ISSN 2077–9879

Eurasian Mathematical Journal

2016, Volume 7, Number 3

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia
the University of Padua

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by
the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

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EMJ: from Scopus Q4 to Scopus Q3 in two years?!

Recently the list was published of all mathematical journals included in 2015 Scopus quartiles Q1 (334 journals), Q2 (318 journals), Q3 (315 journals), and Q4 (285 journals). Altogether 1252 journals.

With great pleasure we inform our readers that the Eurasian Mathematical Journal was included in this list, currently the only mathematical journal in the Republic of Kazakhstan and Central Asia.

It was included in Q4 with the SCImago Journal & Country Rank (SJR) indicator equal to 0,101, and is somewhere at the bottom of the Q4 list. With this indicator the journal shares places from 1240 to 1248 in the list of all 2015 Scopus mathematical journals. Nevertheless, this may be considered to be a good achievement, because Scopus uses information about journals for the three previous years, i. e. for years 2013-2015, and the EMJ is in Scopus only from the first quarter of year 2015.

The SJR indicator is calculated by using a sophisticated formula, taking into account various characteristics of journals and journals publications, in particular the average number of weighted citations received in the selected year by the documents published in the selected journal in the three previous years. This formula and related comments can be viewed on the web-page

$$http://www.scimagojr.com/journalrank.php?category = 2601\&area = 2600\&page = 1\&total_size = 373$$

(Help/Journals/Understand tables and charts/Detailed description of SJR.)

In order to enter Q3 the SJR indicator should be greater than 0,250. It looks like the ambitious aim of entering Q3 in year 2017 is nevertheless realistic due to recognized high level of the EMJ.

We hope that all respected members of the international Editorial Board, reviewers, authors of our journal, representing more than 35 countries, and future authors will provide high quality publications in the EMJ which will allow to achieve this aim.

On behalf of the Editorial Board of the EMJ
V.I. Burenkov, E.D. Nursultanov, T.Sh. Kalmenov,
R. Oinarov, M. Otelbaev, T.V. Tararykova, A.M. Temirkhanova

VICTOR IVANOVICH BURENKOV

(to the 75th birthday)

On July 15, 2016 was the 75th birthday of Victor Ivanovich Burenkov, editor-in-chief of the Eurasian Mathematical Journal (together with V.A. Sadovnichy and M. Otelbaev), director of the S.M. Nikol'skii Institute of Mathematics, head of the Department of Mathematical Analysis and Theory of Functions, chairman of Dissertation Council at the RUDN University (Moscow), research fellow (part-time) at the Steklov Institute of Mathematics (Moscow), scientific supervisor of the Laboratory of Mathematical Analysis at the Russian-Armenian (Slavonic) University (Yerevan, Armenia), doctor of physical and mathematical sciences (1983), professor (1986), honorary professor of the L.N. Gumilyov Eurasian National University (Astana, Kazakhstan, 2006), honorary doctor of the Russian-Armenian (Slavonic) University (Yerevan, Armenia, 2007), honorary member of staff of the University of Padua (Italy, 2011), honorary distinguished professor of the Cardiff School of Mathematics (UK, 2014), honorary professor of the Aktobe Regional State University (Kazakhstan, 2015).

V.I. Burenkov graduated from the Moscow Institute of Physics and Technology (1963) and completed his postgraduate studies there in 1966 under supervision of the famous Russian mathematician academician S.M. Nikol'skii.

He worked at several universities, in particular for more than 10 years at the Moscow Institute of Electronics, Radio-engineering, and Automation, the RUDN University, and the Cardiff University. He also worked at the Moscow Institute of Physics and Technology, the University of Padua, and the L.N. Gumilyov Eurasian National University.

He obtained seminal scientific results in several areas of functional analysis and the theory of partial differential and integral equations. Some of his results and methods are named after him: Burenkov's theorem of composition of absolutely continuous functions, Burenkov's theorem on conditional hypoellipticity, Burenkov's method of mollifiers with variable step, Burenkov's method of extending functions, the Burenkov-Lamberti method of transition operators in the problem of spectral stability of differential operators, the Burenkov-Guliyevs conditions for boundedness of operators in Morrey-type spaces. On the whole, the results obtained by V.I. Burenkov have laid the groundwork for new perspective scientific directions in the theory of functions spaces and its applications to partial differential equations, the spectral theory in particular.

