteqdot \{\nu_j \mid j \in J\}$. Moreover let $b \in \mathfrak{E}$ and $\{b_\alpha\}_{\alpha\in D}$ a net in $\mathfrak{E}$. Then $(1) \Leftarrow (2) \Leftarrow (3) \Leftrightarrow (4)$ where*

1. $\lim_{\alpha\in D} b_\alpha = b$;
2. $(\exists U \in Op(X) \mid U \ni \pi(b))(\exists \sigma \in \Gamma_U(\pi))(\sigma \circ \pi(b) = b)$ *such that* $\lim_{\alpha\in D} \pi(b_\alpha) = \pi(b)$ *and* $(\forall j \in J)(\lim_{\alpha\in D} \nu_j(b_\alpha - \sigma(\pi(b_\alpha))) = 0)$;
3. $(\exists U' \in Op(X) \mid U' \ni \pi(b))(\exists \sigma' \in \Gamma_U(\pi) \mid \sigma' \circ \pi(b) = b)$ *and* $(\forall U \in Op(X) \mid U \ni \pi(b))(\forall \sigma \in \Gamma_U(\pi) \mid \sigma \circ \pi(b) = b)$ *we have* $\lim_{\alpha\in D} \pi(b_\alpha) = \pi(b)$ *and* $(\forall j \in J)(\lim_{\alpha\in D} \nu_j(b_\alpha - \sigma(\pi(b_\alpha))) = 0)$;
4. $(\exists U' \in Op(X) \mid U' \ni \pi(b))(\exists \sigma' \in \Gamma_U(\pi))(\sigma' \circ \pi(b) = b)$ *and* $\lim_{\alpha\in D} b_\alpha = b$.

Moreover if $\mathfrak{V}$ is locally full then $(1) \Leftrightarrow (4)$.

Proof. Clearly $(3) \Rightarrow (2)$. (2) is equivalent to say that $(\exists U \in Op(X) \mid U \ni \pi(b))(\exists \sigma \in \Gamma_U(\pi))(\sigma \circ \pi(b) = b)$ such that $(\forall V \in Op(X) \mid \pi(b) \in V \subseteq U)(\exists \alpha(V) \in D)(\forall \alpha \geq \alpha(V))(\pi(b_\alpha) \in V)$ and $(\forall j \in J)(\forall \varepsilon > 0)(\exists \alpha(V) \in D)(\forall \alpha \geq \alpha(j, \varepsilon))(\nu_j(b_\alpha - \sigma(\pi(b_\alpha))) < \varepsilon)$. Set $\alpha(V, j, \varepsilon) \in D$ such that $\alpha(V, j, \varepsilon) \geq \alpha(V), \alpha(j, \varepsilon)$ which there exists D being directed, thus we have $(\forall V \in Op(X) \mid \pi(b) \in V \subseteq U)(\forall j \in J)(\forall \varepsilon > 0)(\exists \alpha(V, j, \varepsilon) \in D)$ such that $(\forall \alpha \geq \alpha(V, j, \varepsilon))(\nu_j(b_\alpha - \sigma(\pi(b_\alpha))) < \varepsilon)$ and $\pi(b_\alpha) \in V$. Thus (1) follows by applying $1.5.VII$ of [10]. Finally by applying $1.5.VII$ of [10] (4) (respectively (1) if $\mathfrak{V}$ is locally full) is equivalent to $(\exists U' \in Op(X) \mid U' \ni \pi(b))(\exists \sigma' \in \Gamma_U(\pi))(\sigma' \circ \pi(b) = b)$ and $(\forall \sigma \in \Gamma_U(\pi) \mid \sigma \circ \pi(b) = b)(\forall j \in J)(\forall \varepsilon > 0)(\forall V \in Op(X) \mid \pi(b) \in V \subseteq U)(\exists \overline{\alpha} \in D)(\forall \alpha \geq \overline{\alpha})$ we have $\pi(b_\alpha) \in V$ and $\nu_j(b_\alpha - \sigma(\pi(b_\alpha))) < \varepsilon$ which is (3). □

Theorem 3.1. *Let $\mathfrak{V} = \langle\langle \mathfrak{E}, \tau\rangle, \pi, X, \mathfrak{N}\rangle$ be a bundle of $\Omega-$spaces, $W \subseteq X$ and indicate $\mathfrak{N} = \{\nu_j \mid j \in J\}$. Moreover let $f \in \mathfrak{E}^W$, $x_\infty \in W$. Then $(1) \Leftarrow (2) \Leftrightarrow (3) \Leftarrow (4) \Leftrightarrow (5) \Leftrightarrow (6)$ where*

1. f *is continuous in* x_∞;
2. $(\exists U \in Op(X) \mid U \ni x_\infty)(\exists \sigma \in \Gamma_U(\pi))(\sigma(x_\infty) = f(x_\infty))$ *such that* $\nu_j \circ (f - \sigma \circ \pi \circ f) : W \cap U \to \mathbb{R}$ *and* $\pi \circ f : W \to X$ *are continuous in* x_∞ *for all* $j \in J$;
3. $\pi \circ f : W \to X$ *is continuous in* x_∞ *and* $(\exists U \in Op(X) \mid U \ni x_\infty)(\exists \sigma \in \Gamma_U(\pi))(\sigma(x_\infty) = f(x_\infty))$ *such that*
$$(\forall j \in J)(\lim_{y \to x_\infty, y \in W\cap U} \nu_j(f(y) - \sigma \circ \pi \circ f(y)) = 0);$$
4. $(\exists U' \in Op(X) \mid U' \ni x_\infty)(\exists \sigma' \in \Gamma_U(\pi))(\sigma'(x_\infty) = f(x_\infty))$ *and* $(\forall U \in Op(X) \mid U \ni x_\infty)(\forall \sigma \in \Gamma_U(\pi) \mid \sigma(x_\infty) = f(x_\infty))$ *we have* $\nu_j \circ (f - \sigma) : W \cap U \to \mathbb{R}$ *and* $\pi \circ f : W \to X$ *are continuous in* x_∞ *for all* $j \in J$;

5. $\pi \circ f : W \to X$ *is continuous in* x_∞ *and* $(\exists U' \in Op(X) \,|\, U' \ni x_\infty)(\exists \sigma' \in \Gamma_U(\pi))(\sigma'(x_\infty) = f(x_\infty))$ *and* $(\forall U \in Op(X) \,|\, U \ni x_\infty)(\forall \sigma \in \Gamma_U(\pi) \,|\, \sigma(x_\infty) = f(x_\infty))$ *we have*
$$(\forall j \in J)(\lim_{y \to x_\infty, y \in W \cap U} \nu_j(f(y) - \sigma \circ \pi \circ f(y)) = 0);$$

6. $(\exists U' \in Op(X) \,|\, U' \ni x_\infty)(\exists \sigma' \in \Gamma_U(\pi))(\sigma'(x_\infty) = f(x_\infty))$ *and* f *is continuous at* x_∞.

Moreover if $\mathfrak{V}$ *is locally full then* (1) $\Leftrightarrow$ (6) *and if it is full we can choose* $U = X$ *and* $U' = X$.

Proof. (1) is equivalent to say that for each net $\{x_\alpha\}_{\alpha \in D} \subset W$ such that $\lim_{\alpha \in D} x_\alpha = x_\infty$ in W, we have $\lim_{\alpha \in D} f(x_\alpha) = f(x_\infty)$ in $\mathfrak{E}$. Similarly (2) is equivalent to say that for each net $\{x_\alpha\}_{\alpha \in D} \subset W$ such that $\lim_{\alpha \in D} x_\alpha = x_\infty$ in W, we have $\lim_{\alpha \in D} \pi \circ f(x_\alpha) = \pi \circ f(x_\infty)$ and $(\forall j \in J)(\lim_{\alpha \in D} \nu_j \circ (f - \sigma \circ \pi \circ f)(x_\alpha) = \nu_j \circ (f - \sigma \circ \pi \circ f)(x_\infty))$. Thus (1) $\Leftarrow$ (2) follows by the corresponding one in Proposition 3.1 with the positions $(\forall \alpha \in D)(b_\alpha \doteqdot f(x_\alpha))$ and $b \doteqdot f(x_\infty)$. Similarly (1) $\Leftarrow$ (5) follows by (1) $\Leftarrow$ (3) of Proposition 3.1. Finally (5) $\Rightarrow$ (6) follows by (5) $\Rightarrow$ (1), while if (6) is true then $\pi \circ f$ is continuous at x_∞ indeed π is continuous, then (5) follows by the implication (4) $\Rightarrow$ (3) of Proposition 3.1 with the positions $(\forall \alpha \in D)(b_\alpha \doteqdot f(x_\alpha))$ and $b \doteqdot f(x_\infty)$. □

Corollary 3.1. *Let* $\mathfrak{V} = \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \mathfrak{N} \rangle$ *be a bundle of* $\Omega-$*spaces,* $W \subseteq X$ *and indicate* $\mathfrak{N} = \{\nu_j \,|\, j \in J\}$. *Moreover let* $f \in \prod_{x \in W} \mathfrak{E}_x$ *and* $x_\infty \in W$. *Then* (1) $\Leftarrow$ (2) $\Leftrightarrow$ (3) $\Leftarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) *where*

1. f *is continuous in* x_∞;

2. $(\exists U \in Op(X) \,|\, U \ni x_\infty)(\exists \sigma \in \Gamma_U(\pi))(\sigma(x_\infty) = f(x_\infty))$ *such that* $\nu_j \circ (f - \sigma) : W \cap U \to \mathbb{R}$ *is continuous in* x_∞ *for all* $j \in J$;

3. $(\exists U \in Op(X) \,|\, U \ni x_\infty)(\exists \sigma \in \Gamma_U(\pi))(\sigma(x_\infty) = f(x_\infty))$ *such that*
$$(\forall j \in J)(\lim_{y \to x_\infty, y \in W \cap U} \nu_j(f(y) - \sigma(y)) = 0);$$

4. $(\exists U' \in Op(X) \,|\, U' \ni x_\infty)(\exists \sigma' \in \Gamma_U(\pi))(\sigma'(x_\infty) = f(x_\infty))$ *and* $(\forall U \in Op(X) \,|\, U \ni x_\infty)(\forall \sigma \in \Gamma_U(\pi) \,|\, \sigma(x_\infty) = f(x_\infty))$ *we have that* $\nu_j \circ (f - \sigma) : W \cap U \to \mathbb{R}$ *is continuous in* x_∞ *for all* $j \in J$;

5. $(\exists U' \in Op(X) \,|\, U' \ni x_\infty)(\exists \sigma' \in \Gamma_U(\pi))(\sigma'(x_\infty) = f(x_\infty))$ *and* $(\forall U \in Op(X) \,|\, U \ni x_\infty)(\forall \sigma \in \Gamma_U(\pi) \,|\, \sigma(x_\infty) = f(x_\infty))$ *we have*
$$(\forall j \in J)(\lim_{y \to x_\infty, y \in W \cap U} \nu_j(f(y) - \sigma(y)) = 0).$$

6. $(\exists U' \in Op(X) \,|\, U' \ni x_\infty)(\exists \sigma' \in \Gamma_U(\pi))(\sigma'(x_\infty) = f(x_\infty))$ *and* f *is continuous at* x_∞

If $\mathfrak{V}$ *is locally full then* (1) $\Leftrightarrow$ (6) *and if it is full we can choose* $U = X$ *and* $U' = X$.

Proof. By Theorem 3.1 and $\pi \circ f = Id$. □

Proposition 3.2. *Let $\mathfrak{V}$ be full and such that there exists a linear space E such that for all $x \in X$ there exists a linear subspace $E_x \subseteq E$ such that $\mathfrak{E}_x = \{x\} \times E_x$, and that* [7]

$$\{\mathfrak{t}_v : X \ni x \mapsto (x, v) \in \mathfrak{E}_x \,|\, v \in \bigcap_{x \in X} E_x\} \subset \Gamma(\pi),$$

If $f_0 \in \prod_{x \in X} E_x$ and $f \in \prod_{x \in X} \mathfrak{E}_x$ such that $f(x) = (x, f_0(x))$ for all $x \in X$ and $f_0(x_\infty) \in \bigcap_{x \in X} E_x$, then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3), where

1. *f is continuous at x_∞*
2. *$(\exists U \in Op(X) \,|\, U \ni x_\infty)(\exists \sigma \in \mathcal{C}_b(U, E))(\sigma(x_\infty) = f(x_\infty))$ such that for all $j \in J$*
$$\lim_{z \to x_\infty, z \in W \cap U} \nu_j^z(f(z) - \sigma(z)) = 0;$$
3. *for all $j \in J$*
$$\lim_{z \to x_\infty, z \in W \cap U} \nu_j^z((z, f_0(z)) - (z, f(x_\infty))) = 0.$$

Proof. By Corollary 3.1 (1) $\Leftrightarrow$ (2). Let (3) hold then (2) is true by setting $\sigma = \mathfrak{t}_{f(x_\infty)} \upharpoonright U$. Let (2) hold then $\nu_j^z((z, f_0(z)) - (z, f(x_\infty))) \leq \nu_j^z((z, f_0(z)) - \sigma(z)) + \nu_j^z(\sigma(z) - \mathfrak{t}_{f(x_\infty)}(z))$, thus (3) follows by (2) and by Corollary 3.1 applied to the continuous map $\mathfrak{t}_{f(x_\infty)} \upharpoonright U$. □

Corollary 3.2. *Let $\mathfrak{V} \doteqdot \langle\langle \mathfrak{E}, \tau\rangle, \pi, X, \mathfrak{N}\rangle$ be a bundle of $\Omega-$spaces, $W \subseteq X$ and indicate $\mathfrak{N} = \{\nu_j \,|\, j \in J\}$. Moreover let $f, g \in \prod_{x \in W} \mathfrak{E}_x$ and $x_\infty \in W$. Then if $\mathfrak{V}$ locally full or ν_j is continuous $\forall j \in J$, then (1) $\rightarrow$ (2) where*

1. *$f(x_\infty) = g(x_\infty)$ and f and g are continuous in x_∞;*
2. *$(\exists U \in Op(X) \,|\, x_\infty \in U)$ such that*
$$(\forall j \in J)(\lim_{y \to x_\infty, y \in W \cap U} \nu_j(f(y) - g(y)) = 0).$$

Moreover if $\mathfrak{V}$ is full we can choose $U = X$.

