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EMJ: from Scopus Q4 to Scopus Q3 in two years?!

Recently the list was published of all mathematical journals included in 2015 Scopus quartiles Q1 (334 journals), Q2 (318 journals), Q3 (315 journals), and Q4 (285 journals). Altogether 1252 journals.

With great pleasure we inform our readers that the Eurasian Mathematical Journal was included in this list, currently the only mathematical journal in the Republic of Kazakhstan and Central Asia.

It was included in Q4 with the SCImago Journal & Country Rank (SJR) indicator equal to 0,101, and is somewhere at the bottom of the Q4 list. With this indicator the journal shares places from 1240 to 1248 in the list of all 2015 Scopus mathematical journals. Nevertheless, this may be considered to be a good achievement, because Scopus uses information about journals for the three previous years, i. e. for years 2013-2015, and the EMJ is in Scopus only from the first quarter of year 2015.

The SJR indicator is calculated by using a sophisticated formula, taking into account various characteristics of journals and journals publications, in particular the average number of weighted citations received in the selected year by the documents published in the selected journal in the three previous years. This formula and related comments can be viewed on the web-page

$$http://www.scimagojr.com/journalrank.php?category = 2601\&area = 2600\&page = 1\&total_size = 373$$

(Help/Journals/Understand tables and charts/Detailed description of SJR.)

In order to enter Q3 the SJR indicator should be greater than 0,250. It looks like the ambitious aim of entering Q3 in year 2017 is nevertheless realistic due to recognized high level of the EMJ.

We hope that all respected members of the international Editorial Board, reviewers, authors of our journal, representing more than 35 countries, and future authors will provide high quality publications in the EMJ which will allow to achieve this aim.

On behalf of the Editorial Board of the EMJ
V.I. Burenkov, E.D. Nursultanov, T.Sh. Kalmenov,
R. Oinarov, M. Otelbaev, T.V. Tararykova, A.M. Temirkhanova

VICTOR IVANOVICH BURENKOV

(to the 75th birthday)

On July 15, 2016 was the 75th birthday of Victor Ivanovich Burenkov, editor-in-chief of the Eurasian Mathematical Journal (together with V.A. Sadovnichy and M. Otelbaev), director of the S.M. Nikol'skii Institute of Mathematics, head of the Department of Mathematical Analysis and Theory of Functions, chairman of Dissertation Council at the RUDN University (Moscow), research fellow (part-time) at the Steklov Institute of Mathematics (Moscow), scientific supervisor of the Laboratory of Mathematical Analysis at the Russian-Armenian (Slavonic) University (Yerevan, Armenia), doctor of physical and mathematical sciences (1983), professor (1986), honorary professor of the L.N. Gumilyov Eurasian National University (Astana, Kazakhstan, 2006), honorary doctor of the Russian-Armenian (Slavonic) University (Yerevan, Armenia, 2007), honorary member of staff of the University of Padua (Italy, 2011), honorary distinguished professor of the Cardiff School of Mathematics (UK, 2014), honorary professor of the Aktobe Regional State University (Kazakhstan, 2015).

V.I. Burenkov graduated from the Moscow Institute of Physics and Technology (1963) and completed his postgraduate studies there in 1966 under supervision of the famous Russian mathematician academician S.M. Nikol'skii.

He worked at several universities, in particular for more than 10 years at the Moscow Institute of Electronics, Radio-engineering, and Automation, the RUDN University, and the Cardiff University. He also worked at the Moscow Institute of Physics and Technology, the University of Padua, and the L.N. Gumilyov Eurasian National University.

He obtained seminal scientific results in several areas of functional analysis and the theory of partial differential and integral equations. Some of his results and methods are named after him: Burenkov's theorem of composition of absolutely continuous functions, Burenkov's theorem on conditional hypoellipticity, Burenkov's method of mollifiers with variable step, Burenkov's method of extending functions, the Burenkov-Lamberti method of transition operators in the problem of spectral stability of differential operators, the Burenkov-Guliyevs conditions for boundedness of operators in Morrey-type spaces. On the whole, the results obtained by V.I. Burenkov have laid the groundwork for new perspective scientific directions in the theory of functions spaces and its applications to partial differential equations, the spectral theory in particular.

More than 30 postgraduate students from more than 10 countries gained candidate of sciences or PhD degrees under his supervision. He has published more than 170 scientific papers. The lists of his publications can be viewed on the portals MathSciNet and MathNet.Ru. His monograph "Sobolev spaces on domains" became a popular text for both experts in the theory of function spaces and a wide range of mathematicians interested in applying the theory of Sobolev spaces.

In 2011 the conference "Operators in Morrey-type Spaces and Applications", dedicated to his 70th birthday was held at the Ahi Evran University (Kirsehir, Turkey). Proceedings of that conference were published in the EMJ 3-3 and EMJ 4-1.

The Editorial Board of the Eurasian Mathematical Journal congratulates Victor Ivanovich Burenkov on the occasion of his 75th birthday and wishes him good health and new achievements in science and teaching!

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NORMAL EXTENSIONS OF LINEAR OPERATORS

B.N. Biyarov

Communicated by K. Ospanov

Key words: formally normal operator, normal operator, correct restriction, correct extension.

AMS Mathematics Subject Classification: 47Axx, 47A05; 47B15.

Abstract. Let L_0 be a densely defined minimal linear operator in a Hilbert space H. We prove that if there exists at least one correct extension L_S of L_0 with the property $D(L_S) = D(L_S^*)$, then we can describe all correct extensions L with the property $D(L) = D(L^*)$. We also prove that if L_0 is formally normal and there exists at least one correct normal extension L_N, then we can describe all correct normal extensions L of L_0. As an example, the Cauchy-Riemann operator is considered.

1 Introduction

Let us state some definitions, notation, and terminology.

In a Hilbert space H, we consider a linear operator L with the domain $D(L)$ and range $R(L)$. By the *kernel* of the operator L we mean the set

$$\operatorname{Ker} L = \big\{ f \in D(L) : \; Lf = 0 \big\}.$$

Definition 1. An operator L is called a *restriction* of an operator L_1, and L_1 is called an *extension* of an operator L, briefly $L \subset L_1$, if:

1) $D(L) \subset D(L_1)$,

2) $Lf = L_1 f$ for all $f \in D(L)$.

