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EMJ: from Scopus Q4 to Scopus Q3 in two years?!

Recently the list was published of all mathematical journals included in 2015 Scopus quartiles Q1 (334 journals), Q2 (318 journals), Q3 (315 journals), and Q4 (285 journals). Altogether 1252 journals.

With great pleasure we inform our readers that the Eurasian Mathematical Journal was included in this list, currently the only mathematical journal in the Republic of Kazakhstan and Central Asia.

It was included in Q4 with the SCImago Journal & Country Rank (SJR) indicator equal to 0,101, and is somewhere at the bottom of the Q4 list. With this indicator the journal shares places from 1240 to 1248 in the list of all 2015 Scopus mathematical journals. Nevertheless, this may be considered to be a good achievement, because Scopus uses information about journals for the three previous years, i. e. for years 2013-2015, and the EMJ is in Scopus only from the first quarter of year 2015.

The SJR indicator is calculated by using a sophisticated formula, taking into account various characteristics of journals and journals publications, in particular the average number of weighted citations received in the selected year by the documents published in the selected journal in the three previous years. This formula and related comments can be viewed on the web-page

$$http://www.scimagojr.com/journalrank.php?category = 2601\&area = 2600\&page = 1\&total_size = 373$$

(Help/Journals/Understand tables and charts/Detailed description of SJR.)

In order to enter Q3 the SJR indicator should be greater than 0,250. It looks like the ambitious aim of entering Q3 in year 2017 is nevertheless realistic due to recognized high level of the EMJ.

We hope that all respected members of the international Editorial Board, reviewers, authors of our journal, representing more than 35 countries, and future authors will provide high quality publications in the EMJ which will allow to achieve this aim.

On behalf of the Editorial Board of the EMJ
V.I. Burenkov, E.D. Nursultanov, T.Sh. Kalmenov,
R. Oinarov, M. Otelbaev, T.V. Tararykova, A.M. Temirkhanova

VICTOR IVANOVICH BURENKOV

(to the 75th birthday)

On July 15, 2016 was the 75th birthday of Victor Ivanovich Burenkov, editor-in-chief of the Eurasian Mathematical Journal (together with V.A. Sadovnichy and M. Otelbaev), director of the S.M. Nikol'skii Institute of Mathematics, head of the Department of Mathematical Analysis and Theory of Functions, chairman of Dissertation Council at the RUDN University (Moscow), research fellow (part-time) at the Steklov Institute of Mathematics (Moscow), scientific supervisor of the Laboratory of Mathematical Analysis at the Russian-Armenian (Slavonic) University (Yerevan, Armenia), doctor of physical and mathematical sciences (1983), professor (1986), honorary professor of the L.N. Gumilyov Eurasian National University (Astana, Kazakhstan, 2006), honorary doctor of the Russian-Armenian (Slavonic) University (Yerevan, Armenia, 2007), honorary member of staff of the University of Padua (Italy, 2011), honorary distinguished professor of the Cardiff School of Mathematics (UK, 2014), honorary professor of the Aktobe Regional State University (Kazakhstan, 2015).

V.I. Burenkov graduated from the Moscow Institute of Physics and Technology (1963) and completed his postgraduate studies there in 1966 under supervision of the famous Russian mathematician academician S.M. Nikol'skii.

He worked at several universities, in particular for more than 10 years at the Moscow Institute of Electronics, Radio-engineering, and Automation, the RUDN University, and the Cardiff University. He also worked at the Moscow Institute of Physics and Technology, the University of Padua, and the L.N. Gumilyov Eurasian National University.

He obtained seminal scientific results in several areas of functional analysis and the theory of partial differential and integral equations. Some of his results and methods are named after him: Burenkov's theorem of composition of absolutely continuous functions, Burenkov's theorem on conditional hypoellipticity, Burenkov's method of mollifiers with variable step, Burenkov's method of extending functions, the Burenkov-Lamberti method of transition operators in the problem of spectral stability of differential operators, the Burenkov-Guliyevs conditions for boundedness of operators in Morrey-type spaces. On the whole, the results obtained by V.I. Burenkov have laid the groundwork for new perspective scientific directions in the theory of functions spaces and its applications to partial differential equations, the spectral theory in particular.

More than 30 postgraduate students from more than 10 countries gained candidate of sciences or PhD degrees under his supervision. He has published more than 170 scientific papers. The lists of his publications can be viewed on the portals MathSciNet and MathNet.Ru. His monograph "Sobolev spaces on domains" became a popular text for both experts in the theory of function spaces and a wide range of mathematicians interested in applying the theory of Sobolev spaces.

In 2011 the conference "Operators in Morrey-type Spaces and Applications", dedicated to his 70th birthday was held at the Ahi Evran University (Kirsehir, Turkey). Proceedings of that conference were published in the EMJ 3-3 and EMJ 4-1.

The Editorial Board of the Eurasian Mathematical Journal congratulates Victor Ivanovich Burenkov on the occasion of his 75th birthday and wishes him good health and new achievements in science and teaching!

