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TYNYSBEK SHARIPOVICH KAL'MENOV

(to the 70th birthday)

On May 5, 2016 was the 70th birthday of Tynysbek Sharipovich Kal'menov, member of the Editorial Board of the Eurasian Mathematical Journal, general director of the Institute of Mathematics and Mathematical Modeling of the Ministry of Education and Science of the Republic of Kazakhstan, laureate of the Lenin Komsomol Prize of the Kazakh SSR (1978), doctor of physical and mathematical sciences (1983), professor (1986), honoured worker of science and technology of the Republic of Kazakhstan (1996), academician of the National Academy of Sciences (2003), laureate of the State Prize in the field of science and technology (2013).

T.Sh. Kal'menov was born in the South-Kazakhstan region of the Kazakh SSR. He graduated from the Novosibirsk State University (1969) and completed his postgraduate studies there in 1972.

He obtained seminal scientific results in the theory of partial differential equations and in the spectral theory of differential operators.

For the Lavrentiev-Bitsadze equation T.Sh. Kal'menov proved the criterion of strong solvability of the Tricomi problem in the L_p-spaces. He described all well-posed boundary value problems for the wave equation and equations of mixed type within the framework of the general theory of boundary value problems.

He solved the problem of existence of an eigenvalue of the Tricomi problem for the Lavrentiev-Bitsadze equation and the general Gellerstedt equation on the basis of the new extremum principle formulated by him.

T.Sh. Kal'menov proved the completeness of root vectors of main types of Bitsadze-Samarskii problems for a general elliptic operator. Green's function of the Dirichlet problem for the polyharmonic equation was constructed. He established that the spectrum of general differential operators, generated by regular boundary conditions, is either an empty or an infinite set. The boundary conditions characterizing the volume Newton potential were found. A new criterion of well-posedness of the mixed Cauchy problem for the Poisson equation was found.

On the whole, the results obtained by T.Sh. Kal'menov have laid the groundwork for new perspective scientific directions in the theory of boundary value problems for hyperbolic equations, equations of the mixed type, as well as in the spectral theory.

More than 50 candidate of sciences and 9 doctor of sciences dissertations have been defended under his supervision. He has published more than 120 scientific papers. The list of his basic publications can be viewed on the web-page

$$https://scholar.google.com/citations?user=Zay4fxkAAAAJ\&hl=ru\&authuser=1$$

The Editorial Board of the Eurasian Mathematical Journal congratulates Tynysbek Sharipovich Kal'menov on the occasion of his 70th birthday and wishes him good health and new creative achievements!

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INTEGRAL REPRESENTATION OF FUNCTIONS AND EMBEDDING THEOREMS FOR MULTIANISOTROPIC SPACES IN THE THREE-DIMENSIONAL CASE

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Key words: integral representation, embedding theorem, multianisotropic spaces, completely regular polyhedron.

AMS Mathematics Subject Classification: 46E35.

Abstract. In this paper we obtain a special integral representation of functions with a set of multi-indices and use it to prove embedding theorems for multianisotropic spaces in three-dimensional case.

1 Introduction

The embedding theorems for multianisotropic spaces in two-dimensional case can be found in [4]. In this paper we prove embedding theorems for the Sobolev multianisotropic spaces in three-dimensional space when the completely regular polyhedron has one anisotropicity vertex. The obtained results generalize embedding theorems for isotropic and anisotropic spaces in [1-3, 5-9] (for an overview of the history of the problem and related results see [2]), and in the case of anisotropic spaces coincide with known theorems, although a method of integral representation of functions is applied, which is based on the usage of special multianisotropic kernels.

2 Multianisotropic kernels and their properties

Let $\mathbb{R}^3$ be the three-dimensional space, $\mathbb{Z}^3_+$ the set of all three-dimensional multi-indices, i.e. $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}^3_+$, if $\alpha_1, \alpha_2, \alpha_3$ are non-negative integers. For $\xi, \eta \in \mathbb{R}^3$, $\alpha \in \mathbb{Z}^3_+$, $t > 0$ let $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3, \xi^\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha_2}\xi_3^{\alpha_3}$, $t^\eta = (t^{\eta_1}, t^{\eta_2}, t^{\eta_3})$, $D_k = \frac{1}{i}\frac{\partial}{\partial x_k}, k = 1, 2, 3,$ and let $D^\alpha = D_1^{\alpha_1}D_2^{\alpha_2}D_3^{\alpha_3}$ be the weak derivative of order α.

