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ON FAMILIES OF LINEAR POLYNOMIAL OPERATORS GENERATED BY RIESZ KERNELS[2]

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Abstract. Families of linear polynomial operators generated by the Riesz kernels are studied. Sharp ranges of convergence are found in many cases. It is shown that the approximation error is equivalent to the polynomial K-functional related to the apropricate power of the Laplace operator, if the family converges.

Introduction

The present paper deals with the approximation properties of the families of linear polynomial operators defined by

$$\mathcal{R}_{n;\lambda}^{(\alpha,\beta)}(f;x) = (2n+1)^{-d} \cdot \sum_{k=0}^{2n} f(t_n^k + \lambda) \cdot R_n^{(\alpha,\beta)}(x - t_n^k - \lambda)\,,\ n \in \mathbb{N}_0\,, \tag{2.1}$$

where

$$R_0^{(\alpha,\beta)}(h) \;=\; 1\,,\ R_n^{(\alpha,\beta)}(h) = \sum_{|\,k\,| \le n} \left(1 - \frac{|\,k\,|^\beta}{n^\beta}\right)^{\alpha} \cdot e^{ikh}\,,\ n \in \mathbb{N}\,, \tag{2.2}$$

are the Riesz kernels with indices $\alpha \ge 0$ and $\beta > 0$ in L_p-spaces of 2π-periodic functions of d variables for $0 < p \le +\infty$. In (2.1) and (2.2) x, k, λ are d-dimensional vectors, $|\,k\,| = (k_1^2 + \ldots + k_d^2)^{1/2}$, $kh = k_1h_1 + \ldots k_dh_d$ and

$$t_n^k = \frac{2\pi k}{2n+1}\,,\ k \in \mathbb{Z}^d\,;\quad \sum_{k=0}^{2n} = \sum_{k_1=0}^{2n} \ldots \sum_{k_d=0}^{2n}\ .$$

The corresponding Riesz means which are given by

$$\mathcal{R}_n^{(\alpha,\beta)}(f;x) = (2\pi)^{-d} \int\limits_{\mathbb{T}^d} f(x+h) \cdot R_n^{(\alpha,\beta)}(h)\ dh\,,\ n \in \mathbb{N}\,, \tag{2.3}$$

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where $\mathbb{T}^d = [0, 2\pi)^d$ is the d-dimensional torus, are classical objects of both harmonic analysis and approximation theory. They were intensively studied by many mathematicians (see, e.g. [2], [9], [17], [20]). In particular, the Riesz means (2.3) converge in L_p for all $1 \leq p \leq +\infty$ independently on β , provided that $\alpha > (d-1)/2$ (see [9]). For further results and details we refer also to [8] (Chapters 3, 10), [19] (Chapter 7), an [21] (Chapter 8).

For $\beta = 2$ the kernels (2.2) are called the Bochner-Riesz kernels. In this case the approximative properties of methods (2.1) and (2.3) have been systematically studied in [16]. In particular, we proved that for α exceeding the critical index $(d-1)/2$ the approximation error of the families (2.1) with $\beta = 2$ averaged with respect to the parameter λ is equivalent to the approximation error of the means (2.3) if $1 \leq p \leq +\infty$. Moreover, family (2.1) converges in $L_p(\mathbb{T}^{2d})$ if and only if $p > 2d/(d+2\alpha+1)$ and in this case its approximation error in L_p is equivalent to the K-functional K_Δ if $1 \leq p \leq +\infty$ and to its polynomial version $\widetilde{K}_\Delta$ if $2d(d+2\alpha+1) < p \leq +\infty$ which are related to the Laplace operator Δ and defined as

$$K_\Delta(f,\delta)_p \;=\; \inf_{g:\, \Delta g \,\in\, L_p} \{ \| f-g \|_p \,+\, \delta^2 \, \| \Delta g \|_p \} \,, \;\; \delta \geq 0, \; f \in L_p \,, \tag{2.4}$$

$$\widetilde{K}_\Delta(f,\delta)_p \;=\; \inf_{T \,\in\, \mathcal{T}_{1/\delta}} \{ \| f-T \|_p \,+\, \delta^2 \, \| \Delta T \|_p \} \,, \;\; \delta > 0, \; f \in L_p \,, \tag{2.5}$$

respectively, where

$$\mathcal{T}_\sigma \;=\; \left\{ T(x) = \sum_{k \in \mathbb{Z}^d} c_k \, e^{ikx} \;\; : c_{-k} = \overline{c_k}, \; |\, k \,| \equiv (k_1^2 + \ldots + k_d^2)^{1/2} \leq \sigma \right\} \tag{2.6}$$

is the space of all real-valued trigonometric polynomials of (spherical) order $\sigma \geq 0$. It was shown in [6] that the K-functional given in (2.4) is identical 0 for $0 < p < 1$. Therefore, in this case it can not be used for a characterization of any approximation process. It follows from the results in [3], [4] and [6] that the quantities given in (2.4) and (2.5) are equivalent for $1 \leq p \leq +\infty$. However, note that (2.5) makes sense for all $0 < p \leq +\infty$. Let us also mention that in the case $1 \leq p \leq +\infty$ the equivalence of the approximation error of (2.3) with $\beta = 2$ and the K-functional given in (2.4) was proved in [5].

The family (2.1) is a special case of the general construction

$$\mathcal{L}^{(\varphi)}_{\sigma;\lambda}(f;x) = (2[\sigma]+1)^{-d} \cdot \sum_{\nu=0}^{2[\sigma]} f\left(t^\nu_{[\sigma]} + \lambda\right) \cdot W_\sigma(\varphi)\left(x - t^\nu_{[\sigma]} - \lambda\right) \;, \tag{2.7}$$

where

$$W_0(\varphi)(h) \equiv 1 \,, \;\; W_\sigma(\varphi)(h) = \sum_{k \in \mathbb{Z}^d} \varphi\left(\frac{k}{\sigma}\right) e^{ikh} \,, \;\; \sigma > 0 \;, \tag{2.8}$$

and the generator of the kernels $\varphi(\xi)$ is a complex-valued continuous function defined on $\mathbb{R}^d$ with compact support contained in the unit ball $D_1 = \{\xi \,:\, |\,\xi\,| \leq 1\}$ and satisfying the conditions $\varphi(0) = 1$ and $\varphi(-\xi) = \overline{\varphi(\xi)}$ for each $\xi \in \mathbb{R}^d$. Clearly, the Riesz kernels are generated by $\varphi_{\alpha,\beta}(\xi) = \left(1 - |\,\xi\,|^{\beta}\right)_{+}^{\alpha}$, where $a_+ = \max(a,0)$. In contrast to the classical methods of trigonometric approximation as, for instance, means of type (2.3) the method of approximation by families (2.6) is comparatively new (see e.g. [12], [13]) and it is relevant for all $0 < p \le +\infty$. Its systematical study was continued in [1], [14] and further papers.