More than 30 postgraduate students from more than 10 countries gained candidate of sciences or PhD degrees under his supervision. He has published more than 170 scientific papers. The lists of his publications can be viewed on the portals MathSciNet and MathNet.Ru. His monograph "Sobolev spaces on domains" became a popular text for both experts in the theory of function spaces and a wide range of mathematicians interested in applying the theory of Sobolev spaces.

In 2011 the conference "Operators in Morrey-type Spaces and Applications", dedicated to his 70th birthday was held at the Ahi Evran University (Kirsehir, Turkey). Proceedings of that conference were published in the EMJ 3-3 and EMJ 4-1.

The Editorial Board of the Eurasian Mathematical Journal congratulates Victor Ivanovich Burenkov on the occasion of his 75th birthday and wishes him good health and new achievements in science and teaching!

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 7, Number 3 (2016), 33 – 40

ELLIPTIC DIFFERENTIAL-DIFFERENCE EQUATIONS WITH INCOMMENSURABLE SHIFTS OF ARGUMENTSTS

E.P. Ivanova

Communicated by V.I. Burenkov

Key words: differential-difference equation, strong ellipticity, boundary value problem, incommensurable argument's shifts.

AMS Mathematics Subject Classification: 47G40.

Abstract. This article deals with the problem of strong ellipticity of differential-difference equations with incommensurable argument's shifts.

1 Introduction

This article examines the boundary value problems for differential-difference equations containing the incommensurable shifts of spatial variables in the senior members. The theory of boundary problems for elliptic differential-difference equations with integer or commensurable shifts of arguments in senior members was constructed in the works by A.L. Skubachevskii [1]. These problems have important applications in plasma theory and the theory of multilayered plates and shells [1], [2]. Boundary value problems for differential-difference equations with incommensurable shifts were studied only in the one-dimensional case [1], [4]. The investigation of such problems is complicated due to a number of factors. Firstly it is the violation of the smoothness of solutions. Solutions of boundary value problems for differential-difference equations with commensurable shifts retain the smoothness in some subdomains [1], but solutions of boundary value problems with incommensurable shifts can have everywhere dense set of points of discontinuity of the derivative (see Example 3.10, [1]). Secondly, the difficulties in verifying the conditions of positive definiteness of difference operators with incommensurable shifts acting on bounded domains. For difference operators with constant coefficients and commensurable shifts a criterion of their positive definiteness was obtained [1], for difference operators with incommensurable shifts only sufficient conditions are known, namely the positivity of the scalar symbol depending on the coefficients of the operator. Since this symbol does not take into account the properties and the size of the domain, these conditions are redundant and far from being necessary. Moreover even small perturbations of shifts may violate their commensurability, therefore the known conditions of strong ellipticity of differential-difference operators with commensurable shifts are unstable relative to these shifts. Continuous dependence of solutions of boundary value problems for functional differential equations of shifts the first time investigated by L.E. Rossovskii [3]. This article is devoted to finding conditions of strong ellipticity for differential-difference equations with incommensurable argument's shifts in bounded domains. We obtain strong ellipticity conditions, which take into account the shape and the size of the domain, stable with respect to small perturbations of argument's shifts.

2 The main result

Consider the boundary value problem for the differential-differential equation

$$-\Delta(R^1u(x)+R^2u(x)) = f(x) \quad (x\in Q), \tag{2.1}$$

$$u(x)=0 \quad (x\notin Q). \tag{2.2}$$

Here Q is a boundary domain in $\mathbb{R}^n$ with a piecewise-smooth boundary and $f\in L_2(Q)$. The difference operators $R^1: L_2(\mathbb{R}^n)\to L_2(\mathbb{R}^n)$, $R^2: L_2(\mathbb{R}^n)\to L_2(\mathbb{R}^n)$ look as follows

$$R^1u(x) = a_0u(x) + \sum_{h\in M_1} a_h(u(x+h)+u(x-h)) \quad (a_h\in\mathbb{R}), \tag{2.3}$$

$$R^2u(x) = b_0u(x) + \sum_{p\in M_2} b_p(u(x+p)+u(x-p)) \quad (b_p\in\mathbb{R}). \tag{2.4}$$

Here M_1 is a finite set of vectors with commensurable coordinates, M_2 is also a finite set of vectors with commensurable coordinates, meanwhile the coordinates of the vectors h are not commensurable with the coordinates of the vectors p.