Proof. The statement is trivial in the case of continuiuty of all the ν_j. Whereas if $\mathfrak{V}$ is locally full by (1) $\rightarrow$ (5) of Corollary 3.1 we have $(\exists U \in Op(X))(\exists \sigma \in \Gamma_U(\pi))(\sigma(x_\infty) = f(x_\infty) = g(x_\infty))$ such that

$$(\forall j \in J)(\lim_{y \to x_\infty, y \in W \cap U} \nu_j(f(y) - \sigma(y)) = \lim_{y \to x_\infty, y \in W \cap U} \nu_j(g(y) - \sigma(y)) = 0).$$

Therefore

$$\lim_{y \to x_\infty, y \in W \cap U} \nu_j(f(y) - g(y)) \leq \lim_{y \to x_\infty, y \in W \cap U} \nu_j(f(y) - \sigma(y)) + \lim_{y \to x_\infty, y \in W \cap U} \nu_j(g(y) - \sigma(y)) = 0.$$

□

[7] An example is when $\mathfrak{V}$ is the trivial bundle.

Corollary 3.3. *Let* $\langle\langle\mathfrak{E},\tau\rangle,\pi,X,\mathfrak{N}\rangle$ *be a bundle of* $\Omega-$*spaces,* $W\in Op(X)$ *and indicate* $\mathfrak{N}=\{\nu_j\,|\,j\in J\}$*. Moreover let* $f\in\prod_{x\in W}^{b}\mathfrak{E}_x$*. Then* $(1)\Leftarrow(2)\Leftrightarrow(3)\Leftarrow(4)\Leftrightarrow(5)$ *where*

1. $f\in\Gamma_W(\pi)$*;*
2. $$(\forall x\in\ W)(\exists\,U_x\in Op(X)\,|\,U_x\ni x)(\exists\,\sigma_x\in\Gamma_{U_x}(\pi))(\sigma_x(x)=f(x))$$
   such that $\nu_j\circ(f-\sigma_x)$ *is continuous in* x*,* $\forall j\in J$*;*
3. $$(\forall x\in\ W)(\exists\,U_x\in Op(X)\,|\,U_x\ni x)(\exists\,\sigma_x\in\Gamma_{U_x}(\pi))(\sigma_x(x)=f(x))$$
   such that $(\forall j\in J)(\lim_{y\to x,y\in W\cap U_x}\nu_j(f(y)-\sigma_x(y))=0)$*;*
4. $$(\forall x\in\ W)(\exists\,U'_x\in Op(X)\,|\,U'_x\ni x)(\exists\,\sigma'_x\in\Gamma_{U_x}(\pi))(\sigma'_x(x)=f(x))$$
   and
   $$(\forall\,U_x\in Op(X)\,|\,U_x\ni x)(\forall\sigma_x\in\Gamma_{U_x}(\pi)\,|\,\sigma_x(x)=f(x))$$
   we have that $\nu_j\circ(f-\sigma_x)$ *is continuous in* x *for all* $x\in W$ *and* $j\in J$*;*
5. $$(\forall x\in\ W)(\exists\,U'_x\in Op(X)\,|\,U'_x\ni x)(\exists\,\sigma'_x\in\Gamma_{U_x}(\pi))(\sigma'_x(x)=f(x))$$
   and
   $$(\forall x\in\ W)(\forall\,U_x\in Op(X)\,|\,U_x\ni x)(\forall\sigma_x\in\Gamma_{U_x}(\pi)\,|\,\sigma_x(x)=f(x))$$
   we have $(\forall j\in J)(\lim_{y\to x,y\in W\cap U_x}\nu_j(f(y)-\sigma_x(y))=0)$*.*

Proof. By Corollary 3.1. □

4 Direct Sum of Bundles of $\Omega-$spaces

The aim of this section is to extend in Definition 2 the standard construction of direct sum of Banach bundles to bundles of $\Omega-$spaces. In order to do this in Theorem 4.1 we find a suitable directed set of seminorms inducing the product topology on the direct sum of a finite family of locally convex spaces. Then since Lemma 4.1 we can apply the general construction given in Definition 15 to the objects defined in Definition 1. Finally the factorization property of the convergence in any direct sum of bundles of $\Omega-$spaces presented in Corollary 4.1, shows that our definition extends the product topology and more in general it extends the usual definition of direct sum of Banach bundles.

Theorem 4.1. *Let* $\{\langle E_i,\nu_i\rangle\}_{i=1}^n$ *be a family of lcs where* $\nu_i=\{\nu_{i,l_i}\,|\,l_i\in L_i\}$ *is a fundamental directed set of seminorms of* E_i*. Let us set for all* $i=1,...,n$*,* $l_i\in L_i$ *and* $\rho\in\prod_{i=1}^n L_i$

$$\begin{cases}\hat{\nu}_{i,l_i}\doteqdot\nu_{i,l_i}\circ\mathrm{Pr}_i\\ \hat{\mu_\rho}\doteqdot\sum_{k=1}^n\hat{\nu}_{k,\rho_k},\end{cases}$$

where $\mathrm{Pr}_i:\prod_{k=1}^n E_k\ni x\mapsto x_i\in E_i$*.*

Then $\hat{\mu} \doteqdot \{\hat{\mu}_\rho \mid \rho \in \prod_{i=1}^n L_i\}$ *is a directed set of seminorms on* $\bigoplus_{i=1}^n E_i$*. Moreover by setting*

$$\begin{cases} \mathcal{B}(\mathbf{0}) \doteqdot \{W_\varepsilon^\rho \mid \varepsilon, \rho \in \prod_{i=1}^n L_i\} \\ W_\varepsilon^\rho \doteqdot \{x \in \bigoplus_{i=1}^n E_i \mid \hat{\mu}_\rho(x) < \varepsilon\}, \end{cases}$$

we have that $\mathcal{B}(\mathbf{0})$ *is a base of the filter of the neighbourhoods of* $\mathbf{0}$ *with respect to the unique locally convex topology* τ *on* $\bigoplus_{i=1}^n E_i$ *generated by* $\hat{\mu}$*. In other words*

$$\mathfrak{F}_{\mathcal{B}(\mathbf{0})}^{\bigoplus_{i=1}^n E_i} = \mathcal{I}_{\mathbf{0}}^\tau.$$

Finally we have $\tau = \tau_0 = \tau_b = \tau_l = \tau_{\mathfrak{l}}$*.*

Proof. Only in this proof we set $I \doteqdot \{1, ..., n\}$, $L \doteqdot \prod_{i \in I} L_i$ and $E^\oplus \doteqdot \bigoplus_{i=1}^n E_i$. Due to the fact that $n < \infty$ we know that $\prod_{i=1}^n E_i = E^\oplus$ so by [11] §4.3. the set $\{\prod_{i=1}^n U_i \mid U_i \in \mathfrak{U}_i\}$ is a $\mathbf{0}-$basis for the box topology on $E^\oplus$ if $\mathfrak{U}_i$ is a $\mathbf{0}-$basis for the topology on E_i. Moreover ν_i is directed so by $II.3$ of [2] we can choose

$$\begin{aligned} \mathfrak{U}_i &= \{V(\nu_{i,l_i}, \varepsilon) \mid \varepsilon > 0, l_i \in L_i\}, \\ V(\nu_{i,l_i}, \varepsilon) &\doteqdot \{x_i \in E_i \mid \nu_{i,l_i}(x_i) < \varepsilon\}. \end{aligned} \tag{4.1}$$

Thus if we set

$$\begin{cases} \mathcal{B}_1(\mathbf{0}) \doteqdot \{U_\eta^\rho \mid \eta \in (\mathbb{R}_0^+)^n, \rho \in L\}, \\ U_\eta^\rho \doteqdot \{x \in E^\oplus \mid (\forall i \in I)(\hat{\nu}_{i,\rho_i}(x) < \eta_i)\}; \end{cases} \tag{4.2}$$

then $\mathcal{B}_1(\mathbf{0})$ is a $\mathbf{0}-$basis for the topology τ_0. Moreover $U_\varepsilon^\rho = \bigcap_{i=1}^n V(\hat{\nu}_{i,\rho_i}\eta_i)$ so if we set

$$\mathcal{G}(\mathbf{0}) \doteqdot \Big\{\bigcap_{s \in M} V(\hat{\nu}_s \varepsilon_M(s)) \mid M \in \mathcal{P}_\omega\big(\bigcup_{i \in I}\{i\} \times L_i\big), \, \varepsilon_M : M \to \mathbb{R}_0^+\Big\},$$

then by (4.2) $\mathcal{B}_1(\mathbf{0}) \subseteq \mathcal{G}(\mathbf{0})$. Moreover by applying $II.3$ of [2], $\mathcal{G}(\mathbf{0})$ is a basis of a filter thus

$$\mathfrak{F}_{\mathcal{B}_1(\mathbf{0})}^{E^\oplus} \subseteq \mathfrak{F}_{\mathcal{G}(\mathbf{0})}^{E^\oplus}.$$

Now for all $M \in \mathcal{P}_\omega\left(\bigcup_{i \in I}\{i\} \times L_i\right)$ we have $M = \bigcup_{i \in I} M_i$ with $M_i \doteqdot M \cap (\{i\} \times L_i) = \{i\} \times Q_i$ for some $Q_i \in \mathcal{P}_\omega(L_i)$. Hence $\forall M \in \mathcal{P}_\omega\left(\bigcup_{i \in I}\{i\} \times L_i\right)$ and $\forall \varepsilon_M : M \to \mathbb{R}_0^+$

$$\begin{aligned} T \doteqdot \bigcap_{s \in M} V(\hat{\nu}_s, \varepsilon_M(s)) &= \bigcap_{i \in I} \bigcap_{s \in M_i} V(\hat{\nu}_s, \varepsilon_M(s)) \\ &= \bigcap_{i \in I} \bigcap_{l_i \in Q_i} \{x \in E^\oplus \mid x_i \in V(\nu_{i,l_i}, \varepsilon_M(i, l_i)) \\ &= \bigcap_{i \in I} \Big\{x \in E^\oplus \mid x_i \in \bigcap_{l_i \in Q_i} V(\nu_{i,l_i}, \varepsilon_M(i, l_i))\Big\}. \end{aligned}$$

Moreover we know that $\mathfrak{U}_i$ is a basis of a filter on E_i thus for any $i \in I$ there exists $\lambda_i > 0$ and $k_i \in L_i$ such that

$$V(\nu_{i,k_i}, \lambda_i) \subseteq \bigcap_{l_i \in Q_i} V(\nu_{i,l_i}, \varepsilon_M(i, l_i)),$$

hence

$$\mathcal{G}(\mathbf{0}) \ni T \supseteq \bigcap_{i\in I}\{x \in E^{\oplus} \mid x_i \in V(\nu_{i,k_i}, \lambda_i)\}$$
$$= \bigcap_{i\in I} V(\hat{\nu}_{i,k_i}, \lambda_i) \in \mathcal{B}_1(\mathbf{0}).$$

Therefore by a well-known property of filters $\mathfrak{F}^{E^{\oplus}}_{\mathcal{G}(\mathbf{0})} \subseteq \mathfrak{F}^{E^{\oplus}}_{\mathcal{B}_1(\mathbf{0})}$ then

$$\mathfrak{F}^{E^{\oplus}}_{\mathcal{G}(\mathbf{0})} = \mathfrak{F}^{E^{\oplus}}_{\mathcal{B}_1(\mathbf{0})}. \tag{4.3}$$

By applying $II.3$ of [2] we know that $\mathfrak{F}^{E^{\oplus}}_{\mathcal{G}(\mathbf{0})}$ is the $\mathbf{0}-$neighbourhood's filter with respect to the locally convex topology generated by the family of seminorms $\{\nu_s \mid s \in \bigcup_{i\in I}\{i\} \times L_i\}$ thus by (4.2) and (4.3)

$$\Big\{\nu_s \mid s \in \bigcup_{i\in I}\{i\} \times L_i\Big\} \text{ is a fss for } \tau_0. \tag{4.4}$$

Now $\hat{\mu}$ is a set of seminorms on $E^{\oplus}$. Let $\rho^1, \rho^2 \in L$ then by the hypothesis that ν_i is directed, for all $i \in I$ there exists $\rho_i \in L_i$ such that $\rho_i \geq \rho^1, \rho^2$ thus $\hat{\mu}_\rho \geq \hat{\mu}_{\rho^1}, \hat{\mu}_{\rho^2}$, hence $\hat{\mu}$ is directed. Therefore setting

$$\begin{cases} \mathcal{B}(\mathbf{0}) \doteqdot \{W^{\rho}_{\varepsilon} \mid \varepsilon > 0, \rho \in L\} \\ W^{\rho}_{\varepsilon} \doteqdot \{x \in E^{\oplus} \mid \hat{\mu}_\rho(x) < \varepsilon\} \end{cases}$$

by applying $II.3$ of [2]

$$\mathcal{B}(\mathbf{0}) \text{ is the } \mathbf{0}-\text{basis for the topology generated by } \hat{\mu}. \tag{4.5}$$

Now $(\forall (k, l_k) \in \bigcup_{i\in I}\{i\} \times L_i)(\exists\, \rho \in L)(\hat{\nu}_{k,l_k} \leq a\hat{\mu}_\rho)$ indeed keep any ρ s.t. $\rho(k) = l_k$. While $(\forall \rho \in I)(m \in \mathbb{N})(\exists\, s_1, ..., s_m \in \bigcup_{i\in I}\{i\} \times L_i)(\exists\, a > 0)(\hat{\mu}_\rho \leq a \sup_r \hat{\nu}_{s_r})$ indeed it is sufficient to set $m = n$, $a = n$ and $s_i = (i, \rho_i)$ for all $i \in I$. Therefore by applying Corollary 1 $II.7$ of [2] and by (4.5) and (4.4) we have that $\hat{\mu}$ is a directed fss for the topology τ_0 hence the part of the statement concerning τ_0 follows. By Prop. 2, §3, Ch 4 of [11] we know that $\tau_0 = \tau_b = \tau_l$. Finally $\tau_{\mathfrak{l}} = \tau_l$ by the fact that $\tau_{\mathfrak{l}}$ is the finest locally convex topology among those which are coarser than τ_l, §6, Ch 6 of [11], and the just now shown fact that τ_l is locally convex being equal to τ_0 which is generated by $\hat{\mu}$. □

Notation 1. In the remaining of the present Section 4 we let $\{\mathfrak{V}_i\}_{i=1}^n$ be a family of **full** bundles of $\Omega-$spaces. Here $\mathfrak{V}_i = \langle\langle \mathfrak{E}_i, \tau_i\rangle, \pi_i, X, \mathfrak{N}_i\rangle$, $\mathfrak{N}_i = \{\nu_{i,l_i} \mid l_i \in L_i\}$ moreover $\mathfrak{N}^x_i \doteqdot \{\nu^x_{i,l_i} \mid l_i \in L_i\}$, with $\nu^x_{i,l_i} \doteqdot \nu_{i,l_i} \upharpoonright (\mathfrak{E}_i)_x$ and $(\mathfrak{E}_i)_x \doteqdot \overset{-1}{\pi_i}(x)$ for all $i = 1, ..., n$ and $x \in X$.