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ORDER OF THE ORTHOPROJECTION WIDTHS OF THE ANISOTROPIC NIKOL'SKII–BESOV CLASSES IN THE ANISOTROPIC LORENTZ SPACE

K.A. Bekmaganbetov, Ye. Toleugazy

Communicated by E.D. Nursultanov

Key words: orthoprojection width, anisotropic Lorentz space, anisotropic Nikol'skii–Besov class.

AMS Mathematics Subject Classification: 41A30, 42B35.

Abstract. In this paper we estimate the order of the orthoprojection widths of the anisotropic Nikol'skii–Besov classes in the anisotropic Lorentz space.

1 Introduction

Let $\{u_i(\mathbf{x})\}_{i=1}^M$ be an orthonormal system of functions in $L_2(\mathbb{T}^n)$ and V be a normed spaces. For each $f \in V$ we consider the approximations $\sum_{i=1}^M (f, u_i)u_i(\mathbf{x})$ that is the orthogonal projections of the function f onto the subspace of $L_2(\mathbb{T}^n)$ generated by the system $\{u_i(\mathbf{x})\}_{i=1}^M$. If a functional class $F \subset V$, then the quantity

$$d_M^{\perp}(F, V) = \inf_{\{u_i(\mathbf{x})\}_{i=1}^M} \sup_{f \in F} \left\| f(\cdot) - \sum_{i=1}^M (f, u_i)u_i(\cdot) \right\|_V \tag{1.1}$$

is called the orthoprojection width of this class in the space V. The width $d_M^{\perp}(F, V)$ was introduced by V.N. Temlyakov (see [13]).

Simultaneously we shall investigate the quantities $d_M^B(F, V)$, also considered by V.N. Temlyakov (see [13]) which are defined in the following way

$$d_M^B(F, V) = \inf_{G \in \mathfrak{L}_M(B)_V} \sup_{f \in F \cap D(G)} \| f(\cdot) - Gf(\cdot) \|_V . \tag{1.2}$$

Here $\mathfrak{L}_M(B)_V$ is the set of all linear operators satisfying the following conditions:

a) the domains $D(G)$ of these operators contain all trigonometric polynomials, and their ranges contained in a subspace of the space V of dimension M;

b) there exists $B \geq 1$ such that for all vectors $\mathbf{k} = (k_1, \dots, k_n)$ the following inequality

$$\left\| Ge^{i(\mathbf{k},\mathbf{x})} \right\|_{L_2(\mathbb{T}^n)} \leq B$$

is satisfied.

It is easy to verify that

$$d_M^B(F,V) \le d_M^{\perp}(F,V). \tag{1.3}$$

Consequently, estimates below for $d_M^B(F,V)$ are also estimates below for $d_M^{\perp}(F,V)$ and conversely estimates above for $d_M^{\perp}(F,V)$ are estimates above $d_M^B(F,V)$. We shall use this fact in the proofs of statements below.

If $V = L_q(\mathbb{T}^n)$ and F is one of the isotropic classes $W^r_{p,\alpha}(\mathbb{T}^n)$, $H^r_p(\mathbb{T}^n)$ or $B^r_{p,\theta}(\mathbb{T}^n)$ widths (2.1) and (2.2) were investigated in the works by V.N. Temlyakov [13, 15], E.M. Galeev [8, 9], A.V. Andrianov and V.N. Temlyakov [3], A.S. Romanyuk [11, 12].

If $V = L_{\mathbf{q}\theta}(\mathbb{T}^n)$ is the anisotropic Lorentz space and F is one of the anisotropic classes $H^{\mathbf{r}}_{\mathbf{p}\theta}(\mathbb{T}^n)$ or $B^{\mathbf{r}\tau}_{\mathbf{p}\theta}(\mathbb{T}^n)$ these widths is some particular cases were investigated by G.A. Akishev [2].

In this paper we strengthen G.A. Akishev's result [2].

2 Main result

Theorem 2.1. *Let $\mathbf{0} < \alpha = (\alpha_1, \ldots, \alpha_n) < \infty$, $\mathbf{1} < \mathbf{p} = (p_1, \ldots, p_n) < \mathbf{q} = (q_1, \ldots, q_n) < \infty$, $\mathbf{1} \le \theta = (\theta_1, \ldots, \theta_n), \tau = (\tau_1, \ldots, \tau_n), \mathbf{r} = (r_1, \ldots, r_n) \le \infty$, $\alpha_{j_0} - 1/p_{j_0} + 1/q_{j_0} = \min\{\alpha_j - 1/p_j + 1/q_j : j = 1, \ldots, n\} > 0$, $D = \{j = 1, \ldots, n : \alpha_j - 1/p_j + 1/q_j = \alpha_{j_0} - 1/p_{j_0} + 1/q_{j_0}\}$, $j_1 = \min\{j : j \in D\}$ and $q_j = q_{j_0}$, $\theta_j \le \tau_j$ for all $j \in D$. Then*

$$d_M^{\perp}(B^{\alpha\tau}_{\mathbf{pr}}(\mathbb{T}^n), L_{\mathbf{q}\theta}(\mathbb{T}^n)) \asymp d_M^B(B^{\alpha\tau}_{\mathbf{pr}}(\mathbb{T}^n), L_{\mathbf{q}\theta}(\mathbb{T}^n)) \asymp$$

$$\asymp M^{-\left(\alpha_{j_0} - 1/p_{j_0} + 1/q_{j_0}\right)} (\log M)^{(|D|-1)(\alpha_{j_0} - 1/p_{j_0} + 1/q_{j_0}) + \sum_{j \in D\setminus\{j_1\}} (1/\theta_j - 1/\tau_j)}, \tag{2.1}$$

where $|D|$ is the number of elements in the set D.