Definition 2. A linear closed operator L_0 in a Hilbert space H is called *minimal* if there exists a bounded inverse operator L_0^{-1} on $R(L_0)$ and $R(L_0) \neq H$.

Definition 3. A linear closed operator $\widehat{L}$ in a Hilbert space H is called *maximal* if $R(\widehat{L}) = H$ and $\operatorname{Ker} \widehat{L} \neq \{0\}$.

Definition 4. A linear closed operator L in a Hilbert space H is called *correct* if there exists a bounded inverse operator L^{-1} defined on all of H.

Definition 5. We say that a correct operator L in a Hilbert space H is a *correct extension* of a minimal operator L_0 (*correct restriction* of a maximal operator $\widehat{L}$) if $L_0 \subset L$ ($L \subset \widehat{L}$).

Definition 6. We say that a correct operator L in a Hilbert space H is a *boundary correct* extension of a minimal operator L_0 with respect to a maximal operator $\widehat{L}$ if L is simultaneously a correct restriction of the maximal operator $\widehat{L}$ and a correct extension of the minimal operator L_0, that is, $L_0 \subset L \subset \widehat{L}$.

At the beginning of the 1950s, Vishik [12] extended the theory of self-adjoint extensions of von Neumann–Krein symmetric operators to nonsymmetric operators in Hilbert space.

At the beginning of the 1980s, M. Otelbaev and his disciples proved abstract theorems that allows us to describe all correct extensions of some minimal operator using any single known correct extension in terms of an inverse operator. Such extensions need not be restrictions of a maximal operator. Similarly, all possible correct restrictions of some maximal operator that need not be extensions of a minimal operator were described (see [7]). For convenience, we present the conclusions of these theorems.

Let $\widehat{L}$ be a maximal linear operator in a Hilbert space H, let L be any known correct restriction of $\widehat{L}$, and let K be an arbitrary linear bounded (in H) operator satisfying the following condition:

$$R(K) \subset \operatorname{Ker} \widehat{L}. \tag{1.1}$$

Then the operator L_K^{-1} defined by the formula

$$L_K^{-1} f = L^{-1} f + K f, \tag{1.2}$$

describes the inverse operators to all possible correct restrictions L_K of $\widehat{L}$, i.e., $L_K \subset \widehat{L}$.

Let L_0 be a minimal operator in a Hilbert space H, let L be any known correct extension of L_0, and let K be a linear bounded operator in H satisfying the conditions

a) $R(L_0) \subset \operatorname{Ker} K$,

b) $\operatorname{Ker} (L^{-1} + K) = \{0\}$,

then the operator L_K^{-1} defined by formula (2.2) describes the inverse operators to all possible correct extensions L_K of L_0.

Let L be any known boundary correct extension of L_0, i.e., $L_0 \subset L \subset \widehat{L}$. The existence of at least one boundary correct extension L was proved by Vishik in [12]. Let K be a linear bounded (in H) operator satisfying the conditions

a) $R(L_0) \subset \operatorname{Ker} K$,

b) $R(K) \subset \operatorname{Ker} \widehat{L}$,

then the operator L_K^{-1} defined by formula (2.2) describes the inverse operators to all possible boundary correct extensions L_K of L_0.

Self-adjoint and unitary operators are particular cases of normal operators. A bounded linear operator N in a Hilbert space H is called *normal* if it commutes with its adjoint:

$$N^* N = N N^*.$$

The theory of bounded normal operators is well developed.

Consider an unbounded linear operator A in a Hilbert space H.

Definition 7. A densely defined closed linear operator A in a Hilbert space H is called *formally normal* if

$$D(A) \subset D(A^*), \quad \|Af\| = \|A^* f\| \quad \text{for all } f \in D(A).$$

Definition 8. A formally normal operator A is called *normal* if

$$D(A) = D(A^*).$$

Normal extensions of formally normal operators have been studied by many authors (see [1], [5], [6], [11]). The problems of the existence of a normal extension and the description of the domains of normal extensions of a formally normal operator were considered.

Spectral properties of the correct restrictions and extensions were studied by many authors (see [2]–[4], [8], [9]). A class of operators K that provides the Volterra property, the completeness of root vectors, and the dissipativity of correct restrictions and extensions were described by the author (see [2]–[4]).

This paper is devoted to the description of correct normal extensions in terms of the operator K.

2 Coincidence criterion of $D(L)$ and $D(L^*)$

We consider a densely defined minimal linear operator L_0 in a Hilbert space H. Let M_0 be a minimal operator with $D(M_0) = D(L_0)$ such that the equality $(L_0 u, v) = (u, M_0 v)$ holds for all u, v in $D(L_0)$. Then the maximal operator $\widehat{L} = M_0^*$ is an extension of L_0, and the maximal operator $\widehat{M} = L_0^*$ is an extension of M_0. The following statement is true.

Assertion 2.1. *If there exists a correct extension L_S of the minimal operator L_0 with the property $D(L_S) = D(L_S^*)$, then the operator L_S is a boundary correct extension, i.e., $L_0 \subset L_S \subset \widehat{L}$.*

Proof. From $L_0 \subset L_S$ it follows that $L_S^* \subset L_0^* = \widehat{M}$. From $D(L_S) = D(L_S^*)$ and the fact that $D(M_0) \subset D(L_S^*)$ we have

$$M_0 \subset L_S^* \subset \widehat{M}.$$

Then $L_0 \subset L_S \subset \widehat{L}$. □

Let there be one fixed correct extension L_S of L_0 such that $D(L_S) = D(L_S^*)$. Then we can describe the inverses to all boundary correct extensions L in the following form

$$u = L^{-1}f = L_S^{-1}f + Kf \quad \text{for all } f \in H, \tag{2.1}$$

where K is an arbitrary bounded operator in a Hilbert space H such that

$$R(K) \subset \operatorname{Ker} \widehat{L} \quad \text{and} \quad R(L_0) \subset \operatorname{Ker} K.$$

Each such operator K defines a boundary correct extension and there do not exist other boundary correct extensions.