By a solution of boundary-value problem (2.1)–(2.2) we understand any function $u\in \mathring{H}^1(Q)$, extended by zero to $\mathbb{R}^n\backslash Q$ and satisfying the integral identity

$$\sum_{i=1}^n (R^1u_{x_i}, v_{x_i})_{L_2(Q)} + (R^2u_{x_i}, v_{x_i})_{L_2(Q)} = (f,v)_{L_2(Q)} \quad (f\in L_2(Q)) \tag{2.5}$$

for any $v\in\mathring{H}^1(Q)$. Here $\mathring{H}^1(Q)$ is the Sobolev space of functions $v\in H^1(Q)$ with the zero trace on ∂Q,

$$(u,v)_{\mathring{H}^1(Q)} = \int\limits_Q \sum_{i=1}^n u_{x_i}\bar{v}_{x_i}dx.$$

Equation (2.1) is called *strongly elliptic* if the inequality

$$\sum_{i=1}^n (R^1u_{x_i}, u_{x_i})_{L_2(Q)} + (R^2u_{x_i}, u_{x_i})_{L_2(Q)} \geq c\sum_{i=1}^n (u_{x_i}, u_{x_i})_{L_2(Q)} = c\,\|u\|^2_{\mathring{H}^1(Q)} \tag{2.6}$$

holds for all $u\in C_0^\infty(Q)$, where $c>0$ does not depend on u. Let us obtain the strong ellipticity condition of equation (2.1), expressed in terms of the coefficients of the difference operators.

Introduce the operators

$$R_Q = P_QRI_Q : L_2(Q)\to L_2(Q),\ R^i_Q = P_QR^iI_Q : L_2(Q)\to L_2(Q),\ i=1,2,$$

where $I_Q: L_2(Q)\to L_2(\mathbb{R}^n)$ is the operator of extension of functions in $L_2(Q)$ by zero to $\mathbb{R}^n$, $P_Q: L_2(\mathbb{R}^n)\to L_2(Q)$ is the operator of restriction of functions in $L_2(\mathbb{R}^n)$ to Q.

We study conditions of positive definiteness of the operators $R_Q = R_Q^1 + R_Q^2 : L_2(Q) \to L_2(Q)$ with incommensurable shifts. Since the operators R_Q^1, R_Q^2 with commensurable shifts are considered independently, we use the method developed by A.L. Skubachevskii in [1]. Given the operator R_Q^1 we build a decomposition of the domain Q into disjoint subdomains $Q_r (r = 1, 2, ...)$ such that

1) $\bigcup\limits_r \bar{Q}_r = \bar{Q}$;

2) for every Q_{r_1} and $h \in M_1^{\pm} = \{M_1, -M_1\}$ there is either exists Q_{r_2} such that $Q_{r_2} = Q_{r_1} + h$, or $Q_{r_1} + h \in \mathbb{R}^n \backslash Q$.

Here M_1 is the set of vectors in formula (2.3).

We can divide the decomposition into disjoint classes in the following way: subdomains Q_{r_1}, Q_{r_2} belong to the same class if there exist a set of subdomains $Q_0, Q_1, ..., Q_N$ and a set of vectors $h_1, ..., h_N$ $(h_i \in M_1^{\pm})$, such that $Q_0 = Q_{r_1},\ Q_N = Q_{r_2}$ and $Q_i = Q_{i-1} + h_i,\ i = 1, ..., N$. We denote the subdomains Q_{sl}, where s is the number of a class and l is the number of a subdomain in the s-th class. Each class with number s consists of a finite number $N = N(s)$ of subdomains Q_{sl}.

We denote by $L_2(\bigcup_l Q_{sl})$ the subspace of functions from $L_2(Q)$, vanishing outside $L_2(\bigcup_l Q_{sl})$ $(l = 1, ..., N(s))$. Denote by $P_s : L_2(Q) \to L_2(\bigcup_l Q_{sl})$ the operator of orthogonal projection onto $L_2(\bigcup_l Q_{sl})$. By virtue of Lemma 8.5 in [1] $L_2(\bigcup_l Q_{sl})$ is an invariant subspace of the operator R_Q^1, and $L_2(Q) = \bigoplus\limits_s L_2(\bigcup_l Q_{sl})$.