Note that in paper [2] G.A. Akishev considers the case $\alpha_1 = \ldots = \alpha_\nu < \alpha_{\nu+1} \le \ldots \le \alpha_n$, $p_1 = \ldots = p_n = p$, $q_1 = \ldots = q_n = q$, and correspondigly $|D| = \nu$.

First, we formulate definitions and auxiliary statements. Then the proof of the theorem will be given.

Let $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ be a measurable function on $\mathbb{T}^n = [0, 2\pi)^n$. We denote by $f^*(\mathbf{t}) = f^{*_1, \ldots, *_n}(t_1, \ldots, t_n)$ the function obtained from $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ by applying the non-increasing rearrangement successively with respect to each of the variables $x_1, \ldots, x_n$ (the other variables are assumed to be fixed).

Let the multi-indices $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{r} = (r_1, \ldots, r_n)$ be such that if $1 \le p_j < \infty$, then $1 \le r_j \le \infty$ and, if $p_j = \infty$, then $r_j = \infty$, $j = 1, \ldots, n$. The anisotropic Lorentz space $L_{\mathbf{pr}}(\mathbb{T}^n)$ (see A.P. Blozinsky [7], E.D. Nursultanov [10]) is the set of functions measurable on $\mathbb{T}^n$ for which the quantity

$$\|f\|_{L_{\mathbf{pr}}(\mathbb{T}^n)} = \left(\int_0^{2\pi} \ldots \left(\int_0^{2\pi} \left(t_1^{1/p_1} \ldots t_n^{1/p_n} f^{*_1, \ldots, *_n}(t_1, \ldots, t_n)\right)^{r_1} \frac{dt_1}{t_1}\right)^{r_2/r_1} \ldots \frac{dt_n}{t_n}\right)^{1/r_n}$$

is finite.

If $r = \infty$, the expression $\left(\int_0^{2\pi} (G(s))^r \frac{ds}{s}\right)^{1/r}$ is meant as $\operatorname{esssup}_{0 \le s < 2\pi} G(s)$.

For a function $f \in L_{\mathbf{pr}}(\mathbb{T}^n)$ we write

$$\Delta_{\mathbf{s}}(f, \mathbf{x}) = \sum_{\mathbf{k} \in \rho(\mathbf{s})} a_{\mathbf{k}}(f) e^{i(\mathbf{k},\mathbf{x})},$$

where $\{a_{\mathbf{k}}(f)\}_{\mathbf{k}\in\mathbb{Z}^n}$ are the Fourier coefficients of f with respect to the multiple trigonometric system, $(\mathbf{k},\mathbf{x}) = \sum_{j=1}^n k_j x_j$ is the scalar product, $\rho(\mathbf{s}) = \{\mathbf{k} = (k_1,\dots,k_n) \in \mathbb{Z}^n : [2^{s_i-1}] \le |k_i| < 2^{s_i}, i = 1,\dots,n\}$.

Let $\mathbf{0} < \alpha = (\alpha_1,\dots,\alpha_n) < \infty$ and $\mathbf{1} \le \mathbf{q} = (q_1,\dots,q_n) \le \infty$. The anisotropic Nikol'skii–Besov class $B_{\mathbf{pr}}^{\alpha\theta}(\mathbb{T}^n)$ (see G.A. Akishev [1], K.A. Bekmaganbetov and E.D. Nursultanov [5]) is the set of functions f in $L_{\mathbf{pr}}(\mathbb{T}^n)$ for which the

$$\|f\|_{B_{\mathbf{pr}}^{\alpha\theta}(\mathbb{T}^n)} = \left\| \left\{ 2^{(\alpha,\mathbf{s})} \|\Delta_{\mathbf{s}}(f)\|_{L_{\mathbf{pr}}(\mathbb{T}^n)} \right\}_{\mathbf{s}\in\mathbb{Z}_+^n} \right\|_{l_\theta} \le 1,$$

where $\|\cdot\|_{l_\theta}$ stands for the norm of the discrete space with mixed metric l_θ.

Let $\gamma = (\gamma_1,\dots,\gamma_n)$, $\mathbf{s} = (s_1,\dots,s_n)$, where $\gamma_j > 0$, $s_j \in \mathbb{Z}_+$ for all $j = 1,\dots,n$ and

$$Q(\gamma, N) = \bigcup_{(\mathbf{s},\gamma)\le N} \rho(\mathbf{s}), \quad T_{Q(\gamma,N)} = \left\{ t(\mathbf{x}) = \sum_{\mathbf{k}\in Q(\gamma,N)} b_{\mathbf{k}} e^{i(\mathbf{k},\mathbf{x})} \right\}.$$

The set $Q(\gamma, N)$ is called the stepped cross of order N corresponding to γ.