Recently, some general principles concerning the approximative properties of families given in (2.7) – (2.8) have been established, see e.g. [10], [11]. They enable us to reduce both the problem of their convergence and their interrelations with smoothness to the study of the Fourier transform of some functions constructed with the help of generators of approximation and smoothness (see Section 1 for exact formulations). In particular, now the results in [16] on convergence of the Bochner-Riesz families ($\beta = 2$) can be easily obtained applying this general approach to the formula for the Fourier transform of the generator $\varphi_\alpha(\xi) = (1 - |\,\xi\,|^2)_+^\alpha$ of the Bochner-Riesz kernels (see, e.g., [18] (Chapter 9, § 2.2, pp. 389-390))

$$\widehat{\varphi}_\alpha(x) \;=\; \pi^{-\alpha}\,\Gamma(\alpha+1)\,|\,x\,|^{-\alpha-d/2}\,J_{\alpha+d/2}(|\,x\,|)\,, \tag{2.9}$$

where $J_s(x)$, $s > -1/2$, is the Bessel function of order s. In contrast to this case it seems that there is no explicit formula for the Fourier transform of the generator of the Riesz kernels (2.2) for arbitrary α and β. For this reason an essential part of our paper is devoted to the study of $\widehat{\varphi}_{\alpha,\beta}(\xi)$. In particular, we give an explicit and complete description of the set

$$\mathcal{P}_{\alpha,\beta} \;=\; \left\{\, p \in (0,+\infty] :\; \widehat{\varphi}_{\alpha,\beta} \;\in\; L_p(\mathbb{R}^d)\,\right\} \tag{2.10}$$

for all admissible parameters α and β. As a consequence some shortcomings concerning the asymptotic behavior of $\widehat{\varphi}_{\alpha,\beta}$ arising in the relevant literature can be corrected.

The paper is organized as follows. Section 1 is devoted to notation and preliminaries. We also give formulations of the General Convergence Theorem (GCT) (see [10]) and the General Equivalence Theorem (GET) (see [11]) which will be applied to families (2.1) – (2.2) later on. In Section 2 we study the Fourier transform of the generator of the Riesz kernels $\varphi_{\alpha,\beta}$. In Section 3 we formulate and prove the main results of this paper: necessary and sufficient conditions of convergence of the Riesz families in the case $\alpha > (d-1)/2$ and the equivalence of their approximation error to the polynomial K-functional given by

$$\widetilde{K}_\beta(f,\delta)_p \;=\; \inf_{T\,\in\,\mathcal{T}_{1/\delta}} \{\,\|\,f - T\,\|_p \,+\, \delta^\beta\,\|\,(-\Delta)^{\beta/2}T\,\|_p\}\,,\; \delta > 0,\; f \in L_p\,. \tag{2.11}$$

1 Definitions, notation and general results on approximation by families

$\mathbf{L_p}$-spaces. As usual, $L_p \equiv L_p(\mathbb{T}^d)$, where $0 < p < +\infty$, $\mathbb{T}^d = [0,2\pi)^d$, is the space of measurable real-valued functions f which are 2π-periodic with respect to each variable such that

$$\|\,f\,\|_p \;=\; \left(\int\limits_{\mathbb{T}^d} |f(x)|^p\,dx\right)^{1/p} \;<\; +\infty\;.$$

$C \equiv C(\mathbb{T}^d)$ $(p = +\infty)$ is the space of real valued 2π-periodic continuous functions equipped with the Chebyshev norm

$$\| f \|_\infty \; = \; \max_{x \in \mathbb{T}^d} |f(x)| \; .$$

For the L_p-spaces of non-periodic functions defined on a measurable set $\Omega \subseteq \mathbb{R}^d$ we will use the notation $L_p(\Omega)$.

Often we deal with functions in $L_p(\mathbb{T}^{2d})$ which depend on both the main variable $x \in \mathbb{T}^d$ and the parameter $\lambda \in \mathbb{T}^d$. Let us denote by $\| \cdot \|_p$ or $\| \cdot \|_{p;\,x}$ the $L_p(\mathbb{T}^d)$-norm with respect to x. For the $L_p(\mathbb{T}^d)$-norm with respect to the parameter λ we use the symbol $\| \cdot \|_{p;\,\lambda}$. For shortness, $L_{\overline{p}}$ stands for the space $L_p(\mathbb{T}^{2d})$ equipped with the norm

$$\| \cdot \|_{\overline{p}} \; = \; (2\pi)^{-d/p} \, \| \; \| \cdot \|_{p;\,x} \, \|_{p;\,\lambda} \; . \tag{1.1}$$

Analogously, we use the symbol $\| \cdot \|_{\overline{\infty}}$ for the norm in the space $C(\mathbb{T}^{2d})$. Clearly, L_p with $0 < p < \infty$ and $C(\mathbb{T}^d)$ can be considered as a subspace of $L_{\overline{p}}$ and $C(\mathbb{T}^{2d})$, respectively, where

$$\| f \|_{\overline{p}} \; = \; \| f \|_p \; , \quad f \in L_p \; (f \in C \text{ if } p = \infty) \; . \tag{1.2}$$

by (1.1).

Spaces of trigonometric polynomials. Let σ be a real non-negative number. The space $\mathcal{T}_\sigma$ of all real-valued trigonometric polynomials of (spherical) order σ is defined by (2.6). We denote by $\mathcal{T}_{\sigma,p}$, where $0 < p \le +\infty$, the space $\mathcal{T}_\sigma$, if it is equipped with the L_p-norm and we use the symbol $\mathcal{T}_{\sigma,\overline{p}}$ to denote the subspace of $L_{\overline{p}}$ which consists of functions $g(x,\lambda)$ such that $g(x,\lambda)$ as a function of x belongs to $\mathcal{T}_\sigma$ for almost all λ. Clearly, $\mathcal{T}_{\sigma,p}$ can be considered as a subspace of $\mathcal{T}_{\sigma,\overline{p}}$ with identity of the norms. As we can see, in our notation the line over the index p indicates that we are dealing with functions of $2d$ variables.