Let us equip $D(\widehat{L})$ with the graph norm $||u||_G = (||u||^2 + ||\widehat{L}u||^2)^{1/2}$. Since $\widehat{L}$ is a closed operator, we obtain a Hilbert space with the scalar product

$$(u, v)_G = (u, v) + (\widehat{L}u, \widehat{L}v) \text{ for all } u,\ v \in D(\widehat{L}).$$

Let us denote this space by $G_{\widehat{L}}$. The domain $D(L_S)$ of the correct restriction L_S is a subspace in $G_{\widehat{L}}$. Therefore, there exists a projection operator of $G_{\widehat{L}}$ on the subspace $D(L_S)$. As such a projection operator, we take $L_S^{-1}\widehat{L}$. Then $\Gamma_{L_S} = I - L_S^{-1}\widehat{L}$ of $G_{\widehat{L}}$ is a projection on the subspace $\operatorname{Ker}\widehat{L}$. It is obvious that

$$\operatorname{Ker}\Gamma_{L_S} = D(L_S) \quad \text{and} \quad R(\Gamma_{L_S}) = \operatorname{Ker}\widehat{L}.$$

All boundary correct extensions (2.1) can be represented in the form

$$L^{-1}f = L_S^{-1}f + Kf = L_S^{-1}f + K\widehat{L}L_S^{-1}f = (I + K\widehat{L})L_S^{-1}f \ \text{ for all } f \in H,$$

where I is the identity operator in H. In virtue of $D(L) \subset D(\widehat{L})$, we have

$$\widehat{L}u = f \ \text{ for all } f \in H, \ \ u \in D(L)$$

where

$$D(L) = \{u \in D(\widehat{L}) : \ (I - K\widehat{L})u \in D(L_S)\}.$$

It is easy to see that the operator K defines the domain of L, since (see [3])

$$(I - K\widehat{L})D(L) = D(L_S), \quad (I + K\widehat{L})D(L_S) = D(L), \quad (I - K\widehat{L}) = (I + K\widehat{L})^{-1}.$$

Therefore, all boundary correct extensions L are differed from the fixed boundary correct extension L_S only the domain. The bounded (in $G_{\widehat{L}}$) operator $I - K\widehat{L}$ maps $D(L)$ onto $D(L_S)$ in a one-to-one fashion. Then the domain of L can be defined as follows:

$$D(L) = \{u \in D(\widehat{L}) : \ \Gamma_{L_S}(I - K\widehat{L})u = 0\}.$$

There exists one more representation of the domain of L

$$D(L) = \{u \in D(\widehat{L}) : \ ((I - K\widehat{L})u, L_S^* v) = (\widehat{L}u, v) \text{ for all } v \text{ from } D(L_S^*) \ \}.$$

Similarly we can define

$$\Gamma_{L_S^*} = I - {L_S^*}^{-1}\widehat{M}$$

and

$$D(L^*) = \{u \in D(\widehat{M}) : \ \Gamma_{L_S^*}(I - K^*\widehat{M})u = 0\}.$$

Now we can formulate the following result.

Theorem 2.2. *Let there exist a correct extension L_S of the minimal operator L_0 with $D(L_S) = D(L_S^*)$, then any other correct extension L has the property $D(L) = D(L^*)$ if and only if $L_0 \subset L \subset \widehat{L}$ and the operator K in the formula* (2.1) *satisfies the conditions*

$$R(K) \cup R(K^*) \subset D(\widehat{L}) \cap D(\widehat{M}),$$

and

$$\begin{cases} \Gamma_{L_S}(I - K\widehat{L})u = 0, \\ \Gamma_{L_S}K^*\widehat{M}u = K\widehat{L}u, \quad \text{for all } \ u \in D(\widehat{L}) \cap D(\widehat{M}), \end{cases} \tag{2.2}$$

where $\Gamma_{L_S} = I - L_S^{-1}\widehat{L}$ is the projection defined above.

Proof. Let $D(L) = D(L^*)$. In view of Assertion 2.1, the operators L_S and L turn out to be boundary correct extensions of L_0, i.e., $L_0 \subset L_S \subset \widehat{L}$ and $L_0 \subset L \subset \widehat{L}$. The inverse to the arbitrary boundary correct extension L has form (2.1). Then

$$(L^*)^{-1}g = (L_S^*)^{-1}g + K^*g \quad \text{for all} \quad g \in H.$$

The condition $D(L) = D(L^*)$ is equivalent to

$$L_S^{-1}f + Kf = (L_S^*)^{-1}g + K^*g, \tag{2.3}$$

where for each $f \in H$ there exists $g \in H$ and vice versa, for each $g \in H$ there exists $f \in H$ such that equality (2.3) is satisfied. It follows from (2.3) that

$$R(K^*) \subset D(\widehat{L}) \quad \text{and} \quad R(K) \subset D(\widehat{M}).$$

Then we get

$$R(K) \cup R(K^*) \subset D(\widehat{L}) \cap D(\widehat{M}).$$

By acting on both sides of equality (2.3) by the operator $\widehat{L}$, we obtain

$$f = L_S(L_S^*)^{-1}g + \widehat{L}K^*g, \quad \text{for all} \quad g \in H.$$

By substituting f in (2.3), we obtain the equality

$$L_S^{-1}\widehat{L}K^*g + KL_S(L_S^*)^{-1}g + K\widehat{L}K^*g = K^*g.$$

It follows that

$$(I - L_S^{-1}\widehat{L})K^*g = K\widehat{L}((L_S^*)^{-1} + K^*)g.$$

This means that

$$(I - L_S^{-1}\widehat{L})K^*g = K\widehat{L}(L^*)^{-1}g.$$

If ${L^*}^{-1}g$ is replaced by u, then

$$(I - L_S^{-1}\widehat{L})K^*\widehat{M}u = K\widehat{L}u, \quad u \in D(L^*).$$

Since $D(L) = D(L^*)$ we obtain $\Gamma_{L_S}K^*\widehat{M}u = K\widehat{L}u$ for all $u \in D(L)$. This is equivalent to condition (2.2).