Let $E_{\gamma,N}(f)_{L_{\mathbf{pr}}(\mathbb{T}^n)}$ be the best approximation of f by polynomials in $T_{Q(\gamma,N)}$ in the metric of the anisotropic Lorentz space $L_{\mathbf{pr}}(\mathbb{T}^n)$ and

$$E_{\gamma,N}\left(B_{\mathbf{pr}}^{\alpha\tau}(\mathbb{T}^n)\right)_{L_{\mathbf{q}\theta}(\mathbb{T}^n)} = \sup_{\|f\|_{B_{\mathbf{pr}}^{\alpha\tau}(\mathbb{T}^n)}\le 1} E_{\gamma,N}(f)_{L_{\mathbf{q}\theta}(\mathbb{T}^n)}$$

is the order of approximation of the anisotropic Nikol'skii–Besov classes in the metric of the anisotropic Lorentz space.

Lemma 2.1 ([4]). *Let $\mathbf{0} < \alpha = (\alpha_1,\dots,\alpha_n) < \infty$, $\mathbf{1} < \mathbf{p} = (p_1,\dots,p_n) < \mathbf{q} = (q_1,\dots,q_n) < \infty$, $\mathbf{1} \le \theta = (\theta_1,\dots,\theta_n), \tau = (\tau_1,\dots,\tau_n), \mathbf{r} = (r_1,\dots,r_n) \le \infty$, $\alpha_{j_0} + 1/q_{j_0} - 1/p_{j_0} = \min\{\alpha_j + 1/q_j - 1/p_j : j = 1,\dots,n\}$ and $\alpha_{j_0} + \frac{1}{q_{j_0}} - \frac{1}{p_{j_0}} > 0$, $\gamma_j = (\alpha_j + 1/q_j - 1/p_j)/(\alpha_{j_0} + 1/q_{j_0} - 1/p_{j_0})$, $1 \le \gamma_j' \le \gamma_j$, $j = 1,\dots,n$.*

Then

$$E_{\gamma',N}\left(B_{\mathbf{pr}}^{\alpha\tau}(\mathbb{T}^n)\right)_{L_{\mathbf{q}\theta}(\mathbb{T}^n)} \asymp 2^{-\left(\alpha_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}}\right)N} N^{\sum_{j\in A\setminus\{j_1\}}\left(\frac{1}{\theta_j}-\frac{1}{\tau_j}\right)_+},$$

where $A = \{j = 1,\dots,n : \gamma_j' = \gamma_j\}$, $j_1 = \min\{j : j \in A\}$ and $(a)_+ = \max(a, 0)$.

Lemma 2.2 ([6], particular case of Theorem 1). *Let $\mathbf{1} < \mathbf{q} = (q_1,\dots,q_n) < \infty$, $\mathbf{1} \le \theta = (\theta_1,\dots,\theta_n) \le \infty$ and $\zeta = \max\{1/(q_j\gamma_j) : j = 1,\dots,n\}$, $B = \{j = 1,\dots,n : 1/(q_j\gamma_j) = \zeta\}$, $j_1 = \min\{j : j \in B\}$.*

Then for any trigonometric polynomial $t \in T_{Q(\gamma,N)}$ the following inequality holds

$$\|t\|_{L_\infty(\mathbb{T}^n)} \le C 2^{\zeta N} N^{\sum_{j\in B\setminus\{j_1\}} 1/\theta_j'} \|t\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)}.$$

Lemma 2.3 ([14], Lemma 3.3.1). *Let A be a linear operator defined on the set of all trigonometric polynomials such that*

$$A\left(e^{i(\mathbf{k},\mathbf{x})}\right)=\sum_{m=1}^{M}a_m^{\mathbf{k}}\psi_m(\mathbf{x}),$$

for arbitrary $\mathbf{k}\in\mathbb{Z}^n$, where $a_m^{\mathbf{k}}$ are given numbers and $\{\psi_m(\mathbf{x})\}_{m=1}^M$ is a given system such that $\|\psi_m\|_{L_2(\mathbb{T}^n)}\leq 1$, $m=1,\ldots,M$. Then

$$\min_{\mathbf{x}\in\mathbb{T}^n}ReA(t(\cdot-\mathbf{y}))(\mathbf{x})\Big|_{\mathbf{y}=\mathbf{x}}\leq\left(M\sum_{m=1}^{M}\sum_{\mathbf{k}}\left|a_m^{\mathbf{k}}\hat{t}(\mathbf{k})\right|^2\right)^{1/2}$$

for any trigonometric polynomial $t(\mathbf{x})=\sum_{\mathbf{k}}\hat{t}(\mathbf{k})e^{i(\mathbf{k},\mathbf{x})}$.