Families of linear polynomial operators. The family $\{ \mathcal{L}^{(\varphi)}_{\sigma;\,\lambda} \}$ given in (2.7)-(2.8) can be considered as an operator into the space of functions of $2d$ variables

$$\mathbb{L}_\sigma \; : \; L_p \; \longrightarrow \; \mathcal{T}_{\sigma,\overline{p}} \; \subset \; L_{\overline{p}} \, , \; \sigma \ge 0 \; . \tag{1.3}$$

Such an interpretation leads to the following natural definition. The family $\{ \mathcal{L}^{(\varphi)}_{\sigma;\,\lambda} \}$ is called *convergent* (or *converges*) *in* L_p, if for each $f \in L_p$

$$\lim_{\sigma \to +\infty} \| f - \mathcal{L}^{(\varphi)}_{\sigma;\,\lambda}(f) \|_{\overline{p}} \; = \; 0 \; . \tag{1.4}$$

The classical means $\mathcal{F}^{(\varphi)}_\sigma$ of type (2.3) generated by the kernels $W_\sigma(\varphi)(h)$ (see (2.8)) in place of the kernels $R^{(\alpha,\beta)}_n(h)$ are also of type (1.3). In this case the space $\mathcal{T}_{\sigma,\overline{p}}$ can be replaced in this case by its subspace $\mathcal{T}_{\sigma,p}$ and in view of (1.2) the concept of convergence described in (1.4) coincides with the usual L_p-convergence. For more details concerning general operators of type (1.3) we refer to [10].

Fourier transform. The Fourier transform and its inverse are given by

$$\widehat{g}(\xi) = \int\limits_{\mathbb{R}^d} g(x)\, e^{-ix\xi}\, dx \ , \quad g^{\vee}(x) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d} g(\xi)\, e^{ix\xi}\, d\xi \ , \quad g \in L_1(\mathbb{R}^d)\,.$$

Relations up to constants. By "$A \lesssim B$" we denote the relation $A \le cB$, where c is a positive constant independent of $f \in L_p$ (or $f \in C$) and $\sigma \ge 0$. The symbol "$\asymp$" indicates equivalence. It means that $A \lesssim B$ and $B \lesssim A$ simultaneously.

Smoothness. Let $\alpha > 0$. We denote by $\mathcal{H}_\alpha$ the class of complex-valued continuous functions ψ defined on $\mathbb{R}^d$ which are infinitely differentiable on $\mathbb{R}^d \setminus \{0\}$ and satisfy the conditions $\psi(-\xi) = \overline{\psi(\xi)}$ for $\xi \in \mathbb{R}^d$, $\psi(0) = 0$, $\psi(\xi) \ne 0$ for $\xi \in \mathbb{R}^d \setminus \{0\}$. Moreover the function ψ is assumed to be homogeneous of order α, that is,

$$\psi(t\xi) \ = \ t^\alpha \psi(\xi)\,, \ \ \xi \in \mathbb{R}^d\,, \ t > 0\,.$$

Any function $\psi \in \mathcal{H}_\alpha$ determines smoothness in the sense that it generates

- the linear operator of multiplier type

$$\mathcal{D}(\psi) : \, e^{i\nu x} \ \longrightarrow \ \psi(\nu)\, e^{i\nu x}\,, \ \nu \in \mathbb{Z}^d\,, \tag{1.5}$$

- the scale of spaces of "smooth" functions

$$X_p(\psi) \ = \ \{\, g \in L_p : \, \mathcal{D}(\psi) g \in L_p \,\}\,, \ 1 \le p \le +\infty\,, \tag{1.6}$$

- the K-functional

$$K_\psi(f,\delta)_p \ = \ \inf_{g:\, g \in X_p(\psi)} \{\, \|\, f - g \,\|_p \, + \, \delta^\alpha \,\|\, \mathcal{D}(\psi) g \,\|_p \}\,, \ \delta > 0, \ f \in L_p\,, \tag{1.7}$$

- and the corresponding polynomial K-functional

$$\widetilde{K}_\psi(f,\delta)_p \ = \ \inf_{T \in \mathcal{T}_{1/\delta}} \{\, \|\, f - T \,\|_p \, + \, \delta^\alpha \,\|\, \mathcal{D}(\psi) T \,\|_p \}\,, \ \delta > 0, \ f \in L_p\,. \tag{1.8}$$

The classical smoothness concepts as, for instance, classical derivatives, the Laplace operator, Sobolev spaces and related K-functionals are special cases of this general constructions. In particular the operator $(-\Delta)^{\beta/2}$ and the polynomial K-functionals given in (2.11) correspond to $\psi(\xi) = |\,\xi\,|^\beta$. For more details about the general approach to the concept of smoothness and for further examples we refer to [11].

General Convergence Theorem (GCT). We denote by $\mathcal{K}$ the class of generators of families of type (2.7)-(2.8), that is, the set consisting of complex-valued continuous functions φ defined on $\mathbb{R}^d$ having compact support contained in $D_1 = \{\xi : \, |\,\xi\,| \le 1\}$ and satisfying $\varphi(0) = 1$ and $\varphi(-\xi) = \overline{\varphi(\xi)}$ for each $\xi \in \mathbb{R}^d$. For $\varphi \in \mathcal{K}$ we put

$$\mathcal{P}_\varphi \ = \ \{\, p \in (0, +\infty] : \, \widehat{\varphi} \in L_p(\mathbb{R}^d) \,\}\,. \tag{1.9}$$

GCT ([10]). *Let $\varphi \in \mathcal{K}$ and $1 \in \mathcal{P}_\varphi$. Then the family $\{ \mathcal{L}^{(\varphi)}_{\sigma;\lambda} \}$ converges in L_p if and only if $p \in \mathcal{P}_\varphi$. Moreover, for $1 \le p \le +\infty$*

$$\| f - \mathcal{L}^{(\varphi)}_{\sigma;\lambda}(f) \|_{\overline{p}} \asymp \| f - \mathcal{F}^{(\varphi)}_{\sigma}(f) \|_p , \ f \in L_p, \sigma \ge 0 . \tag{1.10}$$

This statement enables us to reduce the convergence problem to the study of the Fourier transform of a generator. In [10] it was applied to the families generated by the kernels of Fejйr, Valleй-Poussin, Rogosinski and Bochner-Riesz.

General Equivalence Theorem (GET). In [11] conditions with respect to φ and ψ providing the equivalence of the approximation error of the family given in (2.7) - (2.8) and a corresponding polynomial K-functional (1.8) have been described. Here we give the formulation of this result.

Henceforth, we write $v(\cdot) \overset{(q,\eta)}{\prec} w(\cdot)$ if the Fourier transform of the function $((\eta v)/w)$ belongs to $L_q(\mathbb{R}^d)$. The notation $v(\cdot) \overset{(q,\eta)}{\asymp} w(\cdot)$ indicates equivalence. It means that $v(\cdot) \overset{(q,\eta)}{\prec} w(\cdot)$ and $w(\cdot) \overset{(q,\eta)}{\prec} v(\cdot)$ simultaneously. A pair of infinitely differentiable functions (η, θ) with compact supports is said to be *a plane resolution of unity* (on the unit ball D_1) if there exists a number ρ, $0 < \rho < 1/2$, such that $\eta(\xi) = 1$ for $|\xi| \le \rho$, $\theta(\xi) = 1$ for $2\rho \le |\xi| \le 1$ and $\eta(\xi) + \theta(\xi) = 1$ for each $\xi \in D_1$.