For a set of multi-indices let $\mathfrak{N}$ be the smallest convex polyhedron containing it. $\mathfrak{N}$ is said to be completely regular if

a) it has vertices at the origin and further vertices on each coordinate axis

b) all components of the outer normals of all two-dimensional non-coordinate faces are positive. For a completely regular polyhedron $\mathfrak{N}$ in $\mathbb{R}^3$ let $\mathfrak{N}_i^2$ $(i = 1, ..., M)$ be the two-dimensional non-coordinate faces. Let $\mu^i (i = 1, \ldots, M)$ be the outer normal of the face $\mathfrak{N}_i^2$, such that the equation of that face is $(\alpha;\mu^i)=1;(i=1,...,M)$. We study the case when the vertices of $\mathfrak{N}$ are multi-index: $\alpha^1=(l_1,0,0)\,,\alpha^2=(0,l_2,0)\,,\alpha^3=(0,0,l_3)\,,\ \alpha^4=(\alpha_1,\alpha_2,\alpha_3)$.

Denote by μ^i $(i=1,2,3)$ the outer normal of $\mathfrak{N}_i^2$, which is the face $\mathfrak{N}_1^2$ passing through all the vertices with the exception of α^i (i=1,2,3), i.e. μ^1 is the outer normal of the face passing through the vertices $\{\alpha^2;\alpha^3;\alpha^4\}$, and so on.

For $\theta>0$ and positive integer k denote

$$P\left(\theta,\xi\right)=\left(\theta\xi_1^{l_1}\right)^{2k}+\left(\theta\xi_2^{l_2}\right)^{2k}+\left(\theta\xi_3^{l_3}\right)^{2k}+\left(\theta\xi_1^{\alpha_1}\xi_2^{\alpha_2}\xi_3^{\alpha_3}\right)^{2k}, \tag{2.1}$$

$$G_0\left(\xi;\theta\right)=e^{-P(\theta,\xi)}, \tag{2.2}$$

$$G_{1,j}\left(\xi,\theta\right)=2k\left(\theta\xi^{\alpha^j}\right)^{2k-1}e^{-P(\theta,\xi)}\ ,j=1,2,3,4. \tag{2.3}$$

It is obvious that for any value of $\theta>0$ $G_0,G_{1,j}\in S,(j=1,2,3,4)$, where $S=S\left(\mathbb{R}^n\right)$ is the Schwartz space of rapidly decreasing functions.

Let $\hat{G}_0(t,\theta)$, $\hat{G}_{1,j}(t,\theta)$ $(j=1,2,3,4)$ be the corresponding Fourier transform of G_0, $G_{1,j}$ $(j=1,2,3,4)$.

Note that $\hat{G}_0\ ,\hat{G}_{1,j}\in S,(j=1,2,3,4)$, because the Fourier transform is an automorphism on S.

Suppose that the inequality $\alpha_1<\alpha_2<\alpha_3<l_3$ holds for $\alpha=\alpha^4=(\alpha_1,\alpha_2,\alpha_3)$. Consider the intersection of planes $\mathfrak{N}_1^2\bigcap\mathfrak{N}_2^2$, i.e. the line passing through the points $\alpha^3=(0;0;l_3)$ and $\alpha^4=(\alpha_1,\alpha_2,\alpha_3)$. The parametric equation of that line is x $=\alpha_1t;\ y=\alpha_2t;\ z=l_3-t\left(l_3-\alpha_3\right)$. Let $\beta=(\beta_1,\beta_2,0)$ be the point of intersection of that line with the XOY plane. Note that $\beta_1=\frac{\alpha_1l_3}{l_3-\alpha_3}$, $\beta_2=\frac{\alpha_2l_3}{l_3-\alpha_3}$, $\beta\in\mathfrak{N}_1^2\bigcap\mathfrak{N}_2^2$ (i.e. $(\beta;\mu^1)=1;(\beta;\mu^2)=1$), also $\beta_1<\beta_2$ which follows from $\alpha_1<\alpha_2$.

Consider the point $\gamma=(\gamma_1,0,0)$, such that $\gamma\in\mathfrak{N}_1^2$. Let N be an even number, such that

all components of multi-indices $N\alpha,N\beta,N\gamma$ are even. (A)

Such N exists, because all components of α,β,γ are rational. Let us prove the following lemma (an analogue of Lemma 2.1 in [4]).