We introduce the isomorphism $U_s : L_2(\bigcup_l Q_{sl}) \to L_2^N(Q_{s1})$ of the Hilbert spaces by the formula

$$(U_s u)_l(x) = u(x + h_{sl})\ (x \in Q_{s1}),$$

where $l = 1, ..., N = N(s)$, h_{sl} is such that $Q_{s1} + h_{sl} = Q_{sl}$ $(h_{s1} = 0)$, $L_2^N(Q_{s1}) = \prod\limits_{l=1}^{N} L_2(Q_{s1})$. By virtue of Lemma 8.6 in [1] the operator $R_{Qs}^1 : L_2^N(Q_{s1}) \to L_2^N(Q_{s1})$, given by $R_{Qs}^1 = U_s R_Q^1 {U_s}^{-1}$, is the operator of multiplication by the matrix $R_s = R_s(x)$ $(x \in Q_{s1})$ of order $N(s) \times N(s)$, the elements of which are calculated by the formula

$$r_{ij}^s(x) = \begin{cases} b_h\ (h = h_{sj} - h_{si} \in M), \\ \quad 0\ (h_{sj} - h_{si} \notin M). \end{cases} \tag{2.7}$$

Let us denote by n_1 the number of different matrices R_{s_ν} $(\nu = 1, ..., n_1)$ relevant to operator R_Q^1.

Lemma 2.1. *(Lemma 8.8 in [1]). The spectrum $\sigma(R_Q^1)$ of the operator R_Q^1 coincides with the union of eigenvalues of the set of matrices $R_{s_\nu} (\nu = 1, ..., n_1)$. Thus R_Q^1 is positive definite if and only if all the matrices $R_{s_\nu} (\nu = 1, ..., n_1)$ are positive definite.*

Denote by $\lambda_{\min}$ the minimal eigenvalue of the entire family of matrices $R_{s_\nu} (\nu = 1, ..., n_1)$. Then, by virtue of Lemma 2.1, we have

$$(R_Q^1 u, u)_{L_2(Q)} \geq \lambda_{\min} (u, u)_{L_2(Q)} \quad (\forall u \in L_2(Q)). \tag{2.8}$$

Introduce the control operators $R^C : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$ and $R_Q^C : L_2(Q) \to L_2(Q)$ defined by the formulas

$$R^C = R^2 + \lambda_{\min} I,\ R_Q^C = P_Q R^C I_Q.$$

Here $I : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$ is the identity operator.

Theorem 2.1. *If the control operator R_Q^C is positive definite, then the operator R_Q is also positive definite.*

Proof. Let operator R_Q^C be positive definite:

$$(R_Q^C u, u)_{L_2(Q)} \geq c_1 (u, u)_{L_2(Q)} \quad (c_1 > 0,\ \forall u \in L_2(Q)). \tag{2.9}$$

Then by virtue of inequalities (2.8), (2.9),

$$\begin{aligned}(R_Q u, u)_{L_2(Q)} &= ((R_Q^1 + R_Q^2)u, u)_{L_2(Q)} = ((R_Q^1 - \lambda_{\min} I u) + (\lambda_{\min} I u + R_Q^2)u, u)_{L_2(Q)} \\ &= ((R_Q^1 - \lambda_{\min} I u, u)_{L_2(Q)} + (R_Q^C u, u)_{L_2(Q)} \\ &\geq (R_Q^C u, u)_{L_2(Q)} \geq c_1 (u, u)_{L_2(Q)} \quad (c_1 > 0,\ \forall u \in L_2(Q)).\end{aligned}$$

□

Remark 1. Since the operator R_Q^C contains only commensurable shifts by construction, we apply the same method of decomposition of the domain Q to study its positive definiteness.

Thus, the study of the spectrum of the operator R_Q^C is reduced to the study of the spectrum of a finite number of the corresponding matrices.