Note that

$$A(t(\cdot-\mathbf{y}))(\mathbf{x})|_{\mathbf{y}=\mathbf{x}}=\sum_{\mathbf{k}}\hat{t}(\mathbf{k})e^{-i(\mathbf{k},\mathbf{x})}\sum_{m=1}^{M}a_m^{\mathbf{k}}\psi_m(\mathbf{x}).$$

Proof of Theorem 1. First, let us prove the upper estimate.

Let $\gamma_j=(\alpha_j-1/p_j+1/q_j)/(\alpha_{j_0}-1/p_{j_0}+1/q_{j_0})$, $j=1,\ldots,n$.

Let us choose a number $N\in\mathbb{N}$ such that $M\asymp 2^N N^{|D|-1}$. Suppose $\gamma_j'=1$ for $j\in D$, and when $j\notin D$ we consider numbers γ_j', satisfying the following property $1<\gamma_j'<\gamma_j$. Then for the step cross $Q(\gamma',N)$ we get $|Q(\gamma',N)|\asymp M$. According to the definition of the orthoprojection width and Lemma 2.1, we get

$$d_M^{\perp}(B_{\mathbf{pr}}^{\alpha\tau}(\mathbb{T}^n),L_{\mathbf{q}\theta}(\mathbb{T}^n))\leq E_{\gamma',N}\left(B_{\mathbf{pr}}^{\alpha\tau}(\mathbb{T}^n)\right)_{L_{\mathbf{q}\theta}(\mathbb{T}^n)}\asymp 2^{-\left(\alpha_{j_0}-1/p_{j_0}+1/q_{j_0}\right)N}N^{\sum_{j\in D\setminus\{j_1\}}(1/\theta_j-1/\tau_j)}. \tag{2.2}$$

Since $M\asymp 2^N N^{|D|-1}$, then

$$2^{-\left(\alpha_{j_0}-1/p_{j_0}+1/q_{j_0}\right)N}N^{\sum_{j\in D\setminus\{j_1\}}(1/\theta_j-1/\tau_j)}$$

$$\asymp M^{-\left(\alpha_{j_0}-1/p_{j_0}+1/q_{j_0}\right)}(\log M)^{(|D|-1)(\alpha_{j_0}-1/p_{j_0}+1/q_{j_0})+\sum_{j\in D\setminus\{j_1\}}(1/\theta_j-1/\tau_j)}.$$

Therefore, from (2.2) the upper estimate in (2.1) follows.

Now, let us prove the lower estimate. Suppose $\mathbf{s_0}=(s_1^0,\ldots,s_n^0)$, where $s_j^0=s_j$ for $j\in D$ and $s_j^0=0$ for $j\notin D$, $\widetilde{\mathbf{s}}=(s_{j_1},\ldots,s_{j_{|D|}})$, $\widetilde{\gamma}=(\gamma_{j_1},\ldots,\gamma_{j_{|D|}})$, here $j_i\in D$, $i=1,\ldots,|D|$ and $j_1<\ldots<j_{|D|}$.

Let us consider the trigonometric polynomial

$$t(\mathbf{x})=N^{-\sum_{j\in D\setminus\{j_1\}}1/\tau_j}\sum_{(\gamma',\mathbf{s_0})=N}2^{-\sum_{j=1}^n(\alpha_j+1-1/p_j)s_j^0}\sum_{\mathbf{k}\in\rho(\mathbf{s_0})}e^{i(\mathbf{k},\mathbf{x})}.$$

According to estimates of the one-dimensional Bernoulli kernels, we obtain

$$t(\mathbf{0})=N^{-\sum_{j\in D\setminus\{j_1\}}1/\tau_j}\sum_{(\gamma',\mathbf{s_0})=N}2^{-\sum_{j=1}^n(\alpha_j+1-1/p_j)s_j^0}\sum_{\mathbf{k}\in\rho(\mathbf{s_0})}1$$

$$\asymp N^{-\sum_{j\in D\setminus\{j_1\}}1/\tau_j}\sum_{(\gamma',\mathbf{s_0})=N}2^{-\sum_{j=1}^{n}(\alpha_j+1-1/p_j)s_j^0}\cdot 2^{\sum_{j=1}^{n}s_j^0}$$

$$=N^{-\sum_{j\in D\setminus\{j_1\}}1/\tau_j}\sum_{(\gamma',\mathbf{s_0})=N}2^{-\sum_{j\in D}(\alpha_j-1/p_j)s_j^0}$$

$$=N^{-\sum_{j\in D\setminus\{j_1\}}1/\tau_j}\sum_{(\widetilde{\mathbf{1}},\widetilde{\mathbf{s}})=N}2^{-\sum_{j\in D}(\alpha_j-1/p_j)s_j}$$

$$=N^{-\sum_{j\in D\setminus\{j_1\}}1/\tau_j}\sum_{(\widetilde{\mathbf{1}},\widetilde{\mathbf{s}})=N}2^{-(\alpha_{j_0}-1/p_{j_0})(\widetilde{\mathbf{1}},\widetilde{\mathbf{s}})}$$

$$\asymp N^{-\sum_{j\in D\setminus\{j_1\}}1/\tau_j}\cdot 2^{-(\alpha_{j_0}-1/p_{j_0})N}N^{|D|-1}=2^{-(\alpha_{j_0}-1/p_{j_0})N}N^{\sum_{j\in D\setminus\{j_1\}}1/\tau_j'}. \tag{2.3}$$