Lemma 2.1. *Let* $\alpha^4=(\alpha_1,\alpha_2,\alpha_3)\in\mathbb{Z}_+^3,\alpha_1<\alpha_2<\alpha_3$ *and* $\theta\in(0;1)$. *Then for any multi-index* $m=(m_1,m_2,m_3)$ *and for any even N satisfying (A) there exist constants* C_i, $(i=0,1,2)$, *such that*

$$\left|D^m\hat{G}_{1,j}\left(t,\theta\right)\right|\leq\frac{\theta^{-\max\limits_{i=1,2,3}\left(\left|\mu^i\right|+(m,\mu^i)\right)}\left(C_0(\ln\theta)^2+C_1\left|\ln\theta\right|+C_2\right)}{1+\theta^{-N}\left(t_1^{N\alpha_1}t_2^{N\alpha_2}t_3^{N\alpha_3}+t_1^{N\beta_1}t_2^{N\beta_2}+t_1^{N\gamma_1}\right)}. \tag{2.4}$$

An analogous inequality is true for $\hat{G}_0\left(t,\theta\right)$.

Proof. As in the proof of Lemma 2.1 in [4], for any multi-index $m=(m_1,m_2,m_3)$ we estimate the following integral:

$$I=\int_0^\infty\int_0^\infty\int_0^\infty\xi_1^{m_1}\xi_2^{m_2}\xi_3^{m_3}e^{-P(\theta,\xi)}d\xi_1d\xi_2d\xi_3.$$

Consider $\frac{\alpha_i}{m_i+1}$ $(i=1,2,3)$. Let i_0 be an index, such that $\max\limits_{i=1,2,3}\frac{\alpha_i}{m_i+1}=\frac{\alpha_{i_0}}{m_{i_0}+1}$. There are 3 possible cases:

1. The maximum is obtained only at one index i_0 , i.e. $\frac{\alpha_i}{m_i+1}<\frac{\alpha_{i_0}}{m_{i_0}+1}$ $i \neq i_0,\ \ i=1,2,3$.
2. The maximum is obtained at two indices (e.g. $i_0 = 2$ Pë $i_0 = 3$), i.e. $\frac{\alpha_1}{m_1+1} < \frac{\alpha_3}{m_3+1}$, $\frac{\alpha_2}{m_2+1} = \frac{\alpha_3}{m_3+1}$
3. All the ratios are equal, i.e. $\frac{\alpha_1}{m_1+1} = \frac{\alpha_2}{m_2+1} = \frac{\alpha_3}{m_3+1}$.

Consider the first case. Let $i_0 = 3$. Note that it suffices to estimate the integral only on the closed upper half-space. By substituting $\xi = \theta^{-\mu^3}\eta$ in I, we have

$$I = \theta^{-\left(|\mu^3|+\left(m,\mu^3\right)\right)} \int_0^\infty \int_0^\infty \int_0^\infty \eta_1^{m_1}\eta_2^{m_2}\eta_3^{m_3} e^{-\eta_1^{2kl_1}} e^{-\eta_2^{2kl_2}} e^{-\eta_1^{2k\alpha_1}\eta_2^{2k\alpha_2}\eta_3^{2k\alpha_3}} d\eta_1 d\eta_2 d\eta_3$$

$$= \theta^{-\left(|\mu^3|+\left(m,\mu^3\right)\right)} \int_0^{+\infty} e^{-\left(\eta_1^{\frac{\alpha_1}{\alpha_3}}\eta_2^{\frac{\alpha_2}{\alpha_3}}\eta_3\right)^{2k\alpha_3}} \left(\eta_1^{\frac{\alpha_1}{\alpha_3}}\eta_2^{\frac{\alpha_2}{\alpha_3}}\eta_3\right)^{m_3} d\left(\eta_1^{\frac{\alpha_1}{\alpha_3}}\eta_2^{\frac{\alpha_2}{\alpha_3}}\eta_3\right)$$

$$\cdot \int_0^\infty \eta_1^{m_1-\frac{\alpha_1}{\alpha_3}m_3-\frac{\alpha_1}{\alpha_3}} e^{-\eta_1^{2kl_1}} d\eta_1 \int_0^\infty \eta_2^{m_2-\frac{\alpha_2}{\alpha_3}m_3-\frac{\alpha_2}{\alpha_3}} e^{-\eta_2^{2kl_2}} d\eta_2$$

$$= C\theta^{-\left(|\mu^3|+\left(m,\mu^3\right)\right)} = C\theta^{-\max\limits_{i=1,2,3}\left(|\mu^i|+\left(m,\mu^i\right)\right)}. \quad (2.5)$$