GET ([11]). *Let $0 < p \le +\infty$, $\widetilde{p} = \min(1,p)$, $\varphi \in \mathcal{K}$, $\varphi(\xi) \ne 1$ for $\xi \ne 0$, $\widehat{\varphi} \in L_{\widetilde{p}}(\mathbb{R}^d)$ and $\psi \in \mathcal{H}_\alpha$ for some $\alpha > 0$. If there exist a plane resolution of unity (η, θ) and a number $k \in \mathbb{N}$ such that $1 - \varphi(\cdot) \overset{(\widetilde{p},\eta)}{\asymp} \psi(\cdot)$ and $(\varphi(\cdot))^k \overset{(\widetilde{p},\theta)}{\prec} 1 - \varphi(\cdot)$ then*

$$\| f - \mathcal{L}^{(\varphi)}_{\sigma;\lambda}(f) \|_{\overline{p}} \asymp \widetilde{\mathcal{K}}_\psi(f, \sigma^{-1})_p , \ f \in L_p, \sigma > 0 . \tag{1.11}$$

In [11] this statement was applied to the study of the approximative behaviour of by methods generated by some classical kernels. In Section 3 we show that it is also applicable to the Riesz families (2.1).

2 Fourier transform of the generator of the Riesz kernels

Following our approach we study now the properties of the Fourier transform of the generators

$$\varphi_{\alpha,\beta}(\xi) \ = \begin{cases} \left(1 - |\xi|^\beta \right)^\alpha & , \quad |\xi| \le 1 \\ \quad 0 & , \quad |\xi| > 1 \end{cases} . \tag{2.1}$$

of the Riesz kernels and we determine the set $\mathcal{P}_{\alpha,\beta}$ defined in (2.10). Henceforth, let us denote by $\mathbb{E}$ the set $\{2k, \, k \in \mathbb{N}\}$ of even integers. Unimportant positive constants independent of x, denoted by c (with subscripts and superscripts) may have different values in different formulas (but not in the same formula).

Theorem 2.1. *For $\alpha \geq 0$, $\beta > 0$ we have*

$$\mathcal{P}_{\alpha,\beta} = \begin{cases} (\,2d/(d+2\alpha+1),\, +\infty\,] &,\quad \alpha \geq 0,\, \beta \in \mathbb{E} \\ (\,2d/(d+2\alpha+1),\, +\infty\,] &,\quad 0 \leq \alpha < \beta + (d-1)/2,\, \beta \notin \mathbb{E} \\ (\,d/(d+\beta),\, +\infty] &,\quad \alpha \geq \beta + (d-1)/2,\, \beta \notin \mathbb{E} \end{cases} . \tag{2.2}$$

Proof. For $\alpha = 0$ the function $\varphi_{\alpha,\beta}$ coincides with the generator of the Bochner-Riesz kernels φ_0. In this case (2.2) directly follows from (2.9) (see also [16]). Henceforth, we suppose $\alpha > 0$.

Let ψ, ψ_0 and ψ_1 be real-valued radial (depending only on $r = |\,\xi\,|$) infinitely differentiable function defined on $\mathbb{R}^d$ satisfying

$$\psi(\xi) = \begin{cases} 1\,, & \xi \in D_1 \\ 0\,, & \xi \notin D_{5/4} \end{cases} ; \; \psi_0(\xi) = \begin{cases} 1\,, & \xi \in D_{1/2} \\ 0\,, & \xi \notin D_{3/4} \end{cases} ; \; \psi_1(\xi) = \psi(\xi) - \psi_0(\xi). \tag{2.3}$$

Here $D_r = \{\,\xi : |\,\xi\,| \leq r\,\}$. Obviously,

$$\varphi_{\alpha,\beta}(\xi) \;=\; \varphi_{\alpha,\beta}\,\psi_0(\xi) \;+\; \varphi_{\alpha,\beta}\,\psi_1(\xi)\,,\; \xi \in \mathbb{R}^d. \tag{2.4}$$

Step 1. *Estimate of* $\widehat{\varphi_{\alpha,\beta}\,\psi_0}$ *for* $\beta \notin \mathbb{E}$.

We put

$$m \;\equiv\; m(d,\beta) \;=\; \left[\,\beta^{-1}(d+2\beta+1)\,\right] \,+\, 1\;. \tag{2.5}$$

Then we have

$$\beta m \;>\; k\;,\quad (\,k \,=\, [\,d/p_*\,] + 1\,,\; p_* \,\equiv\, p_*(d,\beta) \,=\, d/(d+2\beta)\,)\;. \tag{2.6}$$

In view of

$$(\,1-y\,)^\alpha \;=\; \sum_{\nu=0}^{m-1} \frac{(-1)^\nu [\alpha]_\nu}{\nu!}\, y^\nu \;+\; y^m g(y)\,,\; -1 < y < 1\;, \tag{2.7}$$

where $[\alpha]_\nu = \alpha(\alpha-1)\ldots(\alpha-\nu+1)$ and $g(y)$ is analytic on $(-1,1)$, we obtain

$$\varphi_{\alpha,\beta}\,\psi_0(\xi) \;=\; \sum_{\nu=0}^{m-1} \frac{(-1)^\nu [\alpha]_\nu}{\nu!}\, |\,\xi\,|^{\beta\nu}\, \psi_0(\xi) \;+\; |\,\xi\,|^{\beta m} g(|\,\xi\,|^\beta)\, \psi_0(\xi)\,,\; \xi \in \mathbb{R}^d. \tag{2.8}$$

As it was shown in [15] it holds

$$\left| \left(|\,\cdot\,|^{\beta\nu}\psi_0(\cdot)\right)^\wedge(x) \right| \;\leq\; c_1\,(1+|\,x\,|)^{-(d+\beta\nu)},\; x \in \mathbb{R}^d,\; \nu = 1,2,\ldots,m-1, \tag{2.9}$$

$$\left| \left(|\,\cdot\,|^{\beta}\,\psi_0(\cdot)\right)^\wedge(x) \right| \;\geq\; c_2\,|\,x\,|^{-(d+\beta)}\,,\; x \in \Omega(\rho,\,\theta,u_0)\,, \tag{2.10}$$

where

$$\Omega(\rho,\,\theta,\,u_0) \,=\, \left\{\, x = ru : r > \rho\,,\; u \in \mathcal{S}^{d-1},\, \cos\theta \leq (u,u_0) \leq 1\,\right\} \tag{2.11}$$

for some $\rho \geq 1,\ 0 < \theta < \pi/2$ and $u_0 \in \mathcal{S}^{d-1}$. Taking into account that ψ_0 is infinitely differentiable and that it has a compact support we get, in particular, the estimate

$$| \widehat{\psi}_0(x) | \ \leq \ c\,(1 + |\,x\,|)^{-(d+2\beta)}\,,\ \in \mathbb{R}^d\ . \tag{2.12}$$