Definition 1. Define

1. $\mathsf{E}^{\oplus}_x \doteqdot \bigoplus_{i=1}^n (\mathfrak{E}_i)_x$;
2. $\mathfrak{n}^{\oplus}_x \doteqdot \{\hat{\mu}^x_\rho \mid \rho \in \prod_{i=1}^n L_i\}$, where

$$\hat{\mu}^x_\rho = \sum_{i=1}^n \hat{\nu}^x_{i,\rho_i}; \tag{4.6}$$

3. $\mathsf{E}^{\oplus} \doteqdot \{\langle \mathsf{E}_x^{\oplus}, \mathfrak{n}_x^{\oplus}\rangle\}_{x\in X}$;

4. $\mathcal{E}^{\oplus}$ is the linear subspace of $\prod_{x\in X}\mathsf{E}_x^{\oplus}$ generated by the following set

$$\bigcup_{i=1}^{n}\tilde{\Gamma}(\pi_i). \tag{4.7}$$

Here $\mathrm{Pr}_i^x : \mathsf{E}_x^{\oplus} \ni x \mapsto x(i) \in (\mathfrak{E}_i)_x$ while $\hat{\nu}_{i,\rho_i}^x = \nu_{i,\rho_i}^x \circ \mathrm{Pr}_i^x$ and $I_i^x : (\mathfrak{E}_i)_x \to \mathsf{E}_x^{\oplus}$ is the canonical inclusion, i.e. $\mathrm{Pr}_j^x \circ I_i^x = \delta_{i,j}\, Id^x$, finally $\tilde{\Gamma}(\pi_i) \doteqdot \{\tilde{f} \mid f \in \Gamma(\pi_i)\}$, with $\tilde{f}(x) \doteqdot I_i^x(f(x))$.

Notice that $\{\langle (\mathfrak{E}_i)_x, \mathfrak{N}_i^x\rangle\}_{i=1}^n$ for all $x \in X$ is a family of Hlcs where $\mathfrak{N}_i^x$ is a directed family of seminorms defining the topology on $(\mathfrak{E}_i)_x$, for all $i = 1,\dots,n$.

Lemma 4.1. *$\mathcal{E}^{\oplus}$ satisfies $FM(3) - FM(4)$ with respect to $\mathsf{E}^{\oplus}$.*

Proof. I_i^x is a bijective map onto its range whose inverse is $\mathrm{Pr}_i^x \upharpoonright Range(I_i^x)$. Moreover by definition of the product topology Pr_i^x is continuous with respect to the topology τ_0^i on $Range(I_i^x)$ induced by τ_0 [1, Ch.1], while I_i^x is continuous with respect to τ_0^i by [11, § 4.3 Pr.1] and the definition of τ_l. Hence by Theorem 4.1 I_i^x is an isomorphism of the tvs's $\langle (\mathfrak{E}_i)_x, \mathfrak{N}_i^x\rangle$ and $I_i^x((\mathfrak{E}_i)_x)$ as subspace of $\langle \mathsf{E}_x^{\oplus}, \mathfrak{n}_x^{\oplus}\rangle$. Since [10, 1.5.III] and [10, 1.6.viii][8] we deduce that $\{\sigma(x) \mid \sigma \in \Gamma(\pi_i)\}$ is dense in $\langle (\mathfrak{E}_i)_x, \mathfrak{N}_i^x\rangle$. Therefore $\forall i = 1,...,n$ and $\forall x \in X$

$$\{I_i^x(\sigma(x)) \mid \sigma \in \Gamma(\pi_i)\} \text{ is dense in } I_i^x((\mathfrak{E}_i)_x). \tag{4.8}$$

where $I_i^x((\mathfrak{E}_i)_x)$ has to be intended as topological vector subspace of $\langle \mathsf{E}_x^{\oplus}, \mathfrak{n}_x^{\oplus}\rangle$. So by the continuity of the sum on $\langle \mathsf{E}_x^{\oplus}, \mathfrak{n}_x^{\oplus}\rangle$ and the fact that $\mathsf{E}_x^{\oplus}$ is generated as linear space by the set $\bigcup_{i=1}^n I_i^x((\mathfrak{E}_i)_x)$ we can state $\forall x \in X$ that

$$\{F(x) \mid F \in \mathcal{E}^{\oplus}\} \text{ is dense in } \left\langle \mathsf{E}_x^{\oplus}, \mathfrak{n}_x^{\oplus}\right\rangle. \tag{4.9}$$

Namely by (4.8)

$$(\forall v \in \mathfrak{E}^{\oplus})(\forall i = 1,...,n)(\exists\, \{\sigma_{\alpha_i}\}_{\alpha_i \in D_i} \text{ net } \subset \Gamma(\pi_i))$$

such that

$$\begin{aligned} v &= \sum_{i=1}^n I_i^x(\overset{x}{\underset{i}{\mathrm{Pr}}}(v)) = \sum_{i=1}^n \lim_{\alpha_i \in D_i} I_i^x(\sigma_{\alpha_i}(x)) \\ &= \sum_{i=1}^n \lim_{\alpha \in D} w_\alpha^i(x) = \lim_{\alpha \in D} \sum_{i=1}^n w_\alpha^i(x) \\ &= \lim_{\alpha \in D} \sum_{i=1}^n I_i^x(\sigma_{\alpha(i)}(x)), \end{aligned}$$

where $D \doteqdot \prod_{i=1}^n D_i$ while $w_\alpha^i(x) \doteqdot I_i^x(\sigma_{\alpha(i)}(x))$ for all $\alpha \in D$. Moreover $\forall \alpha \in D$

$$\left(X \ni x \mapsto \sum_{i=1}^n I_i^x(\sigma_{\alpha(i)}(x))\right) \in \mathcal{E}^{\oplus}$$

[8] which ensures that the locally convex topology on $(\mathfrak{E}_i)_x$ generated by the set of seminorms $\mathfrak{N}_i^x$ is exactly the topology induced on it by the topology τ_i on $\mathfrak{E}_i$, for all i and $x \in X$.

then (4.9) and $FM(3)$ follow.

Finally $FM(4)$ follows by [10, 1.6.iii] applied to any $\sigma_i \in \Gamma(\pi_i)$ for all $i = 1, ..., n$ indeed $\forall \sigma_i \in \Gamma(\pi_i)$

$$\nu_{i,\rho_i}^{\hat{x}}(\tilde{\sigma}_i(x)) = \nu_{i,\rho_i}^{x} \circ \underset{i}{\overset{x}{\Pr}} \circ I_i^x \circ \sigma_i(x) = \nu_{i,\rho_i}^{x} \circ \sigma_i(x).$$

□

Now we are able to extend to bundles of $\Omega-$spaces, the standard construction of direct sum of Banach bundles. Namely by Theorem 4.1 we know that $\mathfrak{n}_x^{\oplus}$ is a directed set of seminorms on $\mathsf{E}_x^{\oplus}$ inducing on $\mathsf{E}_x^{\oplus}$ the product topology, thus since Lemma 4.1 we can apply Definition 15 and set the following

Definition 2. We call bundle direct sum of the family $\{\mathfrak{V}_i\}_{i=1}^n$ the following bundle of $\Omega-$spaces

$$\bigoplus_{i=1}^{n} \mathfrak{V}_i \doteqdot \mathfrak{V}(\mathsf{E}^{\oplus}, \mathcal{E}^{\oplus}).$$

Remark 1. By Definition 15 and Definition 2

$$\bigoplus_{i=1}^{n} \mathfrak{V}_i = \left\langle \left\langle \mathfrak{E}(\mathsf{E}^{\oplus}), \tau(\mathsf{E}^{\oplus}, \mathcal{E}^{\oplus}) \right\rangle, \pi_{\mathsf{E}^{\oplus}}, X, \mathfrak{n}^{\oplus} \right\rangle$$

where

1. $\mathfrak{E}(\mathsf{E}^{\oplus}) \doteqdot \bigcup_{x \in X} \{x\} \times \mathsf{E}_x^{\oplus}$, $\pi_{\mathsf{E}^{\oplus}} : \mathfrak{E}(\mathsf{E}^{\oplus}) \ni (x, v) \mapsto x \in X$.

2. $\mathfrak{n}^{\oplus} = \{\hat{\mu}_\rho : \mid \rho \in \prod_{i=1}^n L_i\}$, with $\hat{\mu}_\rho : \mathfrak{E}(\mathsf{E}^{\oplus}) \ni (x, v) \mapsto \hat{\mu}_\rho^x(v)$;

3. $\tau(\mathsf{E}^{\oplus}, \mathcal{E}^{\oplus})$ is the topology on $\mathfrak{E}(\mathsf{E}^{\oplus})$ such that for all $(x, v) \in \mathfrak{E}(\mathsf{E}^{\oplus})$

$$\mathcal{I}_{(x,v)}^{\mathfrak{E}(\mathsf{E}^{\oplus})} \doteqdot \mathfrak{F}_{\mathcal{B}^{\oplus}((x,v))}^{\mathfrak{E}(\mathsf{E}^{\oplus})}.$$

Here we recall that $\mathfrak{F}_{\mathcal{B}^{\oplus}((x,v))}^{\mathfrak{E}(\mathsf{E}^{\oplus})}$ is the filter on $\mathfrak{E}(\mathsf{E}^{\oplus})$ generated by the following base of filters

$$\mathcal{B}^{\oplus}((x, v)) \doteqdot \Big\{ T_{\mathsf{E}^{\oplus}}(U, \sigma, \varepsilon, \rho) \,|\, U \in Open(X), \sigma \in \mathcal{E}^{\oplus}, \varepsilon > 0, \rho \in \prod_{i=1}^{n} L_i \\ \,|\, x \in U, \hat{\mu}_\rho^x(v - \sigma(x)) < \varepsilon \Big\},$$

where

$$T_{\mathsf{E}^{\oplus}}(U, \sigma, \varepsilon, \rho) \doteqdot \left\{ (y, w) \in \mathfrak{E}(\mathsf{E}^{\oplus}) \,|\, y \in U, \hat{\mu}_\rho^y(w - \sigma(y)) < \varepsilon \right\}.$$

In what follows we state the factorization property of convergence which proves that our construction of bundle direct sum of a family of bundles of $\Omega-$spaces, extends the standard definition provided in the Banach bundle case.

Corollary 4.1. *Let $f : X \to \mathfrak{E}(\mathsf{E}^{\oplus})$ and $x \in X$. Thus f is continuous in x if and only if $f_0^i : X \to \mathfrak{E}_i$ is continuous in x for all $i = 1, ..., n$, where $f_0 : X \to \bigcup_{z \in X} \mathsf{E}_z^{\oplus}$ such that $\forall z \in X$ $f(z) = (z, f_0(z))$ and*

$$f_0^i(z) \doteqdot \overset{\pi_{\mathsf{E}^{\oplus}}(f(z))}{\underset{i}{\mathrm{Pr}}} \circ f_0(z).$$

In particular $f \in \Gamma(\pi_{\mathsf{E}^{\oplus}})$ if and only if $(X \ni z \mapsto \mathrm{Pr}_i^z \circ f_0(z) \in (\mathfrak{E}_i)_z) \in \Gamma(\pi_i)$, for all $i = 1, ..., n$.

Proof. Since the definition of $\mathcal{E}^{\oplus}$, the request that all the bundles in the family $\{\mathfrak{V}_i\}_{i=1}^n$ are full and the fact that $\mathcal{E}^{\oplus}$ is linearly isomorphic to a subspace of $\Gamma(\pi_{\mathsf{E}^{\oplus}})$ we obtain that, when applied to the bundle direct sum of the family $\{\mathfrak{V}_i\}_{i=1}^n$, the first part of (6) in Theorem 3.1 is satisfied by global sections belonging to $\mathcal{E}^{\oplus}$. Therefore the statement follows since (5) $\Leftrightarrow$ (6) in Theorem 3.1. □

Convention 1. By construction we have that $\Gamma(\pi_{\mathsf{E}^{\oplus}}) \subset \prod_{x \in X} \{x\} \times \mathsf{E}_x^{\oplus}$. In what follows, except contrary mention, we convein to consider with abuse of language in the obvious manner

$$\Gamma(\pi_{\mathsf{E}^{\oplus}}) \subset \prod_{x \in X} \bigoplus_{i=1}^{n} (\mathfrak{E}_i)_x .$$

Similarly for $\Gamma^x(\pi_{\mathsf{E}^{\oplus}})$ for any $x \in X$. Moreover in the case in which for any $i = 1, ..., n$ we have $\mathfrak{V}_i = \mathfrak{V}(\mathsf{E}_i, \mathcal{E}_i)$, with obvious meaning of the symbols we consider

$$\Gamma(\pi_{\mathsf{E}^{\oplus}}) \subset \prod_{x \in X} \bigoplus_{i=1}^{n} (\mathsf{E}_i)_x .$$

5 $(\Theta, \mathcal{E})$ −structure

In Definition 6 we define the concept of $(\Theta, \mathcal{E})$ −structure. In Lemma 5.1 and Corollary 5.1 we characterize basic properties of this structure. In Theorem 5.1 we construct the $(\Theta, \mathcal{E})$ −structure described in Introduction and provide a set of continuous sections which serves as a model to build the general definition. Finally in Proposition 5.1 we provide a characterization of continuous sections related to a suitable $(\Theta, \mathcal{E})$ −structure. In order to construct the structure provided in Definition 6 we need a sequence of steps starting with the following

Definition 3. $\langle X, \mathsf{E}, \mathcal{S} \rangle$ is a map system if

1. X is a set;
2. $\mathsf{E} = \{\langle \mathsf{E}_x, \mathfrak{N}_x \rangle\}_{x \in X}$ is a nice family of Hlcs with $\mathfrak{N}_x \doteqdot \{\nu_j^x \mid j \in J\}$ for all $x \in X$;
3. $(\exists L \neq \emptyset)(\mathcal{S} = \{S_x\}_{x \in X})$ where $S_x \doteqdot \{B_l^x \mid l \in L\} \subseteq \mathit{Bounded}(\mathsf{E}_x)$ and $\bigcup_{l \in L} B_l^x$ is total in E_x for all $x \in X$.