The last relation follows from the convergence of the three integrals and from inequalities $m_1-\frac{\alpha_1}{\alpha_3}m_3-\frac{\alpha_1}{\alpha_3} > -1, m_2-\frac{\alpha_2}{\alpha_3}m_3-\frac{\alpha_2}{\alpha_3} > -1$, which are in turn inferred from $\frac{\alpha_1}{\alpha_3} < \frac{m_1+1}{m_3+1}$, $\frac{\alpha_2}{\alpha_3} < \frac{m_2+1}{m_3+1}$.

Consider the second case. We have $\frac{\alpha_1}{\alpha_3} < \frac{m_1+1}{m_3+1}$, $\frac{\alpha_2}{\alpha_3} = \frac{m_2+1}{m_3+1}$ and thus $\frac{\alpha_1}{\alpha_2} < \frac{m_1+1}{m_2+1}$

The polyhedron $\mathfrak{N}$ is completely regular, thus $\frac{\alpha_1}{l_1} + \frac{\alpha_2}{l_2} + \frac{\alpha_3}{l_3} > 1$ and $\mu_3^3 < \mu_3^2$. We can split I into 8 integrals:

$$I = \int_0^{\theta^{-\mu_1^3}} d\xi_1 \int_0^{\theta^{-\mu_2^3}} d\xi_2 \int_0^{\theta^{-\mu_3^3}} \dots\, d\xi_3 + \int_0^{\theta^{-\mu_1^3}} d\xi_1 \int_0^{\theta^{-\mu_2^3}} d\xi_2 \int_{\theta^{-\mu_3^3}}^\infty \dots\, d\xi_3$$
$$+ \int_0^{\theta^{-\mu_1^3}} d\xi_1 \int_{\theta^{-\mu_2^3}}^\infty d\xi_2 \int_0^{\theta^{-\mu_3^3}} \dots\, d\xi_3 + \int_{\theta^{-\mu_1^3}}^\infty d\xi_1 \int_0^{\theta^{-\mu_2^3}} d\xi_2 \int_0^{\theta^{-\mu_3^3}} \dots\, d\xi_3$$
$$+ \int_0^{\theta^{-\mu_1^3}} d\xi_1 \int_{\theta^{-\mu_2^3}}^\infty d\xi_2 \int_{\theta^{-\mu_3^3}}^\infty \dots\, d\xi_3 + \int_{\theta^{-\mu_1^3}}^\infty d\xi_1 \int_{\theta^{-\mu_2^3}}^\infty d\xi_2 \int_0^{\theta^{-\mu_3^3}} \dots\, d\xi_3$$
$$+ \int_{\theta^{-\mu_1^3}}^\infty d\xi_1 \int_0^{\theta^{-\mu_2^3}} d\xi_2 \int_{\theta^{-\mu_3^3}}^\infty \dots\, d\xi_3 + \int_{\theta^{-\mu_1^3}}^\infty d\xi_1 \int_{\theta^{-\mu_2^3}}^\infty d\xi_2 \int_{\theta^{-\mu_3^3}}^\infty \dots\, d\xi_3 = I_1 + \cdots + I_8.$$

We will estimate each integral separately. The substitution $\xi = \theta^{-\mu^3}\eta$ in I_1 yields $I_1 \leq C\theta^{-\left(|\mu^3|+\left(m,\mu^3\right)\right)}$.

By substituting $\xi = \theta^{-\mu^2}\eta$ in I_2, we get

$$I_2 \leq C\theta^{-\left(|\mu^2|+\left(m,\mu^2\right)\right)} \int_0^\infty d\eta_1 \int_0^\infty d\eta_2 \int_{\theta^{-\mu_3^3}}^\infty \eta_1^{m_1}\eta_2^{m_2}\eta_3^{m_3} e^{-\eta_1^{2kl_1}} e^{-\eta_3^{2kl_3}} e^{-\eta_1^{2k\alpha_1}\eta_2^{2k\alpha_2}\eta_3^{2k\alpha_3}}\, d\eta_3$$