In a straightforward manner one can easily check that the function $|\,\xi\,|^{\beta m} g(|\,\xi\,|^{\beta})$ has continuous (mixed) derivatives up to the order k given in (2.6). By elementary properties of the Fourier transform in combination with (2.5)-(2.6) we find

$$\left| \left(|\,\cdot\,|^{\beta m}\, g(|\,\cdot\,|^{\beta})\,\psi_0(\cdot)\right)^{\wedge}(x) \right| \ \leq \ c\,(1 + |\,x\,|)^{-([d+2\beta]+1)}\,,\ x \in \mathbb{R}^d\ . \tag{2.13}$$

In view of (2.8) we obtain

$$\left| \widehat{\varphi_{\alpha,\beta}\,\psi_0}(x) \right| \ \leq \ c'\,(1 + |\,x\,|)^{-(d+\beta)}\,,\ x \in \mathbb{R}^d\,, \tag{2.14}$$

$$\left| \widehat{\varphi_{\alpha,\beta}\,\psi_0}(x) \right| \ \geq \ c''\,|\,x\,|^{-(d+\beta)}\,,\ x \in \Omega(\rho',\,\theta,\,u_0) \tag{2.15}$$

by (2.9) - (2.13) for some $\rho' > \rho$.

Step 2. *Estimate of* $\widehat{\varphi_{\alpha,\beta}\,\psi_0}$ *for* $\beta \in \mathbb{E}$.

In the case $\beta \in \mathbb{E}$ the behavior of the Fourier transform of the first item in (2.4) is completely different. Indeed, the function $1 - |\,\xi\,|^{\beta}$ becomes a polynomial and, therefore, the function $\varphi_{\alpha,\,\beta}\,\psi_0(\xi)$ is infinitely differentiable and, in particular, we have

$$\left| \widehat{\varphi_{\alpha,\,\beta}\,\psi_0}(x) \right| \ \leq \ c\,(1 + |\,x\,|)^{-(\alpha+d/2+3/2)}\,,\ x \in \mathbb{R}^d\ . \tag{2.16}$$

Step 3. *Estimate of* $\widehat{\varphi_{\alpha,\beta}\,\psi_1}$.

The Fourier transform of the second item in (2.4) can be studied by reduction to the properties of the generator of the Bochner-Riesz kernels $\varphi_{\delta}(\xi) = (1 - |\,\xi\,|^2)_+^{\delta}$ with $\delta > 0$. Using (2.9), the asymptotic formula for the Bessel function and the results on the distribution of its zeros (see, e.g. [18] (Chapter 8)) we obtain the estimates

$$|\,\widehat{\varphi}_{\delta}(x)\,| \ \leq \ c'\,(1 + |\,x\,|)^{-(\delta+d/2+1/2)}\,,\ x \in \mathbb{R}\,, \tag{2.17}$$

$$\widehat{\varphi}_{\delta}(x) \ \geq \ c_{(+)}\,|\,x\,|^{-(\delta+d/2+1/2)}\,,\ x \in \Omega_+\,, \tag{2.18}$$

$$\widehat{\varphi}_{\delta}(x) \ \leq \ -c_{(-)}\,|\,x\,|^{-(\delta+d/2+1/2)}\,,\ x \in \Omega_-\,, \tag{2.19}$$

$$|\,\widehat{\varphi}_{\delta}(x)\,| \ \geq \ c''\,|\,x\,|^{-(\delta+d/2+1/2)}\,,\ x \in \Omega_+ \cup \Omega_-\,, \tag{2.20}$$

where (for the sake of shortness we omit the index δ in our notations, if it does not affect the concepts)

$$\Omega_{\pm} = \bigcup_{k=1}^{+\infty} \left\{\, x \in \mathbb{R}^d : \, a_k^{\pm} \ \leq \ |\,x\,| \ \leq \ b_k^{\pm} \,\right\}, \tag{2.21}$$

$$1 \le a_k^{\pm} < b_k^{\pm} \le a_{k+1}^{\pm}\,;\ \inf_k (b_k - a_k^{\pm}) > 0\,;\ a_k^{\pm} = O(k)\,,\ k \to +\infty\,. \tag{2.22}$$

We shall use the representation

$$\varphi_{\alpha,\beta}\,\psi_1(\xi) = \varphi_\alpha(\xi)\,\psi_1(\xi)\,g_{\alpha,\beta}(\,1 - |\,\xi\,|^2\,)\,,\ \xi \in \mathbb{R}^d\,, \tag{2.23}$$

where

$$g_{\alpha,\beta}(\eta) = \begin{cases} \left(\dfrac{1-(1-\eta)^{\beta/2}}{\eta}\right)^{\alpha} & ,\quad 0 < |\,\eta\,| < 1 \\ (\beta/2)^{\alpha} & ,\quad \eta = 0 \end{cases}\,. \tag{2.24}$$

The function $g_{\alpha,\beta}(\eta)$ is analytic in $(-1,1)$. Taylor expansion at $\eta = 0$ up to the $(s-1)$th term, where

$$s \equiv s(d,\,\alpha) = [\,d/p_0\,] + 2\,,\quad p_0 \equiv p_0(d,\,\alpha) = 2d/(d+2\alpha+3)\,, \tag{2.25}$$

yields

$$\begin{aligned}(\beta/2)^{-\alpha}\,(\varphi_{\alpha,\beta}\,\psi_1)\,(\xi) &= \varphi_\alpha(\xi)\,\psi_1(\xi) + \sum_{\nu=1}^{s-1} g_\nu^{(\alpha,\beta)} \cdot (\varphi_{\alpha+\nu}\,\psi_1)\,(\xi) + \\ &\quad g_s^{(\alpha,\,\beta)} \cdot (\varphi_{\alpha+s}\,\psi_1)\,(\xi)\,h(\xi) = \varphi_\alpha(\xi) + \sum_{\nu=1}^{s-1} \\ &\quad g_\nu^{(\alpha,\,\beta)} \varphi_{\alpha+\nu}(\xi) - \sum_{\nu=0}^{s-1} g_\nu^{(\alpha,\,\beta)}\,(\varphi_{\alpha+\nu}\,\psi_0)\,(\xi) + \\ &\quad g_s^{(\alpha,\beta)}\,(\varphi_{\alpha+s}\,\psi_1)\,(\xi)\,h(\xi) \equiv \varphi_\alpha(\xi) + I(\xi)\,,\end{aligned} \tag{2.26}$$

by (2.23) and (2.4) for $\xi \in \mathbb{R}^d$. Here $g_\nu^{(\alpha,\beta)}$, $\nu = 1,\ldots,s$, are appropriate coefficients and the function h is infinitely differentiable in $D\sqrt{2} \setminus \{0\}$. By (2.17) we get

$$|\,\widehat{\varphi}_{\alpha+\nu}(x)\,| \le c\,(\,1 + |\,x\,|\,)^{-(\alpha+\nu+d/2+1/2)}\,,\ x \in \mathbb{R}^d\,,\ \nu = 0,1,\ldots,s-1\,. \tag{2.27}$$