Definition 4. We say that M is a map pre-bundle relative to $\langle X, Y, \mathsf{E}, \mathcal{S} \rangle$ if

1. $\langle X, \mathsf{E}, \mathcal{S} \rangle$ is a map system;

2. $\mathsf{M} = \{\langle \mathsf{M}_x, \mathfrak{R}_x \rangle\}_{x \in X}$ is a nice family of Hlcs;

3. Y is a Hausdorff topological space and $\forall x \in X$

$$\mathsf{M}_x \subseteq \mathcal{C}\left(Y, \mathcal{L}_{S_x}(\mathsf{E}_x)\right);$$
$$\mathfrak{R}_x = \Big\{ \sup_{(K,j,l) \in \mathcal{O}} q^x_{(K,j,l)} \upharpoonright \mathsf{M}_x \,|\, \mathcal{O} \in \mathcal{P}_\omega\left(Comp(Y) \times J \times L\right) \Big\}.$$

Here we recall that $\mathcal{P}_\omega(A)$ is the set of all finite parts of the set A, $\mathcal{L}_{S_x}(\mathsf{E}_x)$, for all $x \in X$, is the *lcs* of all continuous linear maps $\mathcal{L}(\mathsf{E}_x)$ on E_x with the topology of uniform convergence over the sets in S_x, hence its topology is generated by the following set of seminorms

$$\big\{ p^x_{j,l} : \mathcal{L}(\mathsf{E}_x) \ni \phi \mapsto \sup_{v \in B^x_l} \nu^x_j(\phi(v)) \,|\, l \in L, j \in J \big\}. \tag{5.1}$$

Thus by the totality hypothesis and by [2, Prop. 3, $III.15$] $\mathcal{L}_{S_x}(\mathsf{E}_x)$ is Hausdorff. Finally for all $(K, j, l) \in Comp(Y) \times J \times L$ we set

$$q^x_{(K,j,l)} : \mathcal{C}_c\left(Y, \mathcal{L}_{S_x}(\mathsf{E}_x)\right) \ni f \mapsto \sup_{t \in K} p^x_{j,l}(f(t)). \tag{5.2}$$

Remark 2. By the fact that $\{t\}$ is compact for all $t \in Y$ we have that $\bigcup_{K \in Comp(Y)} K = Y$ thus by the shown fact that $\mathcal{L}_{S_x}(\mathsf{E}_x)$ is Hausdorff we deduce by [1, Proposition (1), §1.2, Ch 10] that $\mathcal{C}_c\left(Y, \mathcal{L}_{S_x}(\mathsf{E}_x)\right)$ is Hausdorff. Moreover by [1, Definition (1), §1.1, Ch 10] and by the fact that (5.1) is a fss on $\mathcal{L}_{S_x}(\mathsf{E}_x)$, we can deduce that $\Big\{ \sup_{(K,j,l) \in \mathcal{O}} q^x_{(K,j,l)} \,|\, \mathcal{O} \in \mathcal{P}_\omega\left(Comp(Y) \times J \times L\right) \Big\}$ is a directed fss on $\mathcal{C}_c\left(Y, \mathcal{L}_{S_x}(\mathsf{E}_x)\right)$. Hence $\langle \mathsf{M}_x, \mathfrak{R}_x \rangle$ is a topological vector subspace of $\mathcal{C}_c\left(Y, \mathcal{L}_{S_x}(\mathsf{E}_x)\right)$ so it is Hausdorff, hence by the construction of $\mathfrak{R}_x$ we can state that $\{\langle \mathsf{M}_x, \mathfrak{R}_x \rangle\}_{x \in X}$ is a nice family of Hlcs in agreement with request (2) in Definition 4.

Next we provide the explicit form of $\mathfrak{V}(\mathsf{M}, \mathcal{M})$.

Remark 3. Let $\mathsf{M} = \{\langle \mathsf{M}_x, \mathfrak{R}_x \rangle\}_{x \in X}$ be a map pre-bundle relative to $\big\langle X, Y, \mathsf{E} = \langle \mathsf{E}_x, \mathfrak{N}_x \rangle_{x \in X}, \mathcal{S} \big\rangle$, moreover let $\mathcal{M}$ satisfy $FM(3) - FM(4)$ with respect to M. Let us denote $\mathfrak{N}_x = \{\nu^x_j \,|\, j \in J\}$ for all $x \in X$ and use the notation in Definition 4. Thus for the bundle $\mathfrak{V}(\mathsf{M}, \mathcal{M})$ generated by the couple $\langle \mathsf{M}, \mathcal{M} \rangle$ we have

1. $\mathfrak{V}(\mathsf{M}, \mathcal{M}) = \langle \langle \mathfrak{E}(\mathsf{M}), \tau(\mathsf{M}, \mathcal{M}) \rangle, \pi_{\mathsf{M}}, X, \mathfrak{R} \rangle$;

2. $\mathfrak{E}(\mathsf{M}) \doteqdot \bigcup_{x \in X} \{x\} \times \mathsf{M}_x$, $\pi_{\mathsf{M}} : \mathfrak{E}(\mathsf{M}) \ni (x, f) \mapsto x \in X$;

3. $\mathfrak{R} = \big\{ \sup_{(K,j,l) \in \mathcal{O}} q_{(K,j,l)} \,|\, \mathcal{O} \in \mathcal{P}_\omega\left(Comp(Y) \times J \times L\right) \big\}$, with $q_{(K,j,l)} : \mathfrak{E}(\mathsf{M}) \ni (x, f) \mapsto q^x_{(K,j,l)}(f)$;

4. $\tau(\mathsf{M}, \mathcal{M})$ is the topology on $\mathfrak{E}(\mathsf{M})$ such that for all $(x, f) \in \mathfrak{E}(\mathsf{M})$

$$\mathcal{I}^{\mathfrak{E}(\mathsf{M})}_{(x,f)} \doteqdot \mathfrak{F}^{\mathfrak{E}(\mathsf{M})}_{\mathcal{B}_{\mathsf{M}}((x,f))}$$

is the neighbourhood's filter of (x, f) with respect to it. Here $\mathfrak{F}^{\mathfrak{E}(\mathsf{M})}_{\mathcal{B}_{\mathsf{M}}((x,f))}$ is the filter on $\mathfrak{E}(\mathsf{M})$ generated by the following filter's base

$$\mathcal{B}_{\mathsf{M}}((x,f)) \doteqdot \{T_{\mathsf{M}}(U,\sigma,\varepsilon,\mathcal{O}) \mid U \in Open(X), \sigma \in \mathcal{M}, \varepsilon > 0,$$
$$\mathcal{O} \in \mathcal{P}_{\omega}(Comp(Y) \times J \times L) \mid x \in U, \sup_{(K,j,l)\in\mathcal{O}} q^{x}_{(K,j,l)}(f - \sigma(x)) < \varepsilon\},$$

where $\forall U \in Open(X), \sigma \in \mathcal{M}, \varepsilon > 0$ and $\forall \mathcal{O} \in \mathcal{P}_{\omega}(Comp(Y) \times J \times L)$

$$T_{\mathsf{M}}(U,\sigma,\varepsilon,\mathcal{O}) \doteqdot \left\{(y,g) \in \mathfrak{E}(\mathsf{M}) \mid y \in U, \sup_{(K,j,l)\in\mathcal{O}} q^{y}_{(K,j,l)}(g - \sigma(y)) < \varepsilon\right\}.$$

Remark 4. Let $\mathsf{M} = \{\langle \mathsf{M}_x, \mathfrak{R}_x\rangle\}_{x\in X}$ be a map pre-bundle relative to $\left\langle X, Y, \mathsf{E} = \langle \mathsf{E}_x, \mathfrak{N}_x\rangle_{x\in X}, \mathcal{S}\right\rangle$, moreover let $\mathcal{M}$ satisfy $FM(3) - FM(4)$ with respect to M. Thus by Remark 12 $\forall U \in Open(X), \sigma \in \mathcal{M}, \varepsilon > 0$ and $\forall \mathcal{O} \in \mathcal{P}_{\omega}(Comp(Y) \times J \times L)$

$$T_{\mathsf{M}}(U,\sigma,\varepsilon,\mathcal{O}) = \bigcup_{y\in U} B_{\mathsf{M}_y,\mathcal{O},\varepsilon}(\sigma(y))$$

where for all $s \in \mathsf{M}_y$

$$B_{\mathsf{M}_y,\mathcal{O},\varepsilon}(s) \doteqdot \left\{(y,f) \in \mathfrak{E}(\mathsf{M})_y \mid \sup_{(K,j,l)\in\mathcal{O}} q^{y}_{(K,j,l)}(f - s) < \varepsilon\right\}.$$

By applying Remark 11 we have the following

Remark 5. Let M be a map pre-bundle relative to $\langle X, Y, \mathsf{E}, \mathcal{S}\rangle$, moreover let $\mathcal{M}$ satisfy $FM(3) - FM(4)$ with respect to M. Thus

1. $\mathfrak{V}(\mathsf{M}, \mathcal{M})$ is a bundle of $\Omega-$spaces;
2. with the notation of Definition 3 $\mathfrak{V}(\mathsf{M}, \mathcal{M})$ is such that
   (a) $\langle \mathfrak{E}(\mathsf{M})_x, \tau(\mathsf{M}, \mathcal{M})\rangle$ as topological vector space is isomorphic to $\langle \mathsf{M}_x, \mathfrak{R}_x\rangle$ for all $x \in X$;
   (b) $\mathcal{M}$ is canonically isomorphic to a linear subspace of $\Gamma(\pi_{\mathsf{M}})$ and if X is compact and $\mathcal{M}$ is a function module, then $\mathcal{M} \simeq \Gamma(\pi_{\mathsf{M}})$.

In Definition 6 we generalize the topology of uniform convergence to bundles $\langle \mathfrak{M}, \rho, X\rangle$ of $\Omega-$spaces, where $\{\mathfrak{M}_x\}_{x\in X}$ is a map pre-bundle relative to $\langle X, Y, \{\mathfrak{E}_x\}_{x\in X}, \mathcal{S}\rangle$ and $\langle \mathfrak{E}, \pi, X\rangle$ is a bundle of $\Omega-$spaces. The aim is to correlate the topology on $\mathfrak{M}$ with that on $\mathfrak{E}$ in order to extend the correlation established in the introduction for the trivial bundle case.

Definition 5.

$$(\bullet) : \prod_{x\in X} (\mathfrak{E}_x)^{\mathfrak{E}_x} \times \prod_{x\in X} \mathfrak{E}_x \to \prod_{x\in X} \mathfrak{E}_x$$

such that for all $F \in \prod_{x\in X} (\mathfrak{E}_x)^{\mathfrak{E}_x}, v \in \prod_{x\in X} \mathfrak{E}_x$ we have

$$(F \bullet w)(x) \doteqdot F(x)(w(x)).$$

Definition 6 ($(\Theta,\mathcal{E})-$structures). We say that $\langle\mathfrak{V},\mathfrak{W},X,Y\rangle$ is a $(\Theta,\mathcal{E})-$structure if

1. $\mathfrak{V}\doteqdot\langle\langle\mathfrak{E},\tau\rangle,\pi,X,\mathfrak{N}\rangle$ is a bundle of $\Omega-$spaces;
2. $\mathcal{E}\subseteq\Gamma(\pi)$;
3. $\Theta\subseteq\prod_{x\in X}Bounded(\mathfrak{E}_x)$;
4. $\forall B\in\Theta$
   (a) $\mathsf{D}(B,\mathcal{E})\neq\emptyset$;
   (b) $\bigcup_{B\in\Theta}\mathcal{B}_B^x$ is total in $\mathfrak{E}_x$ for all $x\in X$;
5. $\mathfrak{W}\doteqdot\langle\langle\mathfrak{M},\gamma\rangle,\rho,X,\mathfrak{R}\rangle$ is a bundle of $\Omega-$spaces such that $\{\langle\mathfrak{M}_x,\mathfrak{R}_x\rangle\}_{x\in X}$ is a map pre-bundle relative to $\langle X,Y,\{\langle\mathfrak{E}_x,\mathfrak{N}_x\rangle\}_{x\in X},\mathcal{S}\rangle$.

Here $\mathcal{S}\doteqdot\{S_x\}_{x\in X}$ and $(\forall B\in\Theta)(\forall x\in X)$

$$\boxed{\begin{cases}\mathsf{D}(B,\mathcal{E})\doteqdot\mathcal{E}\cap\left(\prod_{x\in X}B_x\right)\\ \mathcal{B}_B^x\doteqdot\{v(x)\,|\,v\in\mathsf{D}(B,\mathcal{E})\}\}\\ S_x\doteqdot\{\mathcal{B}_B^x\,|\,B\in\Theta\}.\end{cases}}\tag{5.3}$$

Moreover $\langle\mathfrak{V},\mathfrak{W},X,Y\rangle$ is an invariant $(\Theta,\mathcal{E})-$structure if it is a $(\Theta,\mathcal{E})-$structure such that

$$\left\{F\in\prod_{z\in X}^{b}\mathfrak{M}_z\,|\,(\forall t\in Y)(F_t\bullet\mathcal{E}(\Theta)\subseteq\Gamma(\pi))\right\}=\Gamma(\rho).\tag{5.4}$$

Finally $\langle\mathfrak{V},\mathfrak{W},X,Y\rangle$ is a *compatible* $(\Theta,\mathcal{E})-$*structure* if it is a $(\Theta,\mathcal{E})-$structure such that for all $t\in Y$

$$\Gamma(\rho)_t\bullet\mathcal{E}(\Theta)\subseteq\Gamma(\pi).\tag{5.5}$$

Here

$$\mathcal{E}(\Theta)\doteqdot\bigcup_{B\in\Theta}\mathsf{D}(B,\mathcal{E}),$$

and $S_t\doteqdot\{F_t\,|\,F\in S\}$ and $F_t\in\prod_{x\in X}\mathcal{L}(\mathfrak{E}_x)$ such that $F_t(x)\doteqdot F(x)(t)$, for all $S\subseteq\prod_{x\in X}\mathcal{L}(\mathfrak{E}_x)^Y$ $t\in Y$, and $F\in S$.