Remark 2. Let shifts h and p in the operators R^1, R^2 have small perturbations $h(\varepsilon),\ p(\varepsilon)$. Suppose that for the perturbed operators $R_Q^{1,\varepsilon}, R_Q^{2,\varepsilon}$ with shifts $h + h(\varepsilon),\ p + p(\varepsilon)$ the decomposition of the domain Q consists of the similar classes of domains as for the operators R_Q^1, R_Q^2. Then the conditions for strong ellipticity of the perturbed operator $R_Q^\varepsilon = R_Q^{1,\varepsilon} + R_Q^{2,\varepsilon}$ remain the same. This means that the resulting condition for strong ellipticity is stable with respect to small perturbations of shifts of the arguments.

By Theorem 10.1 in [1] we get the following statement.

Theorem 2.2. *Let the operator R_Q^C be positive definite. Then equation (2.1) is strongly elliptic in Q, and boundary value problem (2.1)–(2.2) has a unique solution $u \in \mathring{H}^1(Q)$ for any function $f \in L_2(Q)$.*

The results can be generalized to the case of equations with several classes of mutually incommensurable shifts of arguments. Consider the boundary value problem

$$-\Delta\Big(\sum_{i=1}^{N} R^i u(x)\Big) = f(x), \quad (x \in Q), \tag{2.10}$$

$$u(x) = 0 \quad (x \notin Q). \tag{2.11}$$

The difference operators $R^i : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$ have the form

$$R^i u(x) = a_0^i u(x) + \sum_{h \in M_i} a_h^i (u(x+h) + u(x-h)) \quad (\ a_h^i \in \mathbb{R}).$$

Here M_i $(i = 1, .., N)$ are the sets of vectors with commensurable coordinates. The coordinates of the vectors from the set M_i are incommensurable with the coordinates of the vectors from the set M_j $(i \neq j)$. Let us obtain the conditions of positive definiteness of the operator $R_Q : L_2(Q) \to L_2(Q)$, $R_Q = P_Q R I_Q$, $R = \sum\limits_{i=1}^{N} R^i$, using Theorem 2.1. We build a decomposition of Q and the corresponding matrices generated by the operator R^1_Q. Then we find the minimal eigenvalue λ^1_{min} of all these matrices. Next, we consider the operator $R^{C,1}_Q : L_2(Q) \to L_2(Q)$,

$$R^{C,1}_Q = P_Q R^{C,1} I_Q, \; R^{C,1} = \lambda^1_{\text{min}} I + \sum_{i=2}^{N} R^i.$$

By virtue of Theorem 2.1, the operator R_Q is positive definite as soon as the operator $R^{C,1}_Q$ is positive definite. In order to study the positive definiteness of the operator $R^{C,1}_Q$, we introduce the operator $\tilde{R}^2_Q : L_2(Q) \to L_2(Q)$ defined by

$$\tilde{R}^2_Q = P_Q \tilde{R}^2 I_Q, \; \tilde{R}^2 = \lambda^1_{\text{min}} I + R^2.$$

We build a decomposition Q for the operator $\tilde{R}^2_Q$ and find minimal eigenvalue λ^2_{min} for all corresponding matrices. Next, we consider the operator $R^{C,2}_Q : L_2(Q) \to L_2(Q)$ defined by

$$R^{C,2}_Q = P_Q R^{C,2} I_Q, \; R^{C,2} = \lambda^2_{\text{min}} I + \sum_{i=3}^{N} R^i.$$

By Theorem 2.1, the positive definiteness of the operator $R^{C,2}_Q$ is sufficient for the positive definiteness of the operator $R^{C,1}_Q$. Continuing this process by induction, we obtain

Corollary 2.1. *If the operator $R^{C,N-1}_Q : L_2(Q) \to L_2(Q)$ defined by*

$$R^{C,N-1}_Q = P_Q R^{C,N-1} I_Q, \; R^{C,N-1} = \lambda^{N-1}_{\text{min}} I + R^N$$

is positive definite, then the operator R_Q is also positive definite.

3 Examples

Example 1.