The functions $\varphi_{\alpha+\nu}\,\psi_0\,,\ \nu = 0,1,\ldots,s-1$, are infinitely differentiable and they have a compact support. Hence, in particular,

$$\left|(\varphi_{\alpha+\nu}\psi_0)^\wedge\,(x)\right| \le c\,(1+|\,x\,|)^{-(\alpha+d/2+3/2)},\ x \in \mathbb{R}^d,\ \nu = 0,1,\ldots,s-1. \tag{2.28}$$

In view of (2.25) one can prove by direct calculation that the function $\varphi_{\alpha+s}\,\psi_1 h$ has continuous (mixed) derivatives up to the order $[d/p_0]+1$. This implies

$$\left|(\varphi_{\alpha+s}\psi_1\,h)^\wedge\,(x)\right| \le c(1+|x|)^{-([\alpha+d/2+3/2]+1)},\ x \in \mathbb{R}^d,\ \nu = 0,1,\ldots,s-1\,. \tag{2.29}$$

By (2.27) for $\nu \ne 0$, (2.28) and (2.29) we find

$$|\,\widehat{I}(x)\,| \le c\,|\,x\,|^{-(\alpha+d/2+3/2)}\,,\ |\,x\,| \ge 1\,, \tag{2.30}$$

for the Fourier transform of the remainder $I(\xi)$ in (2.26). Combining (2.26), (2.30) with (2.17) - (2.20) for $\delta = \alpha$ we obtain the estimates

$$\left|\,\widehat{\varphi_{\alpha,\,\beta}\,\psi_1}(x)\,\right| \le c'\,(\,1 + |\,x\,|\,)^{-(\alpha+d/2+1/2)}\,,\ x \in \mathbb{R}^d\,, \tag{2.31}$$

$$\widehat{\varphi_{\alpha,\beta}\,\psi_1}(x) \;\geq\; c_{(+)}\,|\,x\,|^{-(\alpha+d/2+1/2)}\,,\; x\in\Omega_+\,, \tag{2.32}$$

$$\widehat{\varphi_{\alpha,\beta}\,\psi_1}(x) \;\leq\; -c_{(-)}\,|\,x\,|^{-(\alpha+d/2+1/2)}\,,\; x\in\Omega_-\,, \tag{2.33}$$

$$\left|\,\widehat{\varphi_{\alpha,\beta}\,\psi_1}(x)\right| \;\geq\; c''\,|\,x\,|^{-(\alpha+d/2+1/2)}\,,\; x\in\Omega_+\cup\Omega_-\,, \tag{2.34}$$

where $\Omega_\pm$ are of type (2.21)-(2.22).

We have to consider the 4 cases:

1) $\alpha\,>\,0,\,\beta\in\mathbb{E}$

2) $0\,<\,\alpha\,<\,\beta+(d-1)/2,\,\beta\notin\mathbb{E}$

3) $\alpha\,>\,\beta+(d-1)/2,\,\beta\notin\mathbb{E}$

4) $\alpha\,=\,\beta+(d-1)/2,\,\beta\notin\mathbb{E}$.

In *case 1)* we get

$$|\,\widehat{\varphi}_{\alpha,\beta}(x)\,| \;\leq\; c\,(\,1\,+\,|\,x\,|\,)^{-(\alpha+d/2+1/2)}\,,\; x\in\mathbb{R}^d\,, \tag{2.35}$$

by (2.4), (2.16) and (2.31) and

$$\begin{aligned}|\,\widehat{\varphi}_{\alpha,\beta}(x)\,| \;&\geq\; \left|\,\widehat{\varphi_{\alpha,\beta}\,\psi_1}(x)\right| \;-\; \left|\,\widehat{\varphi_{\alpha,\beta}\,\psi_0}(x)\right| \;\geq\\ &\geq\; |\,x\,|^{-(\alpha+d/2+1/2)}\,\left(c_{(+)}\;-\;2c\,|\,x\,|^{-1}\right)\;\geq\\ &\geq\; c'\,|\,x\,|^{-(\alpha+d/2+1/2)}\,,\; x\in\Omega_+\,,\;|\,x\,|\geq(4c)/c_{(+)}\,.\end{aligned} \tag{2.36}$$

by (2.4), (2.16) and (2.32). In view of (2.21) - (2.22) we obtain

$$c_1\sum_{k=k_0}^{+\infty}\int_{a_k^+}^{b_k^+} r^{-p(\alpha+d/2+1/2)+d-1}\,dr \;\leq\; \|\,\widehat{\varphi}_{\alpha,\beta}\,\|_p^p \;\leq\; c_2\int_1^{+\infty} r^{-p(\alpha+d/2+1/2)+d-1}\,dr \tag{2.37}$$

by (2.35) and (2.36) for $0\,<\,p\,<\,+\infty$ and some $k_0\in\mathbb{N}$. This implies $\mathcal{P}_{\alpha,\beta}=(\,2d/(d+2\alpha+1),+\infty\,]$.