We now prove the converse of this theorem. Let $L_0 \subset L \subset \widehat{L}$ and the operator K in formula (2.1) satisfies the conditions $R(K) \cup R(K^*) \subset D(\widehat{L}) \cap D(\widehat{M})$, and (2.2). Hence, it is easy to see that

$$D(L) \cup D(L^*) \subset D(\widehat{L}) \cap D(\widehat{M}).$$

Since $Lu = f$ for all $u \in D(L)$, we may replace Lu by f in the second equation of condition (2.2). Then

$$\Gamma_{L_S}K^*\widehat{M}L^{-1}f = Kf \quad \text{for all } f \in H.$$

By acting on both sides of this equality by the projection $\Gamma_{L_S^*}$, we obtain

$$K^*\widehat{M}L^{-1}f = (I - (L_S^*)^{-1}\widehat{M})Kf \quad \text{for all } f \in H.$$

Note that
$$K^*L_S^*L_S^{-1}f + K^*\widehat{M}Kf + (L_S^*)^{-1}\widehat{M}Kf = Kf.$$
By adding the bounded operator $L_S^{-1}f$ to both sides, we get
$$(L_S^*)^{-1}L_S^*L_S^{-1}f + K^*L_S^*L_S^{-1}f + K^*\widehat{M}Kf + (L_S^*)^{-1}\widehat{M}Kf = Kf + L_S^{-1}f.$$
It follows that
$$(L_S^*)^{-1}(L_S^*L_S^{-1} + \widehat{M}K)f + K^*(L_S^*L_S^{-1} + \widehat{M}K)f = L^{-1}f \quad \text{for all } f \in H.$$
If we denote by
$$g = L_S^*L_S^{-1}f + \widehat{M}Kf \quad \text{for all } f \in H,$$
then we have
$$(L^*)^{-1}g = L^{-1}f \quad \text{for all } f \in H.$$
It follows that $D(L) \subset D(L^*)$. By acting on both sides of the equations (2.2) by the projection $\Gamma_{L_S^*}$, we get
$$\begin{cases} \Gamma_{L_S^*}(I - K\widehat{L})u = 0, \\ \Gamma_{L_S^*}K\widehat{L}u = K^*\widehat{M}u \quad \text{for all } \ u \in D(\widehat{L}) \cap D(\widehat{M}). \end{cases}$$
By the second equation of this system, we can rewrite this system of equations in the form
$$\begin{cases} \Gamma_{L_S^*}(I - K^*\widehat{M})u = 0, \\ \Gamma_{L_S^*}K\widehat{L}u = K^*\widehat{M}u \quad \text{for all } \ u \in D(\widehat{L}) \cap D(\widehat{M}). \end{cases}$$
The first equation of this system means that u belongs to $D(L^*)$. Then we denote $L^*u = g$. Therefore, $u = (L^*)^{-1}g$ for all g from H. Then the second equation of this system has the form
$$\Gamma_{L_S^*}K\widehat{L}(L^*)^{-1}g = K^*\widehat{M}(L^*)^{-1}g \quad \text{for all } \ g \in H.$$
By acting on both sides of this equality by the projection Γ_{L_S}, we obtain
$$K\widehat{L}(L^*)^{-1}g = (I - (L_S)^{-1}\widehat{L})K^*g \quad \text{for all } g \in H.$$
Note that
$$KL_S(L_S^*)^{-1}g + K\widehat{L}K^*g + L_S^{-1}\widehat{L}K^*g = K^*g.$$
By adding the bounded operator $(L_S^*)^{-1}g$ to both sides, we get
$$L_S^{-1}L_S(L_S^*)^{-1}g + KL_S(L_S^*)^{-1}g + K\widehat{L}K^*g + L_S^{-1}\widehat{L}K^*g = K^*g + (L_S^*)^{-1}g.$$
It follows that
$$L_S^{-1}(L_S(L_S^*)^{-1} + \widehat{L}K^*)g + K(L_S(L_S^*)^{-1} + \widehat{L}K^*)g = (L^*)^{-1}g \quad \text{for all } g \in H.$$
If we denote by
$$f = (L_S(L_S^*)^{-1} + \widehat{L}K^*)g \quad \text{for all } g \in H,$$
then we have
$$L^{-1}f = (L^*)^{-1}g \quad \text{for all } g \in H.$$
It follows that $D(L^*) \subset D(L)$. $\square$

3 Normality criterion of correct extensions

Let L_0 be a formally normal minimal operator in a Hilbert space H. An operator M_0 is the restriction of $L_0^* = \widehat{M}$ to $D(L_0)$. Then $\widehat{L} = M_0^*$ defines the maximal operator such that $L_0 \subset \widehat{L}$. Let there be at least one normal correct extension L_N of the formally normal minimal operator L_0. In view of Assertion 2.1, we have that $L_0 \subset L_N \subset \widehat{L}$, i.e., L_N is the boundary correct extension. Then the inverses to all boundary correct extensions L of L_0 have the form

$$u = L^{-1}f = L_N^{-1}f + Kf \quad \text{for all } f \in H, \tag{3.1}$$

where K is an arbitrary bounded operator in a Hilbert space H that $R(K) \subset \operatorname{Ker}\widehat{L}$ and $R(L_0) \subset \operatorname{Ker} K$. Then the direct operator L acts as

$$\widehat{L}u = f \quad \text{for all } f \in H,$$

on the domain

$$D(L) = \big\{u \in D(\widehat{L}) : \ \Gamma_{L_N}(I - K\widehat{L})u = 0\big\},$$

where the projection $\Gamma_{L_N} = I - L_N^{-1}\widehat{L}$ is a bounded operator in the space $G_{\widehat{L}}$. It is known that

$$\operatorname{Ker}\Gamma_{L_N} = D(L_N) \ \text{ and } \ R(\Gamma_{L_N}) = \operatorname{Ker}\widehat{L}.$$

Theorem 3.1. *Let there exist a correct normal extension L_N of the formally normal minimal operator L_0 in a Hilbert space H. Then any other correct extension L of L_0 is normal if and only if $L_0 \subset L \subset \widehat{L}$ and operator K in formula* (3.1) *satisfies the conditions:*

$$R(K) \cup R(K^*) \subset D(\widehat{L}) \cap D(\widehat{M}),$$

$$\begin{cases} \Gamma_{L_N}(I - K\widehat{L})u = 0, \\ \Gamma_{L_N}K^*\widehat{M}u = K\widehat{L}u \quad \textit{for all } \ u \in D(\widehat{L}) \cap D(\widehat{M}), \end{cases} \tag{3.2}$$

and

$$\widehat{L}K^* = (\widehat{M}K)^*, \tag{3.3}$$

where $\Gamma_{L_N} = I - L_N^{-1}\widehat{L}$ is a projection on Ker $\widehat{L}$.