Remark 6. Let $\langle\mathfrak{V},\mathfrak{W},X,Y\rangle$ be a $(\Theta,\mathcal{E})-$structure. Then for all $x\in X$

$$\mathfrak{R}_x=\{\sup_{(K,j,B)\in O}q_{(K,j,B)}^x\upharpoonright\mathfrak{M}_x\,|\,O\in\mathcal{P}_\omega(Comp(Y)\times J\times\Theta)\}\tag{5.6}$$

where by using the notation of Definition 6 we set $\mathfrak{N}=\{\nu_j^x\,|\,j\in J\}$ and for all $K\in Comp(Y)$,$j\in J$, $B\in\Theta$

$$q_{(K,j,B)}^x:\mathcal{C}_c\left(Y,\mathcal{L}_{S_x}(\mathfrak{E}_x)\right)\ni f_x\mapsto\sup_{t\in K}\sup_{v\in\mathsf{D}(B,\mathcal{E})}\nu_j^x\left(f_x(t)v(x)\right).\tag{5.7}$$

Remark 7. Let $\mathfrak{V} \doteqdot \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \mathfrak{N} \rangle$ be a bundle, $\mathsf{M} = \{\langle \mathsf{M}_x, \mathfrak{R}_x \rangle\}_{x \in X}$ a map pre-bundle relative to $\langle X, Y, \{\langle \mathfrak{E}_x, \mathfrak{N}_x \rangle\}_{x \in X}, \mathcal{S} \rangle$ and $\mathcal{M}$ satisfy $FM(3)-FM(4)$ with respect to M. Then Remark 3 allows us to construct $\mathfrak{W}$ satisfying the condition (5) in Definition 6.

The following characterization of $\mathcal{U} \in \Gamma_U^{x_\infty}(\rho)$ will be basic in the sequel.

Lemma 5.1. *Let $\langle \mathfrak{V}, \mathfrak{W}, X, Y \rangle$ be a $(\Theta, \mathcal{E})-$structure, $x_\infty \in W \subseteq X$ and $\mathcal{U} \in \prod_{x \in W}^b \mathfrak{M}_x$. By using the notation in Definition 6 we have (1) $\Leftarrow$ (2) $\Leftarrow$ (3) $\Leftrightarrow$ (4) moreover if $\mathfrak{W}$ is locally full (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4), finally if $\mathfrak{W}$ is full we can choose $U = X$ in (2) and $U' = X$ in (3) and (4). Here*

1. $\mathcal{U} \in \Gamma_W^{x_\infty}(\rho)$;

2. $(\exists U \in Op(X) \,|\, U \ni x_\infty)(\exists F \in \Gamma_U(\rho))(F(x_\infty) = \mathcal{U}(x_\infty))$ *such that* $(\forall j \in J)(\forall K \in Comp(Y))(\forall B \in \Theta)$

$$\boxed{\lim_{z \to x_\infty, z \in W \cap U} \sup_{t \in K} \sup_{v \in \mathsf{D}(B, \mathcal{E})} \nu_j \left(\mathcal{U}(z)(t)v(z) - F(z)(t)v(z)\right) = 0;} \tag{5.8}$$

3. $(\exists U' \in Op(X) \,|\, U' \ni x_\infty)(\exists F \in \Gamma_{U'}(\rho))(F(x_\infty) = \mathcal{U}(x_\infty))$ *and* $(\forall U \in Op(X) \,|\, U \ni x_\infty)(\forall F \in \Gamma_U(\rho) \,|\, F(x_\infty) = \mathcal{U}(x_\infty))$ *we have* (5.8) $(\forall j \in J)(\forall K \in Comp(Y))(\forall B \in \Theta)$;

4. $(\exists U' \in Op(X) \,|\, U' \ni x_\infty)(\exists F \in \Gamma_{U'}(\rho))(F(x_\infty) = \mathcal{U}(x_\infty))$ *and* $\mathcal{U} \in \Gamma_W^{x_\infty}(\rho)$.

Proof. Since Corollary 3.1 and Definition 4. □

Corollary 5.1. *Let us assume the hypotheses of Lemma 5.1 and that $\mathfrak{W}$ is full. Moreover let $B \in \Theta$ and $v \in \mathsf{D}(B, \mathcal{E})$. Then (1) $\Rightarrow$ (2), where*

1. $\mathcal{U} \in \Gamma_W^{x_\infty}(\rho)$ *and* $\exists F \in \Gamma(\rho)$ *such that* $F(x_\infty) = \mathcal{U}(x_\infty)$ *and* $(\forall t \in Y)(F(\cdot)(t) \bullet v \in \Gamma(\pi))$;

2. $(\forall t \in X)(\mathcal{U}(\cdot)(t) \bullet v \in \Gamma_W^{x_\infty}(\pi))$.

Proof. By the position (1) and by the implication (1) $\Rightarrow$ (3) of Lemma 5.1 and by the fact that the union of all compact subsets of Y is Y, being locally compact, we deduce that $(\exists F \in \Gamma(\rho))(F(x_\infty) = \mathcal{U}(x_\infty))$ such that $(\forall j \in J)(\forall t \in Y)(\forall B \in \Theta)$ and $\forall v \in \mathsf{D}(B, \mathcal{E})$

$$\begin{cases} \lim_{z \to x_\infty, z \in W} \nu_j \left(\mathcal{U}(z)(t)v(z) - F(z)(t)v(z)\right) = 0, \\ F(\cdot)(t) \bullet v \in \Gamma(\pi). \end{cases}$$

Thus the statement follows by implication (3) $\Rightarrow$ (1) of Corollary 3.1. □

Let us conclude this section with two results constructing a $(\Theta, \mathcal{E})-$structure and describing $\Gamma^{x_\infty}(\rho)$ when $\mathfrak{V}$ is trivial.

Lemma 5.2. *Let Z be a normed space X, Y be two topological spaces. Set for all $x \in X$ and $v \in \mathcal{C}_b(X, Z)$*

$$\begin{cases} \mathcal{M} \doteqdot \{F \in \mathcal{C}_b(X, \mathcal{C}_c(Y, \mathcal{L}_s(Z))) \mid (\forall K \in Comp(Y)) \\ \qquad (C(F,K) \doteqdot \sup_{(x,s) \in X \times K} \|F(x)(s)\|_{B(Z)} < \infty)\}, \\ \mathsf{M}_x \doteqdot \overline{\{F(x) \mid F \in \mathcal{M}\}}, \\ \mu^K_{(v,x)} : \mathsf{M}_x \ni G \mapsto \sup_{s \in K} \|G(s)v(x)\|, \\ \mathcal{A}_x \doteqdot \{\mu^K_{(w,x)} \mid K \in Comp(Y), w \in \mathcal{C}_b(X, Z)\}, \\ \mathsf{M} \doteqdot \{\langle \mathsf{M}_x, \mathcal{A}_x \rangle\}_{x \in X}. \end{cases}$$

closure in $\mathcal{C}_c(Y, B_s(Z))$. Then $\mathcal{M}$ satisfies $FM3 - FM4$ with respect to M

Proof. $FM(3)$ is true by construction, let $v \in \mathcal{C}_b(X, Z)$, $K \in Comp(Y)$, $F \in \mathcal{M}$, then

$$\sup_{x \in X} \mu^K_{(v,x)}(F(x)) \leq \sup_{(x,s) \in X \times K} \|F(x)(s)\|_{B(Z)} \sup_{x \in X} \|v(x)\| < \infty.$$

For all $x, x_0 \in X$

$$\mu^K_{(v,x)}(F(x)) \leq C\|v(x) - v(x_0)\| + \sup_{s \in K} \|F(x)(s)v(x_0)\|, \tag{5.9}$$

where $C \doteqdot \sup_{(x,s) \in X \times K} \|F(x)(s)\|_{B(Z)}$. Moreover the map $\mathcal{C}_c(Y, B_s(Z)) \ni f \mapsto \sup_{s \in K} \|f(s)w\| \in \mathbb{R}^+$, for all $w \in Z$ is a continuous seminorm, hence by the continuity of F also the map $X \ni x \mapsto \sup_{s \in K} \|F(x)(s)w\| \in \mathbb{R}^+$ is continuous. So by (5.9) $\overline{\lim}_{x \to x_0} \mu^K_{(v,x)}(F(x)) \leq \sup_{s \in K} \|F(x_0)(s)v(x_0)\| = \mu^K_{(v,x_0)}(F(x_0))$, and by [1, (15), §5.6] we have

$$\overline{\lim_{x \to x_0}} \mu^K_{(v,x)}(F(x)) = \mu^K_{(v,x_0)}(F(x_0)).$$

Therefore by [1, (13), §5.6], [1, Proposition 3, §6.2], and the fact that any map g is *u.s.c.* at a point if and only if $-g$ is *l.s.c.*, we can state that $X \ni x \mapsto \mu^K_{(v,x)}(F(x))$ is *u.s.c.* at x_0 for all $x_0 \in X$, hence it is *u.s.c.*, which is the $FM(4)$ condition. □

Remark 8. Let $\mathfrak{V} \doteqdot \langle\langle \mathfrak{E}, \tau \rangle, \pi, X, \mathfrak{N}\rangle$ be a bundle of $\Omega-$spaces and $\mathcal{E} \subseteq \prod_{x \in X} \mathfrak{E}_x$. Set for all $v \in \prod_{x \in X} \mathfrak{E}_x$

$$\begin{cases} B_v : X \ni x \mapsto \{v(x)\}, \\ \Theta \doteqdot \{B_w \mid w \in \mathcal{E}\} \end{cases}$$

Thus $\Theta \subset \prod_{x \in X} Bounded(\mathfrak{E}_x)$ and $\forall v \in \mathcal{E}$

$$\mathcal{E} \cap \prod_{x \in X} B_v(x) = \{v\}.$$

Therefore for all $v \in \mathcal{E}$, and for all $x \in X$ with the notation of Definition 6

$$\begin{cases} \mathsf{D}(B_v, \mathcal{E}) = \{v\}, \\ \mathcal{B}^x_{B_v} = \{v(x)\}, \\ S_x = \{\{w(x)\} \mid w \in \mathcal{E}\}, \\ \mathcal{E}(\Theta) = \mathcal{E}. \end{cases}$$

By Lemma 5.2 and Definition 15 we can construct the bundle $\mathfrak{V}(\mathsf{M},\mathcal{M})$ generated by the couple $\langle \mathsf{M},\mathcal{M}\rangle$. In the following result we construct a $(\Theta,\mathcal{E})-$structure and describe a subset of $\Gamma^{x_\infty}(\rho)$.

Theorem 5.1. *Let us assume the notation and hypotheses of Lemma 5.2, let $\mathfrak{V}$ be the trivial Banach bundle with constant stalk Z and set $\Theta \doteqdot \{B_v \mid v \in \mathcal{C}_b(X,Z)\}$. Then*

1. *$\langle \mathfrak{V}, \mathfrak{V}(\mathsf{M},\mathcal{M}), X, Y\rangle$ is a $(\Theta, \mathcal{C}_b(X,Z))-$ structure, moreover if X is compact and Y is locally compact then it is compatible;*

2. *Let $f \in \prod_{x\in X} \mathsf{M}_x$ be such that $\sup_{(x,s)\in X\times K} \|f(x)(s)\|_{B(Z)} < \infty$ for all $K \in Comp(Y)$ then (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d), where*

   (a) $f \in \Gamma^{x_\infty}(\pi_{\mathsf{M}})$;

   (b) $(\forall K \in Comp(Y))(\forall v \in \mathcal{C}_b(X,Z))$
$$\lim_{x\to x_\infty} \sup_{s\in K} \|f(x)(s)v(x) - f(x_\infty)(s)v(x)\| = 0$$

   (c) $f : X \to \mathcal{C}_c(Y, B_s(Z))$ *continuous at* x_∞;

   (d) $(\forall K \in Comp(Y))(\forall w \in Z)$
$$\lim_{x\to x_\infty} \sup_{s\in K} \|f(x)(s)w - f(x_\infty)(s)w\| = 0.$$

Proof. By Remark 7 and Lemma 5.2 we have that (5) of Definition 6 follows. $\Gamma(\pi) \simeq \mathcal{C}_b(X,Z)$ hence by Remark 8 the other requests of Definition 6 follow. Thus the first sentence of statement (1). If X is compact by Lemma 5.2 and Remark 11 follows that $\mathcal{M} \simeq \Gamma(\pi_{\mathsf{M}})$, moreover by Remark 8 we have $\mathcal{E}(\Theta) = \mathcal{E}$ and finally $\mathcal{E} \doteq \Gamma(\pi) \simeq \mathcal{C}_b(X,Z)$. Hence the second sentence of statement (1) follows if we show that $\mathcal{M}_t \bullet \mathcal{C}_b(X,Z) \subseteq \mathcal{C}_b(X,Z)$. To this end fix $v \in \mathcal{C}_b(X,Z)$, $F \in \mathcal{M}$, $s \in Y$ and K_s a compact neighbourhood of s, which there exists by the hypothesis that Y is locally compact. Then we have for all $x, x_0 \in X$

$$\begin{aligned}&\|F(x)(s)v(x) - F(x_\infty)(s)v(x_0)\| \leq \\ &C(F,K_s)\|v(x) - v(x_0)\| + \|\left(F(x)(s) - F(x_0)(s)\right)v(x_0)\|\end{aligned} \tag{5.10}$$

By considering that $F \in \mathcal{C}_b(X, \mathcal{C}_c(Y, B_s(Z)))$ and that $s \in K_s$ we have that $\lim_{x\to x_0} \|(F(x)(s) - F(x_0)(s))v(x_0)\| = 0$. Hence by (5.10) we deduce that $F_s \bullet v$ is continuous at x_0, so continuous on X, in particular X being compact it is also $\|\cdot\|_Z-$bounded. Thus $F_s \bullet v \in \mathcal{C}_b(X,Z)$ and the second sentence of the statement follows.