In *case 2)* the right-hand side of (2.14) can be estimated by the right-hand side of (2.31). By (2.4), (2.14), (2.31) we find

$$|\,\widehat{\varphi}_{\alpha,\beta}(x)\,| \;\leq\; c\,(\,1\,+\,|\,x\,|\,)^{-(\alpha+d/2+1/2)}\,,\; x\in\mathbb{R}^d\,, \tag{2.38}$$

and by (2.4), (2.14) and (2.32) the estimates

$$\begin{aligned}|\,\widehat{\varphi}_{\alpha,\beta}(x)\,| \;&\geq\; \left|\,\widehat{\varphi_{\alpha,\beta}\,\psi_1}(x)\right| \;-\; \left|\,\widehat{\varphi_{\alpha,\beta}\,\psi_0}(x)\right| \;\geq\\ &\geq\; |\,x\,|^{-(\alpha+d/2+1/2)}\,\left(c_{(+)}\;-\;2c'\,|\,x\,|^{-\delta}\right)\;\geq\\ &\geq\; c\,|\,x\,|^{-(\alpha+d/2+1/2)}\,,\; x\in\Omega_+\,,\;|\,x\,|\geq\left(\,(4c')/c_{(+)}\,\right)^{1/\delta}\,,\end{aligned} \tag{2.39}$$

where

$$\delta \;=\; (d+\beta) - (\alpha + d/2 + 1/2) \;>\; 0. \tag{2.40}$$

Applying the argument in (2.37) it follows that $\mathcal{P}_{\alpha,\beta} = (\,2d/(d+2\alpha+1), +\infty\,]$ by (2.38) and (2.39).

In *case 3)* the first item in (2.4) is dominating and now the right-hand side of (2.31) can be estimated by the right-hand side of (2.14). By (2.4), (2.14), (2.31) we derive

$$|\,\widehat{\varphi}_{\alpha,\beta}(x)\,| \;\le\; c\,(\,1 \,+\, |\,x\,|\,)^{-(d+\beta)}\,,\; x \in \mathbb{R}^d\,, \tag{2.41}$$

and by (2.4), (2.15) and (2.31) we arrive at

$$\begin{aligned} |\,\widehat{\varphi}_{\alpha,\beta}(x)\,| \;&\ge\; \left|\,\widehat{\varphi_{\alpha,\beta}\,\psi_0}(x)\,\right| \;-\; \left|\,\widehat{\varphi_{\alpha,\beta}\,\psi_1}(x)\,\right| \;\ge \\ &\ge\; |\,x\,|^{-(d+\beta)}\left(c'' \,-\, 2c'\,|\,x\,|^{-|\,\delta\,|}\right) \;\ge\; c\,|\,x\,|^{-(d+\beta)}\,, \\ &\qquad x \in \Omega(\rho\,',\,\theta,\,u_0)\,,\; |\,x\,| \ge (\,(4c')/c''\,)^{1/|\,\delta\,|}\,, \end{aligned} \tag{2.42}$$

where δ given in (2.40) is now negative. Applying the argument in (2.37) with $d+\beta$ in place of $\alpha + d/2 + 1/2$ and $\Omega(\rho\,',\,\theta,\,u_0)$ of type (2.11) in place of Ω_+ we obtain $\mathcal{P}_{\alpha,\beta} = (\,d/(d+\beta), +\infty\,]$ by (2.41) and (2.42).

In *case 4)* we have $d+\beta = \alpha + d/2 + 1/2$ and the right-hand sides of all estimates we applied are equivalent to each other. Now the lower estimate can be obtained by the observation that in view of (2.15) the Fourier transform of the first item in (2.4) does not change its sign in a certain connected unbounded domain, but in view of (2.32) and (2.33) the second item does. Similarly to cases 2) and 3) we get

$$|\,\widehat{\varphi}_{\alpha,\beta}(x)\,| \;\le\; c\,(\,1 \,+\, |\,x\,|\,)^{-(d+\beta)}\,,\; x \in \mathbb{R}^d \tag{2.43}$$

by (2.4), (2.14), (2.31). By (2.15) the Fourier transform of $\varphi_{\alpha,\,\beta}\,\psi_0$ does not change its sign in the domain $\Omega \equiv \Omega(\rho\,',\,\theta,\,u_0)$. If it is positive we obtain

$$\begin{aligned} \widehat{\varphi}_{\alpha,\beta}(x) \;&=\; \widehat{\varphi_{\alpha,\beta}\,\psi_0}(x) \;+\; \widehat{\varphi_{\alpha,\,\beta}\,\psi_1}(x) \;\ge \\ &\ge\; (\,c'' \,+\, c_{(+)}\,)\,|\,x\,|^{-(d+\beta)}\,,\; x \in \Omega \,\cap\, \Omega_+ \end{aligned}$$

applying (2.4), (2.15) and (2.32). Otherwise, we use (2.4), (2.15) and (2.33) to find

$$\begin{aligned} \widehat{\varphi}_{\alpha,\beta}(x) \;&=\; \widehat{\varphi_{\alpha,\beta}\,\psi_0} \;+\; \widehat{\varphi_{\alpha,\beta}\,\psi_1} \;\le \\ &\le\; -(\,c'' \,+\, c_{(-)}\,)\,|\,x\,|^{-(d+\beta)}\,,\; x \in \Omega \,\cap\, \Omega_-\,. \end{aligned}$$

In both cases

$$|\,\widehat{\varphi}_{\alpha,\beta}(x)\,| \;\ge\; c\,|\,x\,|^{-(d+\beta)}\,,\; x \in \Omega \,\cap\, \Omega'\,, \tag{2.44}$$

where Ω' is Ω_+ or Ω_-. Applying the standard argument (see (2.37)) we obtain $\mathcal{P}_{\alpha,\beta} = (\,d/(d+\beta), +\infty\,]$ by (2.43) and (2.44). The proof is complete. □

We give some remarks. As it was mentioned in Introduction, the convergence of Riesz means (2.3) in L_p with $1 \le p \le +\infty$ for $\alpha > (d-1)/2$ and the equivalent statement on the membership of $\widehat{\varphi}_{\alpha,\beta}$ in L_1 for such α can be found in ([9]). A widespread mistake is connected with this result. For example, in [17] (where [9] (p. 215) was quoted) one derives these statements (which themselves are correct) from the estimate

$$|\,\widehat{\varphi}_{\alpha,\beta}(x)\,| \;\le\; c\,(\,1 \,+\, |\,x\,|\,)^{-(\alpha+d/2+1/2)}\,,\; x \in \mathbb{R}^d\,, \tag{2.45}$$

which was quoted to hold for all $\alpha, \beta > 0$. However, estimate (2.45) is not valid for $\alpha > \beta+(d-1)/2$ by (2.42). For this reason some statements as, for instance, Remark 13 in [17] devoted to the case $0 < p < 1$ contain incorrect conjectures. Another example. Apparently, with a reference to the same work [9] by J. Peetre it was incorrectly stated in [7] that the Fourier transform of the function $(\,1 - |\,\xi\,|\,)_+^\alpha$ in the one-dimensional case should behave like $|\,x\,|^{-(\alpha+1)}$ if $|\,x\,| \to +\infty$. However, note that this is true only if $0 < \alpha \le 1$. Fortunately, the author dealt mainly with the case $1 \le p \le +\infty$, where, as it was mentioned above, estimate (2.45) always leads to correct conclusions.