Proof. Let L be a normal correct extension of the formally normal operator L_0. In view of Theorem 2.2, the conditions $L_0 \subset L \subset \widehat{L}$, $R(K) \cup R(K^*) \subset D(\widehat{L}) \cap D(\widehat{M})$ and (3.2) will be satisfied. The normality of L^{-1} follows by the normality of L:

$$L^{-1}(L^*)^{-1} = (L^*)^{-1}L^{-1}.$$

By virtue of (3.1), we obtain

$$(L_N^{-1} + K)((L_N^*)^{-1} + K^*)f = ((L_N^*)^{-1} + K^*)(L_N^{-1} + K)f \quad \text{for all } \ f \in H.$$

It follows that

$$L_N^{-1}K^*f + K(L_N^*)^{-1} + KK^*f = (L_N^*)^{-1}Kf + K^*L_N^{-1}f + K^*Kf. \tag{3.4}$$

By acting on both sides of the equality (3.4) by the operator $\widehat{L}$, we get

$$K^*f = L_N(L_N^*)^{-1}Kf + \widehat{L}K^*L_N^{-1}f + \widehat{L}K^*Kf.$$

By taking conjugates of both sides of the above equality, we have

$$Kf = K^*(L_N(L_N^*)^{-1})^*f + (L_N^*)^{-1}(\widehat{L}K^*)^*f + K^*(\widehat{L}K^*)^*f \quad \text{for all} \;\; f \in H.$$

By acting on both sides by the operator $\widehat{M}$, we obtain

$$\widehat{M}Kf = (\widehat{L}K^*)^*f \quad \text{for all} \;\; f \in H.$$

This is equivalent to

$$\widehat{L}K^* = (\widehat{M}K)^*.$$

Let us prove the converse. Suppose that the conditions of Theorem 3.1 hold. From the conditions $L_0 \subset L \subset \widehat{L}$, $R(K) \cup R(K^*) \subset D(\widehat{L}) \cap D(\widehat{M})$ and (3.2), in view of Theorem 2.2, we have that $D(L) = D(L^*)$. Then for all $f \in H$ there exists $g \in H$ such that $L^{-1}f = (L^*)^{-1}g$. It can be rewritten in the form

$$L_N^{-1}f + Kf = (L_N^*)^{-1}g + K^*g. \tag{3.5}$$

By acting on both sides by the operator $\widehat{M}$, we get

$$g = L_N^*L_N^{-1}f + \widehat{M}Kf.$$

By substituting g in (3.5), we have

$$Kf = (L_N^*)^{-1}\widehat{M}Kf + K^*L_N^*L_N^{-1}f + K^*\widehat{M}Kf \quad \text{for all} \;\; f \in H.$$

Then

$$K^*f = (\widehat{M}K)^*L_N^{-1}f + (L_N^*L_N^{-1})^*Kf + (\widehat{M}K)^*Kf \quad \text{for all} \;\; f \in H. \tag{3.6}$$

Let us show that

$$(L_N^*L_N^{-1})^* = L_N(L_N^*)^{-1}.$$

It is known that if A is a closed operator, B is bounded in H and AB is densely defined in H, then

$$(AB)^* = \overline{B^*A^*},$$

where the overbar denotes the closure of an operator. Note that

$$L_N^*L_N^{-1} \supset L_N^{-1}L_N^*.$$

Then

$$(L_N^*L_N^{-1})^* = \overline{(L_N^*)^{-1}L_N} \subset L_N(L_N^*)^{-1}.$$

Taking into account the fact that $L_N(L_N^*)^{-1}$ is a bounded operator that coincides with $(L_N^*)^{-1}L_N$ on the dense set $D(L_N)$, we obtain that

$$L_N(L_N^*)^{-1} = \overline{(L_N^*)^{-1}L_N} = (L_N^*L_N^{-1})^*.$$

Then, taking into account (3.3), equality (3.6) can be rewritten in the form

$$K^*f = \widehat{L}K^*L_N^{-1}f + L_N(L_N^*)^{-1}Kf + \widehat{L}K^*Kf \quad \text{for all } f \in H.$$

By adding $(L_N^*)^{-1}f$ to both sides of the last equality, we get

$$K^*f + (L_N^*)^{-1}f = L_NL_N^{-1}(L_N^*)^{-1}f + \widehat{L}K^*L_N^{-1}f + L_N(L_N^*)^{-1}Kf + \widehat{L}K^*Kf.$$

It follows that

$$(L^*)^{-1}f = L(L^*)^{-1}L^{-1}f \quad \text{for all } f \in H.$$

Thus

$$L^{-1}(L^*)^{-1}f = (L^*)^{-1}L^{-1}f \quad \text{for all } f \in H.$$

□

The domain of L_S is described as the kernel of the projection $\Gamma_{L_S} = I - L_S^{-1}\widehat{L}$. Sometimes there exists another operator T_{L_S} defined on $D(\widehat{L})$ and with the property $\operatorname{Ker}\Gamma_{L_S} = \operatorname{Ker}T_{L_S}$. Between these operators there is the following relationship

$$T_{L_S}\Gamma_{L_S}v = T_{L_S}(I - L_S^{-1}\widehat{L})v = T_{L_S}v - T_{L_S}L_S^{-1}\widehat{L}v = T_{L_S}v \quad \text{for all } v \in D(\widehat{L}).$$

If we know $T_{L_S}v$, then $\Gamma_{L_S}v$ is uniquely determined as the solution of the homogeneous equation $\widehat{L}(\Gamma_{L_S}v) = 0$ with an inhomogeneous condition

$$T_{L_S}(\Gamma_{L_S}v) = T_{L_S}v.$$

Its unique solvability follows by the correctness of the operator L_S. Therefore, it is not necessary to know the explicit form of the operator L_S^{-1}. In the study of differential operators (see [12]) the operator T_{L_S} is realized in the form of the boundary operator. In such cases we say that the domain is described in terms of the boundary operator. For example, in the case of the Dirichlet problem for a differential equation of elliptic type in $L_2(\Omega)$ the operator T_{L_S} corresponds to the trace operator on the boundary of Ω, i.e., $T_{L_S}u = u\,|_{\partial\Omega}$. Therefore it suffices to know the form of the boundary operator T_{L_S}. Thus we obtain the following