Given an operator $G\in\mathfrak{L}_M(B)_{L_{\mathbf{q}\theta}(\mathbb{T}^n)}$, let us consider the operator

$$A=(S_{\gamma',N}-S_{\gamma',N-1})\,G,$$

where $S_{\gamma,N}$ is the operator of the partial sum of Fourier series corresponding to the stepped cross $Q(\gamma,N)$ and N will be choosen later. Then $A\in\mathfrak{L}_M(B)_{L_{\mathbf{q}\theta}(\mathbb{T}^n)}$ and the range of the operator A is a subspace A_M of the space $T_{Q(\gamma',N)}$ with $\dim(A_M)=\bar{M}\le M$. By the construction of the polynomial t and the fact that the partial sum $S_{\gamma',N}$ is bounded in $L_{\mathbf{q}\theta}(\mathbb{T}^n)$ for any trigonometric polynomial $t\in T_{Q(\gamma',N)}$, then we have

$$\|t-At\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)}=\|\,(S_{\gamma',N}-S_{\gamma',N-1})\,(t-Gt)\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)}\le C_1\|t-Gt\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)}. \tag{2.4}$$

Let $\{\psi_m(\mathbf{x})\}_{m=1}^{\bar{M}}$ be an orthonormal basis in A_M and

$$A\left(e^{i(\mathbf{k},\mathbf{x})}\right)=\sum_{m=1}^{\bar{M}}a_m^{\mathbf{k}}\psi_m(\mathbf{x}).$$

Then

$$\left(\sum_{m=1}^{\bar{M}}\left|a_m^{\mathbf{k}}\right|^2\right)^{1/2}\le B,$$

and for any trigonometric polynomial $t\in T_{Q(\gamma',N)}$, by Parseval's equality and Lemma 2.3, we get

$$\min_{\mathbf{x}\in\mathbb{T}^n}ReA\,(t(\cdot-\mathbf{y}))\,(\mathbf{x})\Big|_{\mathbf{y}=\mathbf{x}}\le\left(\bar{M}\sum_{m=1}^{\bar{M}}\sum_{\mathbf{k}}\left|a_m^{\mathbf{k}}\widehat{t}(\mathbf{k})\right|^2\right)^{1/2}$$

$$=\left(\bar{M}\sum_{\mathbf{k}}|\widehat{t}(\mathbf{k})|^2\sum_{m=1}^{\bar{M}}\left|a_m^{\mathbf{k}}\right|^2\right)^{1/2}\le B\left(M\sum_{\mathbf{k}}|\widehat{t}(\mathbf{k})|^2\right)^{1/2}$$

$$=BN^{-\sum_{j\in D\setminus\{j_1\}}1/\tau_j}\left(M\sum_{(\gamma',\mathbf{s_0})=N}2^{-\sum_{j=1}^{n}2(\alpha_j+1-1/p_j)s_j^0}\sum_{\mathbf{k}\in\rho(\mathbf{s_0})}1\right)^{1/2}$$

$$\asymp N^{-\sum_{j\in D\setminus\{j_1\}} 1/\tau_j} \left(M \sum_{(\gamma',\mathbf{s_0})=N} 2^{-\sum_{j=1}^{n} 2(\alpha_j+1-1/p_j)s_j^0} \cdot 2^{\sum_{j=1}^{n} s_j^0} \right)^{1/2}$$

$$= BN^{-\sum_{j\in D\setminus\{j_1\}} 1/\tau_j} \left(M \sum_{(\gamma',\mathbf{s_0})=N} 2^{-\sum_{j=1}^{n}(2(\alpha_j-1/p_j)+1)s_j^0} \right)^{1/2}$$

$$= BN^{-\sum_{j\in D\setminus\{j_1\}} 1/\tau_j} \left(M \sum_{(\widetilde{\mathbf{1}},\widetilde{\mathbf{s}})=N} 2^{-\sum_{j\in D}(2(\alpha_j-1/p_j)+1)s_j} \right)^{1/2}$$

$$= BN^{-\sum_{j\in D\setminus\{j_1\}} 1/\tau_j} \left(M \sum_{(\widetilde{\mathbf{1}},\widetilde{\mathbf{s}})=N} 2^{-\left(2\left(\alpha_{j_0}-1/p_{j_0}\right)+1\right)(\widetilde{\mathbf{1}},\widetilde{\mathbf{s}})} \right)^{1/2}$$

$$\asymp BM^{1/2} N^{-\sum_{j\in D\setminus\{j_1\}} 1/\tau_j} \left(2^{-(2(\alpha_{j_0}-1/p_{j_0})+1)N} N^{|D|-1}\right)^{1/2}$$

$$= 2^{-(\alpha_{j_0}-1/p_{j_0})N} N^{\sum_{j\in D\setminus\{j_1\}} 1/\tau_j'} \left(\frac{B^2 M}{2^N N^{|D|-1}} \right)^{1/2}. \tag{2.5}$$