Fix $f \in \prod_{x\in X} \mathsf{M}_x$ such that $(\forall K \in Comp(Y))(C(f,K) \doteqdot \sup_{(x,s)\in X\times K} \|f(x)(s)\|_{B(Z)} < \infty)$. $(a) \Leftrightarrow (b)$ follows by Lemma 5.1, the fact that $\mathcal{M} \subseteq \Gamma(\pi_{\mathsf{M}})$ by Remark 11, and by $(H : X \ni x \mapsto f(x_\infty) \in \mathcal{C}_c(Y, B_s(Z))) \in \mathcal{M}$, indeed H it is bounded and continuous being constant, moreover $\sup_{(x,s)\in X\times K} \|H(x)(s)\|_{B(Z)} = \sup_{s\in K} \|f(x_\infty)(s)\|_{B(Z)} < \infty$, for all $K \in Comp(Y)$. $(b) \Rightarrow (d)$ follows by the fact that $(X \ni x \mapsto w \in Z) \in \mathcal{C}_b(X,Z)$, and $(c) \Leftrightarrow (d)$ is trivial. For all $K \in Comp(Y)$, $x \in X$ and $s \in K$

$$\begin{aligned}&\|(f(x)(s) - f(x_\infty)(s))v(x)\| \leq \\ &\|f(x)(s)v(x) - f(x_\infty)(s)v(x_\infty)\| + \|f(x_\infty)(s)v(x_\infty) - f(x_\infty)(s)v(x)\| \leq \\ &\|f(x)(s)(v(x) - v(x_\infty))\| + \|(f(x)(s) - f(x_\infty)(s))v(x_\infty)\| + \|f(x_\infty)(s)(v(x_\infty) - v(x))\| \leq \\ &(\|f(x)(s)\| + \|f(x_\infty)(s)\|)\,\|v(x_\infty) - v(x)\| + \|(f(x)(s) - f(x_\infty)(s))v(x_\infty)\| \leq \\ &2C(f,K)\|v(x_\infty) - v(x)\| + \|(f(x)(s) - f(x_\infty)(s))v(x_\infty)\|.\end{aligned}$$

Hence (d) implies (b). □

Definition 7. Let $\langle \mathfrak{V}, \mathfrak{W}, X, Y\rangle$ be a $(\Theta, \mathcal{E})$ −structure, $Y_0 \subset Y$ and $\mathcal{V} \in \prod_{x\in X} \mathfrak{M}_x$. We say that $\mathcal{V}$ is equicontinuous on Y_0 if and only if $(\forall j \in J)(\exists a > 0)(\exists\, j_1 \in J)(\forall z \in X)(\forall v_z \in \mathfrak{E}_z)$

$$\sup_{t\in Y_0} \nu_j \left(\mathcal{V}(z)(t)v_z\right) \leq a\nu_{j_1}(v_z). \tag{5.11}$$

$\mathcal{V}$ is equicontinuous if and only if it is equicontinuous on Y. $\mathcal{V}$ is pointwise equicontinuous if and only if it is equicontinuous on every point of Y and compactly equicontinuous if and only if it is equicontinuous on every compact of Y.

Note that in case $\mathfrak{V}$ is trivial with costant stalk E then $\mathcal{V}$ is equicontinuous on Y_0 if and only if it is equicontinuous in the standard sense[9] the following set of maps $\{\mathcal{V}_0(z)(t) \in \mathcal{L}(E) \,|\, (z,t) \in X \times Y_0\}$, where $\mathcal{V}_0 \in \left(\mathcal{L}(E)^Y\right)^X$ such that $\mathcal{V}(z) = (z, \mathcal{V}_0(z))$ for all $z \in X$.

Proposition 5.1. *Let $\mathfrak{V}$ be trivial with costant stalk E, $A^0 \in Bounded(E)$, $x_\infty \in X$ and*

$$\begin{cases} \mathcal{E}_0 \subseteq \mathcal{C}_b\left(X, E\right) \\ \mathcal{E}_0 \text{ equicontinuous set at } x_\infty \\ \{(X \ni x \mapsto a \in E) \,|\, a \in A^0\} \subset \mathcal{E}_0. \end{cases} \tag{5.12}$$

Moreover let $\langle \mathfrak{V}, \mathfrak{W}, X, Y\rangle$ be a $(\Theta, \mathcal{E})$ −structure such that for all $x \in X$

$$\mathfrak{M}_x = \mathcal{C}_c\left(Y, \mathcal{L}_{S_x}(\{x\} \times E)\right).$$

and

$$\begin{cases} \mathcal{E} = \prod_{x\in X}\{x\} \times \mathcal{E}_0 \\ \Theta = \{B_{A^0}\} \end{cases}$$

where $B_{A^0}(x) \doteqdot \{x\} \times A^0$, then

$$\begin{cases} S_x = \{x\} \times A^0, \forall x \in X \\ \mathfrak{M}_x \simeq \{x\} \times \mathcal{C}_c\left(Y, \mathcal{L}_{A^0}(E)\right). \\ \mathfrak{M} = \bigcup_{x\in X} \mathfrak{M}_x \simeq \bigcup_{x\in X}\{x\} \times \mathcal{C}_c\left(Y, \mathcal{L}_{A^0}(E)\right) \\ \prod_{x\in X} \mathfrak{M}_x \simeq \prod_{x\in X}\{x\} \times \mathcal{C}_c\left(Y, \mathcal{L}_{A^0}(E)\right) \simeq \mathcal{C}_c\left(Y, \mathcal{L}_{A^0}(E)\right)^X. \end{cases} \tag{5.13}$$

If $\mathfrak{W}$ is full and

$$\{X \ni x \mapsto \mathfrak{t}_f(x) = (x, f) \in \mathfrak{M}_x \,|\, f \in \mathcal{C}_c\left(Y, \mathcal{L}_{A^0}(E)\right)\} \subset \Gamma(\rho),$$

then for all $\mathcal{V} \in \prod_{x\in X}^b \mathfrak{M}_x$, (1) ⇒ (2) and (3) ⇔ (4), where

1. *$\mathcal{V} \in \Gamma^{x_\infty}(\rho)$*
2. *$\mathcal{V}_0 \in \mathcal{C}\left(X, \mathcal{C}_c\left(Y, \mathcal{L}_{A^0}(E)\right)\right)$,*
3. *$\mathcal{V}$ is compactly equicontinuous and $\mathcal{V} \in \Gamma^{x_\infty}(\rho)$*

[9]See for instance [1, Def 1, §2.1, Ch. 10].

4. $\mathcal{V}$ is compactly equicontinuous and $\mathcal{V}_0 \in \mathcal{C}\left(X, \mathcal{C}_c\left(Y, \mathcal{L}_{A^0}(E)\right)\right)$.

Here in (2) – (4) we consider the isomorphism $\prod_{x \in X} \mathfrak{M}_x \simeq \mathcal{C}_c\left(Y, \mathcal{L}_{A^0}(E)\right)^X$, and set $\mathcal{V}_0 \in \mathcal{C}_c\left(Y, \mathcal{L}_{A^0}(E)\right)^X$ such that $\mathcal{V}(x) = (x, \mathcal{V}_0(x))$ for all $x \in X$.

Proof. For all $x \in X$ by (5.3) $\mathcal{B}^x_{B_{A^0}} = \{(x, v_0(x)) \,|\, v_0 \in \mathcal{E}_0, v_0(X) \subseteq A^0\}$ so $\mathcal{B}^x_{B_{A^0}} \subseteq A^0$. Moreover by construction $(X \ni x \mapsto a \in E) \in \mathcal{E}_0$ for all $a \in A^0$, thus $\mathcal{B}^x_{B_{A^0}} = A^0$. Thus the first equality in (5.13) follows, the others are trivial. By Proposition 3.2

$$(1) \Leftrightarrow \lim_{z \to x_\infty} \sup_{t \in K} \sup_{v_0 \in \mathcal{E}_0 \cap B_{A^0}} \nu_j\left((\mathcal{V}_0(z)(t) - \mathcal{V}_0(x_\infty)(t)) v_0(z)\right) = 0. \tag{5.14}$$

Moreover by construction we deduce that $\{(X \ni x \mapsto a \in E) \,|\, a \in A^0\} \subset \mathcal{E}_0 \cap B_{A^0}$, so (2) follows by (1) and (5.14). Let $v_0 \in \mathcal{E}_0$ then for all $z \in X$ and $t \in Y$

$$\begin{aligned}(\mathcal{V}(z)(t) - \mathcal{V}(x_\infty)(t)) v_0(z) = \mathcal{V}(z)(t)(v_0(z) - v_0(x_\infty)) + \\ (\mathcal{V}(z)(t) - \mathcal{V}(x_\infty)(t)) v_0(x_\infty) + \mathcal{V}(x_\infty)(t)(v_0(z) - v_0(x_\infty)).\end{aligned} \tag{5.15}$$

Moreover by the hypothesis of equicontinuity at x_∞ of the set $\mathcal{E}_0$, for all $j \in J$

$$\lim_{z \to x_\infty} \sup_{v_0 \in \mathcal{E}_0} \nu_j(v_0(z) - v_0(x_\infty)) = 0. \tag{5.16}$$

By (5.15) and (5.11) for all $j \in J$ there exists $j_1 \in J$ and $a > 0$ such that for all $z \in X$

$$\begin{aligned}&\sup_{t \in K} \sup_{v_0 \in \mathcal{E}_0 \cap B_{A^0}} \nu_j\left((\mathcal{V}_0(z)(t) - \mathcal{V}_0(x_\infty)(t)) v_0(z)\right) \leq \\ &2a \sup_{v_0 \in \mathcal{E}_0 \cap B_{A^0}} \nu_{j_1}\left(v_0(z) - v_0(x_\infty)\right) + \\ &\sup_{t \in K} \sup_{v_0 \in \mathcal{E}_0 \cap B_{A^0}} \nu_j\left(\mathcal{V}(z)(t) - \mathcal{V}(x_\infty)(t)\right) v_0(x_\infty).\end{aligned} \tag{5.17}$$

Therefore by (5.17), (5.16) and by (4) follows

$$\lim_{z \to x_\infty} \sup_{t \in K} \sup_{v_0 \in \mathcal{E}_0 \cap B_{A^0}} \nu_j\left((\mathcal{V}_0(z)(t) - \mathcal{V}_0(x_\infty)(t)) v_0(z)\right) = 0.$$

Hence (1) follows by (5.14). □

6 Main claim

In this section we state in a precise way the claims outlined in Introduction. The main Claim 6.1 which essentially establishes the existence of $\mathcal{T}$ and $\mathcal{P}$ satisfying (1.1), (1.14) and (1.15). The auxiliary Claim 6.2 which provides $\mathcal{U}$ satisfying (1.13) and then Claim 6.3. Proposition 6.1 provides the main properties of those realizations of the main claim obtained combining realizations of the two auxiliary ones. We anticipate that [19, Theorem 4.2] resolves the main claim in this fashion. In what follows when dealing with bundle direct sums we use the notation provided in Remark 1.

Definition 8. Let $\mathfrak{V}_i \doteqdot \langle\langle\mathfrak{E}_i,\tau_i\rangle,\pi_i,X,\mathfrak{N}_i\rangle$ be a full bundle of $\Omega-$spaces for any $i=1,2$. Then we call set of graph sections relative to $\mathfrak{V}_1$ and $\mathfrak{V}_2$ the set $\mathsf{Gr}(\mathfrak{V}_1,\mathfrak{V}_2)$ of the elements $\langle\mathcal{T},x_\infty,\Phi\rangle$ such that

1. $\mathcal{T}\in\prod_{x\in X} Graph((\mathfrak{E}_1)_x\times(\mathfrak{E}_2)_x)$;
2. $x_\infty\in X$;
3. Φ is a linear subspace of $\Gamma^{x_\infty}(\pi_{\mathsf{E}^\oplus})$;
4. $(\forall x\in X)(\forall\phi\in\Phi)(\phi(x)\in\mathcal{T}(x))$
5. Asymptotic Graph

$$\boxed{\{\phi(x_\infty)\,|\,\phi\in\Phi\}=\mathcal{T}(x_\infty).} \tag{6.1}$$

Definition 9. Let $\mathfrak{V}_i \doteqdot \langle\langle\mathfrak{E}_i,\tau_i\rangle,\pi_i,X,\mathfrak{N}_i\rangle$ be a full bundle of $\Omega-$spaces for any $i=1,2$. Then we call set of pregraph sections relative to $\mathfrak{V}_1$ and $\mathfrak{V}_2$ the set $Pregraph(\mathfrak{V}_1,\mathfrak{V}_2)$ of the elements $\langle\mathcal{T}_0,x_\infty,\Phi\rangle$ such that

1. $x_\infty\in X$;
2. $\mathcal{T}_0\in\prod_{x\in X-\{x_\infty\}} Graph((\mathfrak{E}_1)_x\times(\mathfrak{E}_2)_x)$;
3. Φ is a linear subspace of $\Gamma^{x_\infty}(\pi_{\mathsf{E}^\oplus})$;
4. $(\forall x\in X-\{x_\infty\})(\forall\phi\in\Phi)(\phi(x)\in\mathcal{T}_0(x))$.

We shall see in [18, Lemma 2.1] that it is possible to construct from any suitable pregraph section $\langle\mathcal{T}_0,x_\infty,\Phi\rangle$ a corresponding graph section $\langle\mathcal{T},x_\infty,\Phi\rangle$ such that $\mathcal{T}$ extends $\mathcal{T}_0$, while $\mathcal{T}(x_\infty)$ is defined by (6.1). To this end it is sufficient to show that $\mathcal{T}(x_\infty)\in Graph((\mathfrak{E}_1)_{x_\infty}\times(\mathfrak{E}_2)_{x_\infty})$.