Corollary 3.1. *Let there exist a correct extension L_S of the minimal operator L_0 with $D(L_S) = D(L_S^*)$, then any other correct extension L has the property $D(L) = D(L^*)$ if and only if $L_0 \subset L \subset \widehat{L}$, $R(K) \cup R(K^*) \subset D(\widehat{L}) \cap D(\widehat{M})$ and*

$$T_{L_S}(K^*\widehat{M} - K\widehat{L})u = 0 \quad \textit{for all } u \in D(L), \tag{3.7}$$

where T_{L_S} is the boundary operator corresponding to the fixed correct extension L_S and

$$D(L) = \big\{u \in D(\widehat{L}) :\ T_{L_S}(I - K\widehat{L})u = 0\big\}.$$

Remark 1. By virtue of the one-to-one mapping of $D(L_S)$ onto $D(L)$:

$$v = (I - K\widehat{L})u \ \text{ for all } u \in D(L), \quad u = (I + K\widehat{L})v \ \text{ for all } v \in D(L_S),$$

in practice, sometimes it is more convenient to use the following condition that is equivalent to (3.7):

$$T_{L_S}(K^*\widehat{M} - K\widehat{L} + K^*\widehat{M}K\widehat{L})v = 0 \quad \text{for all } v \in D(L_S). \tag{3.8}$$

It has the practical convenience because $D(L_S)$ is a fixed domain.

Similarly, we can rephrase Theorem 3.1 in the following form

Corollary 3.2. *Let there exist a correct normal extension L_N of the formally normal minimal operator L_0 in a Hilbert space H. Then any other correct extension L of L_0 is normal if and only if $L_0 \subset L \subset \widehat{L}$, $R(K) \cup R(K^*) \subset D(\widehat{L}) \cap D(\widehat{M})$,*

$$T_{L_N}(K^*\widehat{M} - K\widehat{L})u = 0 \quad \textit{for all } u \in D(L), \tag{3.9}$$

and

$$\widehat{L}K^* = (\widehat{M}K)^*, \tag{3.10}$$

where T_{L_N} is a boundary operator corresponding to the fixed correct extension L_N and

$$D(L) = \{u \in D(\widehat{L}) : T_{L_N}(I - K\widehat{L})u = 0\},$$

and K is the operator determining the boundary correct extension L in formula (3.1).

4 Examples

Example 1. We consider the following operator in the Hilbert space $L_2(0,1)$

$$\widehat{L}y \equiv y'' + y' = f, \tag{4.1}$$

to which corresponds the minimal operator L_0 with domain

$$D(L_0) = \{y \in W_2^2(0,1) : \ y(0) = y(1) = y'(0) = y'(1) = 0\}.$$

We define the operator M_0 as the restriction of $\widehat{M}$ on the set $D(L_0)$. Then the action of the operator M_0 has the form

$$\widehat{M}v \equiv v'' - v' = g.$$

We will denote the maximal operators M_0^* and L_0^* by $\widehat{L}$ and $\widehat{M}$, respectively. Then we have

$$L_0 \subset \widehat{L}, \ M_0 \subset \widehat{M} \quad \text{and} \quad D(\widehat{L}) = D(\widehat{M}) = W_2^2(0,1).$$

Let the operator L_N acts as $\widehat{L}$ with domain

$$D(L_N) = \{y \in D(\widehat{L}) : \ y(0) + y(1) = 0, \ y'(0) + y'(1) = 0\}.$$

We take the operator L_N as a fixed correct extensions of L_0. Note that $D(L_N) = D(L_N^*)$ and $L_0 \subset L_N \subset \widehat{L}, \ M_0 \subset L_N^* \subset \widehat{M}$. The inverse operator to L_N has the form

$$y = L_N^{-1}f = \int_0^x (1 - e^{t-x})f(t)dt - \frac{1}{2}\int_0^1 f(t)dt + \frac{e^{1-x}}{1+e}\int_0^1 e^{t-1}f(t)dt.$$

Then Γ_{L_N} is defined as

$$\Gamma_{L_N}y = \frac{y(0) + y(1)}{2} + \left(\frac{1}{2} - \frac{e^{1-x}}{1+e}\right)[y'(0) + y'(1)],$$

and $\Gamma_{L_N^*} = I - (L_N^*)^{-1}\widehat{M}$ has the following form

$$\Gamma_{L_N^*} v = \frac{v(0)+v(1)}{2} + \left(\frac{e^x}{1+e} - \frac{1}{2}\right)[v'(0)+v'(1)].$$

The correct extension L of L_0 with the property $D(L) = D(L^*)$ is a boundary correct extension. Their inverses are described in the following form

$$y = L^{-1}f = L_N^{-1}f + Kf \quad \text{for all } \; f \in L_2(0,1),$$

where K is a bounded linear operator in $L_2(0,1)$ with the properties

$$R(K) \subset \operatorname{Ker}\widehat{L}, \quad R(L_0) \subset \operatorname{Ker} K.$$

In our case, such operators are exhausted by the following operators

$$Kf = \int_0^1 f(t)(\overline{a_{11}} + \overline{a_{12}}e^t)dt + e^{-x}\int_0^1 f(t)(\overline{a_{21}} + \overline{a_{22}}e^t)dt,$$

where $a_{ij},\ i,j = 1,2$ are arbitrary complex numbers. Then

$$K^*f = (a_{11} + a_{12}e^x)\int_0^1 f(t)dt + (a_{21} + a_{22}e^x)\int_0^1 e^{-t}f(t)dt.$$

It is known that the direct operator L acts as $\widehat{L}$ from (4.1) and the domain has the form

$$D(L) = \big\{y \in D(\widehat{L}) : \; \Gamma_{L_N}(I - K\widehat{L})y = 0\big\}.$$

In view of Corollary 3.1, the domain of L can be defined in another way

$$D(L) = \Big\{ y \in D(\widehat{L}) : y(0) + y(1) = (K\widehat{L}y)(0) + (K\widehat{L}y)(1),$$
$$y'(0) + y'(1) = \Big(\frac{d}{dx}K\widehat{L}y\Big)(0) + \Big(\frac{d}{dx}K\widehat{L}y\Big)(1)\Big\}.$$