Further, from (2.3) and (2.5) we obtain

$$t(\mathbf{0}) - \min_{\mathbf{x}\in\mathbb{T}^n} ReA\left(t(\cdot-\mathbf{y})\right)(\mathbf{x})\Big|_{\mathbf{y}=\mathbf{x}} \geq$$

$$\geq 2^{-(\alpha_{j_0}-1/p_{j_0})N} N^{\sum_{j\in D\setminus\{j_1\}} 1/\tau_j'} \left(C_2 - C_3 B \left(\frac{M}{2^N N^{|D|-1}} \right)^{1/2} \right), \tag{2.6}$$

where C_2 is the constant in the lower estimate in (2.3), and C_3 is the constant in the upper estimate in (2.5).

Let us choose N such that

$$\left(C_2 - C_3 B \left(\frac{M}{2^N N^{|D|-1}} \right)^{1/2} \right) \geq C_4 > 0.$$

Then from (2.6) we obtain

$$C_4 2^{-(\alpha_{j_0}-1/p_{j_0})N} N^{\sum_{j\in D\setminus\{j_1\}} 1/\tau_j'} \leq t(\mathbf{0}) - \min_{\mathbf{x}\in\mathbb{T}^n} ReA\left(t(\cdot-\mathbf{y})\right)(\mathbf{x})\Big|_{\mathbf{y}=\mathbf{x}}$$

$$= t(\mathbf{0}) - \min_{\mathbf{x}\in\mathbb{T}^n} Re\left(\sum_{\mathbf{k}} \hat{t}(\mathbf{k}) e^{-i(\mathbf{k},\mathbf{x})} \sum_{m=1}^{M} a_m^{\mathbf{k}} \psi_m(\mathbf{x}) \right)$$

$$= t(\mathbf{0}) - Re\left(\sum_{\mathbf{k}} \hat{t}(\mathbf{k}) e^{-i(\mathbf{k},\mathbf{x_0})} \sum_{m=1}^{M} a_m^{\mathbf{k}} \psi_m(\mathbf{x_0}) \right)$$

$$= Re\left(t(\mathbf{0}) - \sum_{\mathbf{k}} \hat{t}(\mathbf{k}) e^{-i(\mathbf{k},\mathbf{x_0})} \sum_{m=1}^{M} a_m^{\mathbf{k}} \psi_m(\mathbf{x_0}) \right)$$

$$\leq \left| t(\mathbf{0}) - \sum_{\mathbf{k}} \hat{t}(\mathbf{k}) e^{-i(\mathbf{k},\mathbf{x_0})} \sum_{m=1}^{M} a_m^{\mathbf{k}} \psi_m(\mathbf{x_0}) \right|$$

$$\leq \sup_{\mathbf{y}} \left| t(\mathbf{x_0}-\mathbf{y}) - \sum_{\mathbf{k}} \hat{t}(\mathbf{k}) e^{-i(\mathbf{k},\mathbf{y})} \sum_{m=1}^{M} a_m^{\mathbf{k}} \psi_m(\mathbf{x_0}) \right|$$

$$= \sup_{\mathbf{y}} \left| t(\mathbf{x_0}-\mathbf{y}) - A\left(t(\cdot-\mathbf{y})\right)(\mathbf{x_0}) \right|$$

$$\leq \sup_{\mathbf{y}} \left\| t(\mathbf{x}-\mathbf{y}) - A\left(t(\cdot-\mathbf{y})\right)(\mathbf{x}) \right\|_{L_\infty(\mathbb{T}^n)}$$

and consequently there exists $\mathbf{y_0}$ such that

$$\left\| t(\mathbf{x}-\mathbf{y_0}) - A\left(t(\cdot-\mathbf{y_0})\right)(\mathbf{x}) \right\|_{L_\infty(\mathbb{T}^n)} \geq \frac{C_4}{2} 2^{-(\alpha_{j_0}-1/p_{j_0})N} N^{\sum_{j\in D\setminus\{j_1\}} 1/\tau_j'}$$

$$= C_5 2^{-(\alpha_{j_0}-1/p_{j_0})N} N^{\sum_{j\in D\setminus\{j_1\}} 1/\tau_j'}$$

Now, let $\varphi(\mathbf{x}) = t(\mathbf{x}-\mathbf{y_0})$, then

$$\left\| \varphi(\mathbf{x}) - A\varphi(\mathbf{x}) \right\|_{L_\infty(\mathbb{T}^n)} \geq C_5 2^{-(\alpha_{j_0}-1/p_{j_0})N} N^{\sum_{j\in D\setminus\{j_1\}} 1/\tau_j'}.$$