Remark 9. The request that any $\phi\in\Phi$ is a section continuous in x_∞ implies that

$$\begin{cases}\{\lim_{z\to x_\infty}\phi(z)\,|\,\phi\in\Phi\}=\mathcal{T}(x_\infty)\in Graph((\mathfrak{E}_1)_{x_\infty}\times(\mathfrak{E}_2)_{x_\infty})\\ \text{with}\\ \phi(z)\in\mathcal{T}(z)\in Graph((\mathfrak{E}_1)_z\times(\mathfrak{E}_2)_z),\ \forall z\in X-\{x_\infty\},\end{cases}$$

which justifies the name of asymptotic graph given to (6.1). Moreover by setting $X\ni z\mapsto\phi_i(z)\doteqdot\mathrm{Pr}_i^z(\phi(z))$ we have by Corollary 4.1 for all $i=1,2$

$$\begin{cases}\{\lim_{z\to x_\infty}\phi_i(z)\,|\,\phi\in\Phi\}=\mathrm{Pr}_i^{x_\infty}(\mathcal{T}(x_\infty))\\ \text{with}\\ \phi(z)\in\mathcal{T}(z)\in Graph((\mathfrak{E}_1)_z\times(\mathfrak{E}_2)_z),\ \forall z\in X-\{x_\infty\}.\end{cases} \tag{6.2}$$

Finally for $i=1,2$ by Corollary 4.1 and Corollary 3.1 we have $(1_i)\Leftrightarrow(2_i)$

(1_i) $(\exists\,\sigma\in\Gamma(\pi))(\sigma(x_\infty)=\phi_i(x_\infty))$ such that

$$(\forall j\in J)(\lim_{z\to x_\infty}\nu_j(\phi_i(z)-\sigma(z))=0);$$

(2_i) $(\forall \sigma \in \Gamma(\pi) \,|\, \sigma(x_\infty) = \phi_i(x_\infty))$ we have

$$(\forall j \in J)(\lim_{z \to x_\infty} \nu_j(\phi_i(z) - \sigma(z)) = 0).$$

Definition 10. Let $\langle \mathfrak{V}, \mathfrak{D}, X, \{pt\} \rangle$ be a $(\Theta, \mathcal{E})-$structure such that $\mathfrak{V}$ is full and let us denote $\mathfrak{D} \doteqdot \langle \langle \mathfrak{B}, \gamma \rangle, \eta, X, \mathfrak{L} \rangle$. Thus $\Omega \in \Delta \langle \mathfrak{V}, \mathfrak{D}, \Theta, \mathcal{E} \rangle$ if

1. $\Omega \subseteq \mathsf{Gr}(\mathfrak{V}, \mathfrak{V})$;
2. Section of projectors associated with $\langle \mathcal{T}, x_\infty, \Phi \rangle$: $\forall \langle \mathcal{T}, x_\infty, \Phi \rangle \in \Omega$

$$\boxed{(\exists \mathcal{P} \in \Gamma^{x_\infty}(\eta) \cap \prod_{x \in X} \Pr(\mathfrak{E}_x))\,(\forall x \in X)\,(\mathcal{P}(x) T_x \subseteq T_x \mathcal{P}(x))\,.} \tag{6.3}$$

Here $T_x : D_x \subseteq \mathfrak{E}_x \to \mathfrak{E}_x$ is the map such that $\mathcal{T}(x) = Graph(T_x)$, for all $x \in X$.

Claim 6.1 *(MAIN).* Under the assumptions in Definition 10, possibly with $\langle \mathfrak{V}, \mathfrak{D}, X, \{pt\} \rangle$ invariant, find elements in the set

$$\Delta \langle \mathfrak{V}, \mathfrak{D}, \Theta, \mathcal{E} \rangle\,.$$

Definition 11. Let $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ be a $(\Theta, \mathcal{E})-$structure. Let us denote $\mathfrak{V} \doteqdot \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \mathfrak{N} \rangle$ and $\mathfrak{W} \doteqdot \langle \langle \mathfrak{M}, \gamma \rangle, \rho, X, \mathfrak{R} \rangle$. We require that $\mathfrak{V}$ is full, $\{\mathfrak{E}_x\}_{x \in X}$ is a family of sequentially complete Hlcs and $\mathsf{U}(\mathcal{L}_{S_x}(\mathfrak{E}_x)) \subset \mathfrak{M}_x$ for all $x \in X$. Then $\Omega \in \Delta_\Theta \langle \mathfrak{V}, \mathfrak{W}, \mathcal{E}, X, \mathbb{R}^+ \rangle$ if and only if

1. $\Omega \subseteq \mathsf{Gr}(\mathfrak{V}, \mathfrak{V})$;
2. Section of semigroups associated with $\langle \mathcal{T}, x_\infty, \Phi \rangle$: $\forall \langle \mathcal{T}, x_\infty, \Phi \rangle \in \Omega$

   $$\boxed{\exists \mathcal{U}_{\langle \mathcal{T}, x_\infty, \Phi \rangle} \in \Gamma^{x_\infty}(\rho)}$$

   such that $\forall x \in X$

   (a) $\mathcal{U}_{\langle \mathcal{T}, x_\infty, \Phi \rangle}(x)$ is an equicontinuous $(C_0)-$semigroup on $\mathfrak{E}_x$;
   (b) $(\forall x \in X)(\mathcal{T}(x) = Graph(R_x))$.

Here R_x is the infinitesimal generator of the semigroup $\mathcal{U}_{\langle \mathcal{T}, x_\infty, \Phi \rangle}(x) \in \mathcal{C}_c(\mathbb{R}^+, \mathcal{L}_{S_x}(\mathfrak{E}_x))$.

Claim 6.2 *(S).* Under the assumptions in Definition 11, possibly with $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ compatible, find elements in the set

$$\Delta_\Theta \left\langle \mathfrak{V}, \mathfrak{W}, \mathcal{E}, X, \mathbb{R}^+ \right\rangle.$$

Remark 10. Notice that $\forall \langle \mathcal{T}, x_\infty, \Phi \rangle \in \Omega$ there exists only one semigroup section associated with it. Moreover $\mathcal{U}_{\langle \mathcal{T}, x_\infty, \Phi \rangle}$ is characterized by any of the equivalent conditions in Lemma 5.1 with $U = X$ and $Y = \mathbb{R}^+$.

Definition 12. Let $\langle\mathfrak{V},\mathfrak{W},X,\mathbb{R}^+\rangle$ be a $(\Theta,\mathcal{E})-$structure and $\langle\mathfrak{V},\mathfrak{D},X,\{pt\}\rangle$ be a $(\Theta,\mathcal{E})-$structure. Let us denote $\mathfrak{V} \doteqdot \langle\langle\mathfrak{E},\tau\rangle,\pi,X,\mathfrak{N}\rangle$, $\mathfrak{D} \doteqdot \langle\langle\mathfrak{B},\gamma\rangle,\eta,X,\mathfrak{L}\rangle$ and $\mathfrak{W} \doteqdot \langle\langle\mathfrak{M},\gamma\rangle,\rho,X,\mathfrak{R}\rangle$. We require that $\mathfrak{V}$ is full, $\{\mathfrak{E}_x\}_{x\in X}$ is a family of sequentially complete Hlcs and $\mathsf{U}(\mathcal{L}_{S_x}(\mathfrak{E}_x)) \subset \mathfrak{M}_x$ for all $x\in X$. Then $\Psi\in\Delta_\Theta\langle\mathfrak{V},\mathfrak{D},\mathfrak{W},\mathcal{E},X,\mathbb{R}^+\rangle$ if and only if

1. $\Psi\subseteq\bigcup_{z\in X}\Gamma^z(\rho)$;
2. $(\forall\mathcal{U}\in\Psi)(\forall x\in X)$ ($\mathcal{U}(x)$ is an equicontinuous $(C_0)-$semigroup on $\mathfrak{E}_x$);
3. Section of projectors associated with $\mathcal{U}$: $(\forall z\in X)(\forall\mathcal{U}\in\Psi\cap\Gamma^z(\rho))$

$$(\exists\,\mathcal{P}\in\Gamma^z(\eta)\cap\prod_{y\in X}\mathrm{Pr}(\mathfrak{E}_y))(\forall x\in X)(\mathcal{P}(x)H_x\subseteq H_x\mathcal{P}(x)). \tag{6.4}$$

Here H_x is the infinitesimal generator of the semigroup $\mathcal{U}(x)\in\mathcal{C}_c(\mathbb{R}^+,\mathcal{L}_{S_x}(\mathfrak{E}_x))$ for all $x\in X$.

Claim 6.3 *(S-P).* Under the assumptions in Definition 12, possibly with $\langle\mathfrak{V},\mathfrak{W},X,\mathbb{R}^+\rangle$ compatible and $\langle\mathfrak{V},\mathfrak{D},X,\{pt\}\rangle$ invariant, find elements in the set $\Delta_\Theta\langle\mathfrak{V},\mathfrak{D},\mathfrak{W},\mathcal{E},X,\mathbb{R}^+\rangle$.

Claims 6.2 and 6.3 can be used to solve the main claim 6.1 indeed

Proposition 6.1. *Under the notation and request in Definition 12 assume that*

1. $\Omega\in\Delta_\Theta\langle\mathfrak{V},\mathfrak{W},\mathcal{E},X,\mathbb{R}^+\rangle$;
2. $\Psi\in\Delta_\Theta\langle\mathfrak{V},\mathfrak{D},\mathfrak{W},\mathcal{E},X,\mathbb{R}^+\rangle$;
3. $(\forall\langle\mathcal{T},x_\infty,\Phi\rangle\in\Omega)(\mathcal{U}_{\langle\mathcal{T},x_\infty,\Phi\rangle}\in\Psi)$.

Thus $\Omega\in\Delta\langle\mathfrak{V},\mathfrak{D},\Theta,\mathcal{E}\rangle$*, namely* Ω *satisfies the claim 6.1. Moreover*

$$(\forall\langle\mathcal{T},x_\infty,\Phi\rangle\in\Omega)\,(\exists\,\mathcal{P}\in\Gamma^{x_\infty}(\eta))\,(\exists\,\mathcal{U}\in\Gamma^{x_\infty}(\rho))$$

1. $\mathcal{U}(x)$ *is an equicontinuous* $(C_0)-$*semigroup on* $\mathfrak{E}_x$*, for all* $x\in X$*;*
2. $(\forall x\in X)\,(\mathcal{P}(x)\in\mathrm{Pr}(\mathfrak{E}_x))$*;*
3. $(\forall x\in X)(\mathcal{T}(x)=Graph(R_x))$*;*
4. $\forall x\in X$

$$\mathcal{P}(x)R_x\subseteq R_x\mathcal{P}(x).$$

Here R_x *is the infinitesimal generator of the semigroup* $\mathcal{U}(x)\in\mathcal{C}_c(\mathbb{R}^+,\mathcal{L}_{S_x}(\mathfrak{E}_x))$*, for all* $x\in X$*.*

We conclude this chapter by anticipating that [18, Theorem 2.1] resolves Claim 6.2 while [19, Theorem 4.2] resolves Claims 6.1 and 6.3.

7 Appendix

Excluding Definition 14 which is ours, in this appendix we provide some of those definitions essentially present in [10] we need in the work and some simple results concerning them. In this section X is a topological space.

Definition 13 ($FM(3) - FM(4)$ in §5 of [10]). Let $\mathsf{V} \doteqdot \{\langle V_x, \mathcal{A}_x \rangle\}_{x \in X}$ be a nice family of Hlcs with $\mathcal{A}_x \doteqdot \{\mu_j^x\}_{j \in J}$ for all $x \in X$. We say that $\mathcal{G}$ satisfies $FM(3) - FM(4)$ with respect to V if $\mathcal{G}$ is a linear subspace of $\prod_{x \in X}^b \langle V_x, \mathcal{A}_x \rangle$ and

$FM(3)$ $\{f(x) \mid f \in \mathcal{G}\}$ is dense in V_x for all $x \in X$;

$FM(4)$ $X \ni x \mapsto \mu_j^x(f(x))$ is u.s.c. $\forall j \in J$ and $\forall f \in \mathcal{G}$.

We introduce a stronger condition namely we say that $\mathcal{G}$ satisfies $FM(3^*) - FM(4)$ with respect to V if $FM(3^*)$ and $FM(4)$ hold where

$$(\forall x \in X)(\{f(x) \mid f \in \mathcal{G}\} = V_x). \qquad FM(3^*)$$

Definition 14. Let $\mathsf{V}' \doteqdot \{\langle V_x, \mathcal{A}'_x \rangle\}_{x \in X}$ be a family of Hlcs where $\mathcal{A}'_x \doteqdot \{\mu_{j_x}^x\}_{j_x \in J_x}$ is a directed family of seminorms on V_x generating the locally convex topology on it, for all $x \in X$. Then we set

$$\begin{cases} J \doteqdot \prod_{x \in X} J_x; \\ \mu_j^x \doteqdot \mu_{j(x)}^x, \ \forall x \in X, j \in J; \\ \mathcal{A}_x \doteqdot \{\mu_j^x\}_{j \in J}, \ \forall x \in X. \end{cases}$$

Clearly the range of $\mathcal{A}_x$ equals that of $\mathcal{A}'_x$ and $\mathcal{A}_x$ is directed. Thus $\mathsf{V} \doteqdot \{\langle V_x, \mathcal{A}_x \rangle\}_{x \in X}$ is a nice family of Hlcs, called the *nice family of Hlcs associated with* V'.

Definition 15 (Essentially §5.2, §5.3 and Proposition 5.8 of [10]). Let $\mathsf{E} = \{\langle \mathsf{E}_x, \mathfrak{N}_x \rangle\}_{x \in X}$ be a nice family of Hlcs with $\mathfrak{N}_x \doteqdot \{\nu_j^x \mid j \in J\}$ for all $x \in X$. Moreover let $\mathcal{E}$ satisfy $FM(3) - FM(4)$ with respect to E. Since [10, Proposition 5.8] we can define

$$\mathfrak{V}(\mathsf{E}, \mathcal{E})$$

to be the bundle generated by $\langle \mathsf{E}, \mathcal{E} \rangle$, if

1. $\mathfrak{V}(\mathsf{E}, \mathcal{E}) = \langle \langle \mathfrak{E}(\mathsf{E}), \tau(\mathsf{E}, \mathcal{E}) \rangle, \pi_{\mathsf{E}}, X, \mathfrak{N} \rangle$;

2. $\mathfrak{E}(\mathsf{E}) \doteqdot \bigcup_{x \in X} \{x\} \times \mathsf{E}_x$, $\pi_{\mathsf{E}} : \mathfrak{E}(\mathsf{E}) \ni (x, v) \mapsto x \in X$.