First, we will find the correct extensions L such that $D(L) = D(L^*)$. Taking into account Remark 1, let the operator K satisfy condition (3.8). Then we obtain the system of equations:

$$\begin{cases} 4(a_{11} + \overline{a_{11}}) + 2(e+1)\Big[\dfrac{\overline{a_{21}}}{e} + a_{12}\Big]\cdot A = 0, \\[2ex] -4(a_{11} - \overline{a_{11}}) - 2(e+1)(a_{12} - \overline{a_{12}}) - 2\dfrac{e+1}{e}(a_{21} - \overline{a_{21}}) \\ \qquad -\dfrac{(e+1)^2}{e}(a_{22} - \overline{a_{22}}) + \Big[4a_{12} + 2\dfrac{e+1}{e}a_{22}\Big]\cdot A = 0, \\[2ex] -\dfrac{1}{e}\overline{a_{21}} + a_{12} + \dfrac{2}{e}a_{12}\Big[\overline{a_{21}}(e-1) + \overline{a_{22}}\dfrac{e^2-1}{2}\Big] = 0, \\[2ex] -\dfrac{1}{e}[2\overline{a_{21}} + \overline{a_{22}}(1+e)] - 2a_{12} - \dfrac{e+1}{e}a_{22} \\ \qquad -4\dfrac{a_{12}}{e}\Big[\overline{a_{21}} + \overline{a_{22}}\dfrac{e^2-1}{2}\Big] - 2\dfrac{e+1}{e^2}a_{22}\Big[\overline{a_{21}}(e-1) + \overline{a_{22}}\dfrac{e^2-1}{2}\Big] = 0, \end{cases}$$

where

$$A = 2(e-1)\overline{a_{11}} + (e^2-1)\overline{a_{12}} + \frac{e+1}{e}\Big[\overline{a_{21}}(e-1) + \overline{a_{22}}\frac{e^2-1}{2}\Big].$$

Solutions of this system of equations with respect to a_{ij}, $i,j=1,2$, define operators K that guarantees the equality $D(L)=D(L^*)$. They will correspond to the following cases:

$I)\quad D(L) = \Big\{y \in D(\widehat{L}) :\ y(0)=0, \quad y(1)=0\Big\},$

$II)\ D(L) = \Big\{y \in D(\widehat{L}) :\ y(0) = \dfrac{a-i}{a+i}y(1), \quad y'(0) = \dfrac{a-i}{a+i}y'(1), \quad a\in\mathbb{R}\Big\},$

where $\mathbb{R}$ is the space of real numbers,

$III)\ D(L) = \Big\{y \in D(\widehat{L}) :\ ay(0) + \bar{b}y(1) = 0, \quad y(1) = by'(0) + ay'(1),$

$\qquad a\in\mathbb{R}, \quad a\neq 0, \quad b\in\mathbb{C}, \quad |b|^2 = a^2\Big\},$

where $\mathbb{C}$ is the space of complex numbers.

We use the criterion given in Theorem 3.1 to find all correct normal extensions L of the minimal operator L_0. It is easy to verify the formal normality of L_0 and the normality of L_N. The equality $D(L)=D(L^*)$ is necessary for the normality of L. They correspond to three cases of $I)-III)$ described above. Now, if the operator K satisfies (3.3), then the operator L is normal. Condition (3.3) is equivalent to the following

$$a_{21}=0, \quad a_{12}=0.$$

Therefore, the operator K takes the form

$$Kf = \overline{a_{11}}\int_0^1 f(t)dt + \overline{a_{22}}e^{-x}\int_0^1 e^t f(t)dt.$$

Then operators L which act as $\widehat{L}$ from (4.1) on the domain

$$D(L) = \Big\{y \in D(\widehat{L}) :\ y(0) = \frac{a-i}{a+i}y(1), \quad y'(0) = \frac{a-i}{a+i}y'(1), \quad a\in\mathbb{R}\Big\},$$

turn out to be normal correct extensions of the minimal operator L_0. Of the three cases $I)-III)$ only the case $II)$ is suitable.

Example 2. Let us consider in the Hilbert space $L_2(\Omega)$, where $\Omega = \{(x,y):\ 0<x<1,\ 0<y<1\}$, the minimal operator L_0 generated by the Cauchy-Riemann differential operator

$$\widehat{L}u \equiv \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} = f(x,y). \tag{4.2}$$

Then

$$D(L_0) = \big\{u \in W_2^1(\Omega) :\ T_{L_0}u = 0\big\},$$

where T_{L_0} is the boundary operator defined as the trace operator on the boundary of $\partial\Omega$ for functions $u \in W_2^1(\Omega)$.

The action of $\widehat{M}$ will have the form

$$\widehat{M}v \equiv -\frac{\partial v}{\partial x} + i\frac{\partial v}{\partial y} = g(x,y).$$

Domains of the operators $\widehat{L}$ and $\widehat{M}$ have the form

$$D(\widehat{L}) = \{u \in L_2(\Omega) : \ \widehat{L}u \in L_2(\Omega)\},$$

$$D(\widehat{M}) = \{v \in L_2(\Omega) : \ \widehat{M}v \in L_2(\Omega)\},$$

respectively. If we define the boundary operator T_{L_N} in the following way

$$T_{L_N}u = \begin{pmatrix} u(0,y)+u(1,y) \\ u(x,0)+u(x,1) \end{pmatrix} \quad \text{for all} \ \ u \in D(\widehat{L}),$$

then the operator L_N acting as $\widehat{L}$ with the domain

$$D(L_N) = \{u \in D(\widehat{L}) : \ T_{L_N}u = 0\},$$

is a correct extension of L_0. It is easy to verify that L_0 is formally normal, L_N is normal, and moreover $L_0 \subset L \subset \widehat{L}$.