Hence by Lemma 2.2, we get

$$\left\| \varphi(\mathbf{x}) - A\varphi(\mathbf{x}) \right\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)} \geq C_6 2^{-N/q_{j_0}} N^{-\sum_{j\in D\setminus\{j_1\}} 1/\theta_j'} \left\| \varphi(\mathbf{x}) - A\varphi(\mathbf{x}) \right\|_{L_\infty(\mathbb{T}^n)}$$

$$\geq C_7 2^{-\left(\alpha_{j_0}-1/p_{j_0}+1/q_{j_0}\right)N} N^{\sum_{j\in D\setminus\{j_1\}}(1/\theta_j-1/\tau_j)}$$

$$\asymp M^{-\left(\alpha_{j_0}-1/p_{j_0}+1/q_{j_0}\right)} (\log M)^{(|D|-1)(\alpha_{j_0}-1/p_{j_0}+1/q_{j_0})+\sum_{j\in D\setminus\{j_1\}}(1/\theta_j-1/\tau_j)}. \qquad (2.7)$$

Taking into account that the function $f = C_8\varphi$ with some constant C_8, which does not depended on N, belongs to class $B_{\mathbf{pr}}^{\alpha\tau}(\mathbb{T}^n)$, and by (2.4) and (2.7), we obtain

$$\sup_{f\in B_{\mathbf{pr}}^{\alpha\tau}(\mathbb{T}^n)} \|f - Gf\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)} \geq C_8 \|\varphi - G\varphi\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)} \geq C_9 \left\| \varphi(\mathbf{x}) - A\varphi(\mathbf{x}) \right\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)}$$

$$\geq C_{10} M^{-\left(\alpha_{j_0}-1/p_{j_0}+1/q_{j_0}\right)} (\log M)^{(|D|-1)(\alpha_{j_0}-1/p_{j_0}+1/q_{j_0})+\sum_{j\in D\setminus\{j_1\}}(1/\theta_j-1/\tau_j)}.$$

According to arbitrariness of an operator $G \in \mathfrak{L}_M(B)_{L_{\mathbf{q}\theta}(\mathbb{T}^n)}$, we have the lower estimate in (2.1). □

Remark 1. By Lemma 2.1 it follows that the upper estimate of $d_M^{\perp}(B_{\mathbf{pr}}^{\alpha\tau}(\mathbb{T}^n), L_{\mathbf{q}\theta}(\mathbb{T}^n))$ holds without the additional condition $q_j = q_{j_0}$ for $j \in D$.

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References

[1] G.A. Akishev, *Approximation of function classes in spaces with mixed norm.* Sborni. Math. 197 (2006), no. 8, 1121 – 1144.

[2] G.A. Akishev, *The ortho-diameters of Nikol'skii and Besov classes in the Lorentz spaces.* Russian Math. 53 (2009), no. 2, 21 – 29.

[3] A.V. Andrianov, V.N. Temlyakov, *On two methods of generalization of properties of univariate function systems to their tensor product.* Proc. Steklov Inst. Math. 219 (1997), 25 – 35.

[4] K.A. Bekmaganbetov, *About order of approximation of Besov classes in metric of anisotropic Lorentz spaces.* Ufimsk. Mat. Zh. 1 (2009), no. 2, 9 – 16 (in Russian).

[5] K.A. Bekmaganbetov, E.D. Nursultanov, *Embedding theorems for anisotropic Besov spaces* $B^{\alpha\mathbf{q}}_{\mathbf{pr}}([0,2\pi)^n)$. Izvestiya: Math. 73 (2009), no. 4, 655 – 668.

[6] K.A. Bekmaganbetov, E.T. Orazgaliev, *Bernstein - Nikol'skii inequalities and estimates of best approximation in anisotropic Lorentz spaces.* Math. Journal 15 (2015), no. 2, 32 – 42 (in Russian).

[7] A.P. Blozinsky, *Multivariate rearrangements and banach function spaces with mixed norms.* Trans. Amer. Math. Soc. 263 (1981), no. 1, 149 – 167.

[8] E.M. Galeev, *Orders of the orthoprojection widths of classes of periodic functions of one and of several variables.* Math. Notes 43 (1988), no. 2, 110 – 118.

[9] E.M. Galeev, *Approximation of classes of periodic functions of several variables by nuclear operators.* Math. Notes 47 (1990), no. 3, 248 – 254.

[10] E.D. Nursultanov, *On the coefficients of multiple Fourier series in* L_p*-spaces.* Izvestiya: Math. 64 (2000), no. 1, 93 – 120.

[11] A.S. Romanyuk, *Best approximations and widths of classes of periodic functions of several variables.* Sbornik: Math. 199 (2008), no. 2, 253 – 275.

[12] A.S. Romanyuk, *Best trigonometric and bilinear approximations of classes of functions of several variables.* Math. Notes 94 (2013), no. 3, 379 – 391.

[13] V.N. Temlyakov, *Widths of some classes of functions of several variables.* Soviet Math. Dokl. 26 (1982), 619 – 622.

[14] V.N. Temlyakov, *Approximations of functions with bounded mixed derivative.* Proc. Steklov Inst. Math. 178 (1989), 1 – 121.

[15] V.N. Temlyakov, *Estimates for the asymptotic characteristics of classes of functions with bounded mixed derivative or difference.* Proc. Steklov Inst. Math. 189 (1990), 161 – 197.

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