3 Approximation properties of families generated by Riesz kernels

Applying the general principles given in Section 1 in combination with the results of Section 2 we obtain the results on the convergence and the approximation quality of families (2.1) generated by the Riesz kernels given in (2.2).

Theorem 3.1. *Let $0 < p \le +\infty$, $\alpha > (d-1)/2$ and $\beta > 0$. Then the family $\{\,\mathcal{R}_{n;\lambda}^{(\alpha,\beta)}\}$ converges in L_p if and only if $p \in \mathcal{P}_{\alpha,\beta}$ given in* (2.2).

Proof. The condition $\alpha > (d-1)/2$ implies $1 \in \mathcal{P}_{\alpha,\beta}$. Now the statement immediately follows from the GCT and Theorem 2.1. □

Theorem 3.2. *Let $\alpha > (d-1)/2$, $\beta > 0$. It holds for $p \in \mathcal{P}_{\alpha,\beta}$ that*

$$\|\, f - \mathcal{R}_{n;\,\lambda}^{(\alpha,\beta)}(f)\,\|_{\overline{p}} \asymp \widetilde{K}_\beta(f,(n+1)^{-1})_p\,,\; f \in L_p\,,\, n \in \mathbb{N}_0\,, \tag{3.1}$$

where $\widetilde{K}_\beta(f,\delta)_p$ is given in (2.11).

Proof. Henceforth, we put $\varphi(\xi) = \varphi_{\alpha,\beta}(\xi)$, $\psi(\xi) = |\,\xi\,|^\beta$. Clearly, $\varphi \in \mathcal{K}$, $\varphi(\xi) \ne 1$ for $\xi \ne 0$, $\widehat{\varphi} \in L_{\widetilde{p}}(\mathbb{R}^d)$ $(\widetilde{p} = (1,p))$ for $p \in \mathcal{P}_{\alpha,\beta}$ and $\psi \in \mathcal{H}_\beta$. Let (η,θ) be a plane resolution of unity.

Applying (2.7) with $m+1$ instead of m, where m is given in (2.5), we get

$$\begin{aligned}(\varphi\,\eta)(\xi) \;&=\; \eta(\xi) - \alpha|\,\xi\,|^\beta\eta(\xi) \,+\, \sum_{\nu=2}^{m}\frac{(-1)^\nu[\alpha]_\nu}{\nu!}\,|\,\xi\,|^{\beta\nu}\eta(\xi)\,+ \\ &\qquad |\,\xi\,|^{\beta(m+1)}h(|\,\xi\,|^\beta)\,\eta(\xi)\,,\; \xi \in \mathbb{R}^d\,,\end{aligned} \tag{3.2}$$

where $h(y)$ is analytic on $(-1,1)$. By (3.2)

$$((1-\varphi)/\psi)\eta(\xi) = \alpha\eta(\xi) + \sum_{\nu=1}^{m-1} \frac{(-1)^\nu [\alpha]_{\nu+1}}{(\nu+1)!} \,|\,\xi\,|^{\beta\nu}\eta(\xi) - |\,\xi\,|^{\beta m} h(|\,\xi\,|^\beta)\,\eta(\xi)\,,\ \xi \in \mathbb{R}^d\,. \tag{3.3}$$

For $\beta \notin \mathbb{E}$ similarly to (2.14) it follows from (3.3) that

$$|\,(((1-\varphi)/\psi)\eta)^\wedge(x)\,| \le c\,(\,1 + |\,x\,|\,)^{-(d+\beta)}\,,\ x \in \mathbb{R}^d\,. \tag{3.4}$$

This relation is also valid for $\beta \in \mathbb{E}$, because the function $((1-\varphi)/\psi)\eta$ being extended to 0 by continuity is infinitely differentiable. By (2.2) the condition $p \in \mathcal{P}_{\alpha,\beta}$ always implies $p > d/(d+\beta)$. Now applying (3.4) we obtain $1-\varphi(\cdot) \overset{(\tilde{p},\eta)}{\prec} \psi(\cdot)$ for such p. In order to prove the inverse relation we note that the function

$$G(y) = \begin{cases} \dfrac{y}{1-(1-y)^\alpha} & ,\quad 0 < |\,y\,| < 1 \\ \alpha^{-1} & ,\quad y = 0 \end{cases}$$

is analytic on $(-1,1)$. Using its expansion at the point 0 we get for $\xi \in \mathbb{R}^d$

$$(\psi/(1-\varphi))\eta(\xi) = \alpha^{-1}\eta(\xi) + \sum_{\nu=1}^{m-1} c_\nu\,|\,\xi\,|^{\beta\nu}\eta(\xi) + |\,\xi\,|^{\beta m} h_1(|\,\xi\,|^\beta)\,\eta(\xi)\,, \tag{3.5}$$

where $c_\nu \equiv c_\nu(\alpha,\beta)$ are appropriate coefficients and $h_1(y)$ is analytic on $(-1,1)$. Applying the argument above for (3.5) in place of (3.3) we obtain $\psi(\cdot) \overset{(\tilde{p},\eta)}{\prec} 1-\varphi(\cdot)$. Thus, we have the equivalence $1-\varphi(\cdot) \overset{(\tilde{p},\eta)}{\asymp} \psi(\cdot)$.

As it was shown in [16] (see relations (5.19) and (5.21)), the function

$$g_k(y) = \frac{(1-y^2)_+^{\alpha k}}{1-(1-y^2)_+^\alpha}\,,\ y \neq 0\,,$$

where $k \in \mathbb{N}$, has the property that

$$\lim_{y\to 1} g_k^{(s)}(y) = 0\,,\ 1 \le s \le k-1\,.$$

Using this fact and taking into account that $g(y) = |\,y\,|^{\beta/2}$ is infinitely differentiable for $y \neq 0$ one can easily check in a straightforward manner that for the function

$$G_k(y) = g_k\left(|\,y\,|^{\beta/2}\right)\,,\ y \neq 0\,, \tag{3.6}$$

the relations

$$\lim_{y\to 1} G_k^{(s)}(y) = 0\,,\ 1 \le s \le k-1\,, \tag{3.7}$$

also hold. By (3.6), (3.7) the function

$$G_k(|\,\xi\,|)\,\theta(\xi) \;=\; \frac{\varphi^k(\xi)}{1-\varphi(\xi)}\,\theta(\xi) \tag{3.8}$$

has continuous (mixed) derivatives up to the order $k-1$. Hence, by choosing relevant k one can guarantee that its Fourier transform will belong to $L_{\widetilde{p}}(\mathbb{R}^d)$ and the condition $(\varphi(\cdot))^k \overset{(\widetilde{p},\theta)}{\prec} 1-\varphi(\cdot)$ will be valid. Now Theorem 3.2 immediately follows from the GET. □

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