3. $\mathfrak{N} = \{\nu_j \mid j \in J\}$, with $\nu_j : \mathfrak{E}(\mathsf{E}) \ni (x, v) \mapsto \nu_j^x(v)$;

4. $\tau(\mathsf{E}, \mathcal{E})$ is the topology on $\mathfrak{E}(\mathsf{E})$[10] such that for all $(x, v) \in \mathfrak{E}(\mathsf{E})$

$$\mathcal{I}_{(x,v)}^{\tau(\mathsf{E}, \mathcal{E})} \doteqdot \mathfrak{F}_{\mathcal{B}_{\mathsf{E}}((x,v))}^{\mathfrak{E}(\mathsf{E})}.$$

[10] By applying [10, §5.3.] and [1, Ch.1] we know that this topology exists.

Here we recall that $\mathcal{I}_{(x,v)}^{\tau(\mathsf{E},\mathcal{E})}$ is the filter of neighbourhoods of (x,v) with respect to the topology $\tau(\mathsf{E},\mathcal{E})$, while $\mathfrak{F}_{\mathcal{B}((x,v))}^{\mathfrak{E}(\mathsf{E})}$ is the filter on $\mathfrak{E}(\mathsf{E})$ generated by the following base of filters

$$\mathcal{B}_{\mathsf{E}}((x,v)) \doteqdot \{T_{\mathsf{E}}(U,\sigma,\varepsilon,j) \,|\, U \in Open(X), \sigma \in \mathcal{E}, \varepsilon > 0, j \in J, \\ U \ni x, \nu_j^x(v - \sigma(x)) < \varepsilon\},$$

where

$$T_{\mathsf{E}}(U,\sigma,\varepsilon,j) \doteqdot \left\{(y,w) \in \mathfrak{E}(\mathsf{E}) \,|\, y \in U, \nu_j^y(w - \sigma(y)) < \varepsilon\right\}. \tag{7.1}$$

$\mathcal{E}$ is canonically isomorphic to a linear subspace of $\Gamma(\pi_{\mathsf{E}})$ indeed

Remark 11. Let $\mathsf{E} = \{\langle \mathsf{E}_x, \mathfrak{N}_x\rangle\}_{x\in X}$ be a nice family of Hlcs with $\mathfrak{N}_x \doteqdot \{\nu_j^x \,|\, j \in J\}$ for all $x \in X$. Moreover let $\mathcal{E}$ satisfy $FM(3) - FM(4)$ with respect to E, and $\mathfrak{V}(\mathsf{E},\mathcal{E})$ be the bundle generated by the couple $\langle \mathsf{E},\mathcal{E}\rangle$. Thus according to [10, Prps. 5.8, 5.9] we have that

1. $\mathfrak{V}(\mathsf{E},\mathcal{E})$ is a bundle of $\Omega-$spaces;
2. $\mathfrak{V}(\mathsf{E},\mathcal{E})$ is such that
   (a) $\langle \mathfrak{E}(\mathsf{E})_x, \tau(\mathsf{E},\mathcal{E})\rangle$ as topological vector space is isomorphic to $\langle \mathsf{E}_x, \mathfrak{N}_x\rangle$ for all $x \in X$;
   (b) $\mathcal{E}$ is canonically isomorphic [11] to a linear subspace of $\Gamma(\pi_{\mathsf{E}})$ and if X is compact and $\mathcal{E}$ is a function module see [10, § 5.1], then $\mathcal{E} \simeq \Gamma(\pi_{\mathsf{E}})$.

Remark 12. Let E be a nice family of Hlcs and let $\mathcal{E}$ satisfy $FM(3-4)$ with respect to E. Thus for all $U \in Open(X)$, $\sigma \in \mathcal{E}$, $\varepsilon > 0$, $j \in J$

$$T_{\mathsf{E}}(U,\sigma,\varepsilon,j) = \bigcup_{y\in U} B_{\mathsf{E}_y,j,\varepsilon}(\sigma(y))$$

where for all $s \in \mathsf{E}_y$

$$B_{\mathsf{E}_y,j,\varepsilon}(s) \doteqdot \left\{(y,w) \in \mathfrak{E}(\mathsf{E})_y \,|\, \nu_j^y\,(w - s) < \varepsilon\right\}.$$

Definition 16 (Essentially §1.5(II) and §1.5(vii) of [10]). Let $\mathfrak{P} \doteqdot \langle\langle \mathfrak{E},\tau\rangle, p, X, \mathfrak{N}\rangle$ be a locally full bundle of $\Omega-$spaces, and let us denote $\mathfrak{N} \doteqdot \{\nu_j \,|\, j \in J\}$. Set

$$\begin{cases} \mathcal{K}^{loc} \doteqdot \prod_{\alpha\in\mathfrak{E}} \mathcal{K}_\alpha^{loc} \\ \mathcal{K}_\alpha^{loc} \doteqdot \{(U,\sigma_U) \,|\, U \in Op(X), \sigma_U \in \Gamma_U(p) \,|\, p(\alpha) \in U, \sigma_U(p(\alpha)) = \alpha\}. \end{cases}$$

Moreover $\forall \alpha \in \mathfrak{E}$ and $\forall \mathfrak{l} \in \mathcal{K}^{loc}$ set

$$\begin{cases} \mathcal{B}_{\mathfrak{l}}^{loc}(\alpha) \doteqdot \left\{T^{loc}(V,\mathfrak{l}_2(\alpha),\varepsilon,j) \,|\, V \in Op(X), \varepsilon > 0, j \in J \,|\, p(\alpha) \in V \subseteq \mathfrak{l}_1(\alpha)\right\}, \\ T^{loc}(U,\sigma_U,\varepsilon,j) \doteqdot \{\beta \in \mathfrak{E} \,|\, p(\beta) \in U, \nu_j(\beta - \sigma_U(p(\beta))) < \varepsilon\}, \end{cases}$$

$(\forall U \in Op(X))(\forall j \in J)(\forall \varepsilon > 0)(\forall \sigma_U \in \Gamma_U(p))$.

[11] I.e. $\sigma \leftrightarrow f$ if and only if $\sigma(x) = (x, f(x))$

If $\mathfrak{P}$ is a full bundle then we can set

$$\begin{cases}\mathcal{K} \doteqdot \prod_{\alpha\in\mathfrak{E}}\mathcal{K}_\alpha\\ \mathcal{K}_\alpha \doteqdot \{(U,\sigma)\,|\,U\in Op(X),\sigma\in\Gamma(p)\,|\,p(\alpha)\in U,\sigma(p(\alpha))=\alpha\}\,.\end{cases}$$

Moreover $\forall\alpha\in\mathfrak{E}$ and $\forall\mathfrak{l}\in\mathcal{K}$ set

$$\begin{cases}\mathcal{B}_{\mathfrak{l}}(\alpha)\doteqdot\{T(V,\mathfrak{l}_2(\alpha),\varepsilon,j)\,|\,V\in Op(X),\varepsilon>0,j\in J\,|\,p(\alpha)\in V\subseteq\mathfrak{l}_1(\alpha)\}\,,\\ T(U,\sigma,\varepsilon,j)\doteqdot T^{loc}(U,\sigma\restriction U,\varepsilon,j),\end{cases}$$

$(\forall U\in Op(X))(\forall j\in J)(\forall\varepsilon>0)(\forall\sigma\in\Gamma(p))$. Any set $T^{loc}(U,\sigma\restriction U,\varepsilon,j)$ for a fixed $\varepsilon>0$ is called $\varepsilon-$tube.

Remark 13. Notice that $(\forall U\in Op(X))(\forall j\in J)(\forall\varepsilon>0)(\forall\sigma_U\in\Gamma_U(p))$

$$T^{loc}(U,\sigma_U,\varepsilon,j)=\bigcup_{y\in U}B_{\mathfrak{E}_y,j,\varepsilon}(\sigma_U(y))$$

where for all $\gamma\in\mathfrak{E}_y$

$$B_{\mathfrak{E}_y,j,\varepsilon}(\gamma)\doteqdot\left\{\beta\in\mathfrak{E}_y\,|\,\nu_j^y(\beta-\gamma)<\varepsilon\right\}.$$

Corollary 7.1 (Neighbourhood's filter $\mathcal{I}_\alpha^\tau$). *Let $\mathfrak{P}\doteqdot\langle\langle\mathfrak{E},\tau\rangle,p,X,\mathfrak{N}\rangle$ be a bundle of $\Omega-$spaces*

1. *if $\mathfrak{P}$ is locally full $\forall\alpha\in\mathfrak{E}$ and $\forall\mathfrak{l}\in\mathcal{K}^{loc}$ the set $\mathcal{B}_{\mathfrak{l}}^{loc}(\alpha)$ is a basis of a filter moreover*
$$\mathfrak{F}^{\mathfrak{E}}_{\mathcal{B}_{\mathfrak{l}}^{loc}(\alpha)}=\mathcal{I}_\alpha^\tau;$$
2. *if $\mathfrak{P}$ is full or locally full over a completely regular space then $\forall\alpha\in\mathfrak{E}$ and $\forall\mathfrak{l}\in\mathcal{K}$ the set $\mathcal{B}_{\mathfrak{l}}(\alpha)$ is a basis of a filter moreover*
$$\mathfrak{F}^{\mathfrak{E}}_{\mathcal{B}_{\mathfrak{l}}(\alpha)}=\mathcal{I}_\alpha^\tau.$$

Here $\mathcal{I}_\alpha^\tau$ is the neighbourhood's filter of α in the topological space $\langle\mathfrak{E},\tau\rangle$.

Proof. Statement (1) follows by applying [10, §1.5.(*vii*)], while statement (2) follows by statement (1) and the fact that for all $U\in Op(X)$ and $\sigma\in\Gamma(p)$ we have $\sigma\restriction U\in\Gamma_U(p)$. □

In what follows let $\mathsf{E}\doteqdot\{\langle\mathsf{E}_x,\mathfrak{N}_x\rangle\}_{x\in X}$ be a nice family of Hlcs with $\mathfrak{N}_x\doteqdot\{\nu_j^x\}_{j\in J}$ for all $x\in X$. Let $\mathcal{E}$ satisfy $FM(3^*)-FM(4)$ with respect to E.

Definition 17. Set

$$\begin{cases}\mathcal{K}^{\mathcal{E}}\doteqdot\prod_{(x,v)\in\mathfrak{E}}\mathcal{K}^{\mathcal{E}}_{(x,v)}\\ \mathcal{K}^{\mathcal{E}}_{(x,v)}\doteqdot\{(U,f)\,|\,U\in Op(X),f\in\mathcal{E}\,|\,x\in U,f(x)=v\}\,.\end{cases}$$

Moreover $\forall(x,v)\in\mathfrak{E}(\mathsf{E})$ and $\forall\mathfrak{l}\in\mathcal{K}^{\mathcal{E}}$ define

$$\mathcal{B}^{\mathcal{E}}_{\mathfrak{l}}((x,v))=\{T_{\mathsf{E}}(V,\mathfrak{l}_2((x,v)),\varepsilon,j)\,|\,\varepsilon>0,j\in J,V\in Op(X),x\in V\subseteq\mathfrak{l}_1((x,v))\}\,. \quad (7.2)$$

Corollary 7.2 (Neighbourhood's filter $\mathcal{I}_{(x,v)}^{\tau(\mathsf{E},\mathcal{E})}$). *Then $\mathfrak{V}(\mathsf{E},\mathcal{E})$ is a full bundle of $\Omega-$spaces and $\forall(x,v)\in\mathfrak{E}(\mathsf{E})$*

$$\mathfrak{F}_{\mathcal{B}_{\mathfrak{l}}^{\mathcal{E}}((x,v))}^{\mathfrak{E}(\mathsf{E})}=\mathcal{I}_{(x,v)}^{\tau(\mathsf{E},\mathcal{E})}.$$

Proof. By Theorem 5.9. of [10] $\mathcal{E}$ and $\Gamma(p_1)$ are canonically isomorphic as linear spaces, so $\mathfrak{V}(\mathsf{E},\mathcal{E})$ is full by $FM(3^*)$. The statement hence follows by statement (2) of Corollary 7.1. □

The following corollaries provide conditions under which the topologies over two bundle spaces are equal.

Corollary 7.3. *Let $\langle\langle\mathfrak{E},\tau_k\rangle,p_k,X,\mathfrak{N}_k\rangle$ be a full bundle of $\Omega-$spaces or a locally full bundle over a completely regular space X, for $k=1,2$. If $p_1=p_2$ and $\Gamma(p_1)=\Gamma(p_2)$ then $\tau_1=\tau_2$.*

Proof. By statement (2) of Corollary 7.1. □

Corollary 7.4. *Let $\mathfrak{P}_2\doteqdot\langle\langle E,\tau_2\rangle,p_2,X,\mathfrak{N}_2\rangle$ be a bundle of $\Omega-$spaces such that $\pi_{\mathsf{E}}=p_2$. Thus if the following conditions are satisfied*

1. *X is compact,*

2. *$\mathcal{E}$ and $\Gamma(p_2)$ are canonically isomorphic as linear spaces,*

then $\tau(\mathsf{E},\mathcal{E})=\tau_2$.

Proof. By Theorem 5.9. of [10] $\mathcal{E}$ and $\Gamma(\pi_{\mathsf{E}})$ are canonically isomorphic as linear spaces if X is compact, so $\Gamma(\pi_{\mathsf{E}})=\Gamma(p_2)$. Moreover $FM(3^*)$ and the shown fact that $\mathcal{E}$ and $\Gamma(\pi_{\mathsf{E}})$ are canonically isomorphic ensure that $\mathfrak{V}(\mathsf{E},\mathcal{E})$ is a full bundle, thus it is so $\mathfrak{P}_2$ by the equality $\Gamma(\pi_{\mathsf{E}})=\Gamma(p_2)$. Hence the statement follows by Corollary 7.3. □

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