Let L be a correct extension of the minimal operator L_0 generated by Cauchy-Riemann operator (4.2). We are interested in the normal correct extensions. The following three properties is necessary for the normality of correct extensions L in $L_2(\Omega)$:

1) $R(K) \subset W_2^1(\Omega)$, where K is from the description (3.1);

2) $(Kf)(x+iy)$, as the operator K in Example 2 acts from $L_2(\Omega)$ in $\mathrm{Ker}\widehat{L} = \{u \in L_2(\Omega) : \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} = 0\}$;

3) $(K^*f)(x-iy)$, as the operator K^* acts from $L_2(\Omega)$ in $\mathrm{Ker}\widehat{M} = \{v \in L_2(\Omega) : -\frac{\partial v}{\partial x} + i\frac{\partial v}{\partial y} = 0\}$. Indeed, it follows by Assertion 2.1 that $L_0 \subset L \subset \widehat{L}$ from it follows that $R(L_0) \subset \mathrm{Ker}K$, this implies, that $R(K^*) \subset \mathrm{Ker}\widehat{M}$.

The first property follows by

Assertion 4.1. *The domain of any normal correct extension L of the minimal operator L_0 generated by differential operator* (4.2) *has the property*

$$D(L) \subset W_2^1(\Omega).$$

Proof. This follows by Theorem 2 of Plesner and Rohlin (see [10]). Now we formulate this theorem: "For each pair of adjoint normal operators A and A^* there exists one and only one pair of self-adjoint operators A_1 and A_2, satisfying the condition

$$A = A_1 + iA_2, \quad A^* = A_1 - iA_2,$$

where the operators A_1 and A_2 commute". □

From conditions (3.9) and (3.10) we obtain that the operators K for which the correct boundary extension L will be normal.

It follows by Assertion 4.1 that L_N^{-1}, K, and L^{-1} are compact operators in $L_2(\Omega)$. This implies that the normal correct extension L of L_0 is an operator with discrete spectrum. Hence we have that L has a complete orthonormal system of eigenfunctions.

For clarity, the check of normality by Theorem 3.1 is carried out in the special following case. Let K will be an integral operator of the form

$$K_1 f = \int_0^1 \int_0^1 \mathcal{K}_1(x, y;\, \xi, \eta) f(\xi, \eta) d\xi d\eta.$$

It follows from properties 1) and 2) that

$$K_2 f = \int_0^1 \int_0^1 \mathcal{K}_2(x + iy,\, \xi + i\eta) f(\xi, \eta) d\xi d\eta.$$

From condition (3.3) of Theorem 3.1, we get that

$$K f = \int_0^1 \int_0^1 \mathcal{K}(x - \xi + i(y - \eta)) f(\xi, \eta) d\xi d\eta.$$

By using condition (3.2) of Theorem 3.1 for the operator K, we obtain all normal correct extensions. We will not give this condition on the kernel $\mathcal{K}(x - \xi + i(y - \eta))$, because it is too cumbersome to write.

To demonstrate the mechanism of checking condition (3.2), we consider the special case when

$$\mathcal{K}(x - \xi + i(y - \eta)) = a e^{i\pi(x - \xi + i(y - \eta))},$$

where $a \in \mathbb{C}$ is a complex number of the form $a = a_1 + i a_2$. Then condition (3.2) is equivalent to

$$2a_2 + (a_1^2 + a_2^2)(e^{\pi} - e^{-\pi}) = 0.$$

There are two kinds of solutions of this equation:

$$I. \quad a_1 = 0, \quad a_2 = \frac{2}{e^{-\pi} - e^{\pi}};$$

$$II. \quad a_2 = \frac{-1 \pm \sqrt{1 - [a_1(e^{\pi} - e^{-\pi})]^2}}{e^{\pi} - e^{-\pi}}, \quad \text{where} \quad |a_1| \le \frac{1}{e^{\pi} - e^{-\pi}}, \quad a_1 \ne 0.$$

Then in the case II, the correct extension corresponding to the following boundary problem

$$\widehat{L}u \equiv \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} = f(x,y) \quad \text{for all} \;\; f \in L_2(\Omega),$$

$$D(L) = \Big\{ u \in W_2^1(\Omega): \; u(0,y) + u(1,y) = 0, \quad 0 \le y \le 1,$$

$$u(x,0) + u(x,1) = ia(e^{\pi}+1)\int_0^1 e^{i\pi(x-\xi)}u(\xi,1)d\xi$$

$$-\, ia(e^{-\pi}+1)\int_0^1 e^{i\pi(x-\xi)}u(\xi,0)d\xi, \quad 0 \le x \le 1 \Big\}$$

is normal, where $a = a_1 + ia_2$, and in the case I, the correct extension corresponding to the boundary problem

$$D(L) = \Big\{ u \in W_2^1(\Omega): \; u(0,y) + u(1,y) = 0,$$

$$u(x,0) + u(x,1) = 2\int_0^1 e^{i\pi(x-\xi)}u(\xi,1)d\xi \Big\},$$

is normal.

All normal correct extensions L have a compact inverse operator because of $D(L) \subset W_2^1(\Omega)$. Therefore, their eigenfunctions create an orthonormal basis in $L_2(\Omega)$. In the particular case when

$$\mathcal{K}(x,y;\,\xi,\eta) = \frac{2i}{e^{-\pi}-e^{\pi}} \cdot e^{i\pi(x-\xi+i(y-\eta))},$$

we obtain the orthonormal basis in the following form:

$$u_{k,n}(x,y) = \begin{cases} e^{2n\pi iy+i\pi x}, & n = 0, \pm 1, \pm 2, \dots \\ e^{(2k+1)\pi ix+(2n+1)\pi iy}, & k = \pm 1, \pm 2, \dots, \; n = 0, \pm 1, \pm 2, \dots \end{cases}$$

and the corresponding eigenvalues

$$\lambda_{k,n} = \begin{cases} i\pi - 2n\pi, & n = 0, \pm 1, \pm 2, \dots \\ (2k+1)\pi i - (2n+1)\pi, & k = \pm 1, \pm 2, \dots, \; n = 0, \pm 1, \pm 2, \dots . \end{cases}$$

Thus, this method allows us to check the normality of an unbounded operator. Preliminary it is necessary to clarify the question of the existence of at least one normal extension. For the existence of a normal extension it is required that the minimal operator must be formally normal.

Remark 2. If in Example 2 the square Ω is replaced by the unit circle, then the minimal operator L_0 will not be formally normal. Thus in this case, there are no normal extensions of L_0 in $L_2(\Omega)$.

Remark 3. When the minimal operator L_0 is symmetric and the fixed operator L_N is self-adjoint then the conditions of Theorem 3.1 are equivalent to $K = K^*$ and we get all self-adjoint correct extensions.

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