Consider the boundary value problem

$$-\Delta(R^1 u(x) + R^2 u(x)) = f(x) \quad (x \in Q), \tag{3.1}$$

$$u(x) = 0 \quad (x \notin Q). \tag{3.2}$$

Here $Q = (0,2) \times (0,2) \in \mathbb{R}^2$, $x = (x_1, x_2)$, $R^1, R^2 : L_2(\mathbb{R}^2) \to L_2(\mathbb{R}^2)$ are difference operators:

$$R^2 u(x) = b_0 u(x_1, x_2) + b_1(u(x_1+1, x_2) + u(x_1-1, x_2)) + b_2(u(x_1, x_2+1) + u(x_1, x_2-1)),$$

$$R^1 u(x) = a_\tau(u(x_1+\tau, x_2) + u(x_1-\tau, x_2)).$$

Here τ is irrational, $\frac{2}{3}<\tau<1$. Let us denote $\theta=2-2\tau$, then $0<\theta<\tau$. For the operator $R_Q^1: L_2(Q)\to L_2(Q),\ R_Q^1=P_QR^1I_Q$ we construct the corresponding decomposition of Q composed of two classes of subdomains. The first class consists of $Q_{11}=(0,\theta)\times(0,2),\ Q_{12}=(\tau,\tau+\theta)\times(0,2),\ Q_{13}=(2\tau,2\tau+\theta)\times(0,2)$; the second class is $Q_{21}=(\theta,\tau)\times(0,2),\ Q_{12}=(\tau+\theta,2\tau)\times(0,2)$. By formula (2.7), to the first class of subdomains there corresponds multiplication by the matrix R_1^1 :

$$R_1^1=\begin{pmatrix}0_0 & a_\tau & 0\\ a_\tau & 0 & a_\tau\\ 0 & a_\tau & 0\end{pmatrix}.$$

Its eigenvalues are $\lambda_1=\sqrt{2}a_\tau,\ \lambda_2=-\sqrt{2}a_\tau,\ \lambda_3=0$, where the minimal eigenvalue is $\lambda_{\min}^1=-\sqrt{2}\,|a_\tau|$. To the second class of subdomains there corresponds multiplication by the matrix R_2^1 :

$$R_2^1=\begin{pmatrix}0 & a_\tau\\ a_\tau & 0\end{pmatrix}, \tag{3.3}$$

whose eigenvalues are $\lambda_4=a_\tau,\ \lambda_5=-a_\tau$. Here the minimal eigenvalue of the two matrices is $\lambda_{\min}=-\sqrt{2}\,|a_\tau|$. Let us denote $\tilde{b}_0=b_0+\lambda_{\min}=b_0-\sqrt{2}\,|a_\tau|$. The control difference operator R^C has the form

$$R^Cu(x)=\tilde{b}_0u(x_1,x_2)+b_1(u(x_1+1,x_2)+u(x_1-1,x_2))+b_2(u(x_1,x_2+1)+u(x_1,x_2-1)).$$

By virtue of Theorem 2.1, the positive definiteness of the operator $R_Q^C=P_QR^CI_Q$ ensures the strong ellipticity of equation (3.1). This operator generates the decomposition of Q into subdomains $G_1=(0,1)\times(0,1),\ G_2=(1,2)\times(0,1),\ G_3=(0,1)\times(1,2),\ G_{4.}=(1,2)\times(1,2)$. By virtue of Lemma 2.1, the operator R_Q^C is positive definite if and only if the matrix

$$R^2=\begin{pmatrix}\tilde{b}_0 & b_1 & b_2 & 0\\ b_1 & \tilde{b}_0 & 0 & b_2\\ b_2 & 0 & \tilde{b}_0 & b_1\\ 0 & b_2 & b_1 & \tilde{b}_0\end{pmatrix}$$

is positive definite. It is easy to see that the conditions for this are $\tilde{b}_0>0,\ \tilde{b}_0>|b_1|,\ \ \tilde{b}_0>|b_1|+|b_2|,\ \left(\tilde{b}_0\right)^2>|b_1|^2+|b_2|^2$, and finally

$$b_0-\sqrt{2}\,|a_\tau|-|b_1|-|b_2|>0. \tag{3.4}$$

If condition (3.4) is satisfied, then by Theorem 2.1 equation (3.1) is strongly elliptic and by Theorem 2.2 there exists a unique solution $u\in\mathring{H}^1(Q)$ of problem (3.1)–(3.2).

Note that sufficient conditions for strong ellipticity of equation (3.1) obtained on the basis of the symbol

$$A_R(\xi)=(b_0+2a_\tau cos(\tau\xi_1)+2b_1cos(\xi_1)+2b_2cos(\xi_2))(\xi_1^2+\xi_2^2)$$

look as follows:

$$b_0-2\,|a_\tau|-2\,|b_1|-2\,|b_2|>0$$

(Theorem 9.4 in [1]).

Consider the case $1 < \tau < 2$. The decomposition of Q consists of two classes of subdomains: the first class $Q_{11} = (2-\tau, \tau) \times (0,2)$. The action of the operator R_Q^1 corresponds to multiplication by 0. The second class : $Q_{21} = (0, 2-\tau) \times (0,2)$, $\quad Q_{22} = (\tau, 2) \times (0,2)$. This class corresponds to multiplication by the matrix R_2^1, defined by formula (3.3). The minimal eigenvalue is $\lambda_{\min} = -|a_\tau|$. In this case, a necessary and sufficient condition of positive definiteness control operator R_Q^C is the condition

$$b_0 - |a_\tau| - |b_1| - |b_2| > 0. \tag{3.5}$$

Thus, if condition (3.4) is satisfied, equation (3.1) is uniformly relatively to $\tau \in \left(\frac{2}{3}, 2\right)$ strongly elliptic.

Examine the possibility to pass to the limit as $\tau \to 1$ in problem (3.1)–(3.2). Consider the boundary value problem (for $\tau = 1$)

$$\begin{gathered} -\Delta(R^{lim} u_{lim}(x)) = f(x) \quad (x \in Q), \\ u_{lim}(x) = 0 \quad (x \notin Q) \end{gathered} \tag{3.6}$$

with the difference operator

$$R^{lim} u(x) = b_0 u(x_1, x_2) + (b_1 + a_\tau)(u(x_1+1, x_2) + u(x_1-1, x_2))$$

$$+ b_2(u(x_1, x_2+1) + u(x_1, x_2-1)).$$

The operator R^{lim} contains only commensurable shifts in the arguments, and the action of the operator $R_Q^{lim} = P_Q R^{lim} I_Q$ corresponds to multiplication by the matrix R_1^{lim}:

$$R_1^{lim} = \begin{pmatrix} b_0 & b_1 + a_\tau & b_2 & 0 \\ b_1 + a_\tau & b_0 & 0 & b_2 \\ b_2 & 0 & b_0 & b_1 + a_\tau \\ 0 & b_2 & b_1 + a_\tau & b_0 \end{pmatrix}.$$

The condition $b_0 - |a_\tau + b_1| - |b_2| > 0$ equivalent to its positive definiteness is less stringent than condition (3.4) of positive definiteness of the control operator R_Q^C. Therefore, Theorem 2.2 guarantees the existence of a unique solution $u_{lim} \in \mathring{H}^1(Q)$ of boundary value problem (3.6). Using the methods of [3], it can be shown that $u_\tau \to u_{lim}$ as $\tau \to 1$ in the norm of the space $\mathring{H}^1(Q)$.

Acknowledgments

The author is grateful to A.L. Skubachevskii and L.E. Rossovskii for their attention to the research and useful discussion of the results.

This research was financially supported by the Ministry of Education and Science of the Russian Federation (the programme of improving the competitiveness of RUDN University among the world's leading research and education centers in the 2016-2020, agreement 02.a03.21.0008) and by the Russian Foundation for Basic Research (grant 14-01-00265).

References

[1] A.L. Skubachevskii, *Elliptic functional-differential equations and applications. In Operator Theory: Advances and Applications.* Birkhäuser Verlag, Basel, 1997.

[2] A.L. Skubachevskii, *Bifurcation of periodic solutions for nonlinear parabolic functional differential equations arising in optoelectronics.* Nonlinear Analysis. 32 (1998), no. 2, 261–278.

[3] L.E. Rossovskii, *Elliptic functional-differential equations with compression and stretching of the arguments of unknown function.* Sovremennaya matematika. Fundamental'nye napravleniya. 54 (2014), 3–138. (in Russian)

[4] E.P. Ivanova, *Continuous dependence of solutions of boundary value problems for differential-difference equations on shifts of arguments.* Sovremennaya matematika. Fundamental'nye napravleniya. 59 (2016), 74–96. (in Russian)

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Received: 10.06.2016