$$= C\theta^{-\left(\left|\mu^2\right|+\left(m,\mu^2\right)\right)}\int_0^\infty \eta_1^{m_1-\frac{\alpha_1}{\alpha_2}m_2-\frac{\alpha_1}{\alpha_2}}e^{-\eta_1^{2kl_1}}d\eta_1\int_{\theta^{-\mu_3^3+\mu_3^2}}^\infty \frac{e^{-\eta_3^{2kl_3}}}{\eta_3}d\eta_3$$

$$\cdot\int_0^\infty \left(\eta_1^{\frac{\alpha_1}{\alpha_2}}\eta_2\eta_3^{\frac{\alpha_3}{\alpha_2}}\right)^{m_2}e^{-\left(\eta_1^{\frac{\alpha_1}{\alpha_2}}\eta_2\eta_3^{\frac{\alpha_3}{\alpha_2}}\right)^{2k\alpha_2}}d(\eta_1^{\frac{\alpha_1}{\alpha_2}}\eta_2\eta_3^{\frac{\alpha_3}{\alpha_2}}) \le \theta^{-\left(\left|\mu^2\right|+\left(m,\mu^2\right)\right)}\left(C_1\left|\ln\theta\right|+C_2\right).$$

The first integral converges, because $\frac{\alpha_1}{\alpha_3} < \frac{m_1+1}{m_3+1}$. By substituting $t = \eta_1^{\frac{\alpha_1}{\alpha_2}}\eta_2\eta_3^{\frac{\alpha_3}{\alpha_2}}$, it follows that the third integral is convergent. The second integral can be estimated via $(C_1|\ln\theta| + C_2)$.

By substituting $\xi = \theta^{-\mu_3}\eta$ in I_3, we get

$$I_3 = \theta^{-\left(\left|\mu^3\right|+\left(m,\mu^3\right)\right)}\int_0^1 d\eta_1\int_0^\infty d\eta_2\int_0^1 (\eta_1^{m_1}\eta_2^{m_2}\eta_3^{m_3}e^{-\eta_1^{2kl_1}}e^{-\eta_2^{2kl_2}}e^{-\eta_1^{2k\alpha_1}\eta_2^{2k\alpha_2}\eta_3^{2k\alpha_3}})d\eta_3$$

$$\le C\theta^{-\left(\left|\mu^3\right|+\left(m,\mu^3\right)\right)}.$$

We substitute $\xi = \theta^{-\mu_3}\eta$ in I_4 and estimate it similarly to I_3.

By substituting $\xi = \theta^{-\mu^3}\eta$ in I_5, we get

$$I_5 \le C\theta^{-\left(\left|\mu^3\right|+\left(m,\mu^3\right)\right)}\int_0^1 \eta_1^{m_1-\frac{\alpha_1}{\alpha_3}m_2-\frac{\alpha_1}{\alpha_3}}e^{-\eta_1^{2kl_1}}d\eta_1\int_1^\infty \frac{e^{-\eta_2{}^{2kl_2}}}{\eta_2}d\eta_2\int_0^\infty t^{m_3}e^{-t^{2k\alpha_3}}dt$$

$$\le C\theta^{-\left(\left|\mu^3\right|+\left(m,\mu^3\right)\right)}.$$

By substituting $\xi = \theta^{-\mu^2}\eta$ in I_6, we get

$$I_6 \le C\theta^{-\left(\left|\mu^2\right|+\left(m,\mu^2\right)\right)}\int_0^\infty \eta_1^{m_1-\frac{\alpha_1}{\alpha_2}m_2-\frac{\alpha_1}{\alpha_2}}e^{-\eta_1^{2kl_1}}d\eta_1\int_1^\infty \frac{e^{-\eta_3{}^{2kl_3}}}{\eta_3}d\eta_3\int_0^\infty t^{m_2}e^{-t^{2k\alpha_2}}dt$$

$$\le \theta^{-\left(\left|\mu^2\right|+\left(m,\mu^2\right)\right)}\left(C_1\left|\ln\theta\right|+C_2\right).$$

By substituting $\xi = \theta^{-\mu^3}\eta$ in I_7, we get

$$I_7 \le C\theta^{-\left(\left|\mu^3\right|+\left(m,\mu^3\right)\right)}\int_0^1 \eta_1^{m_1-\frac{\alpha_1}{\alpha_3}m_2-\frac{\alpha_1}{\alpha_3}}e^{-\eta_1^{2kl_1}}d\eta_1\int_1^\infty \frac{e^{-\eta_2{}^{2kl_2}}}{\eta_2}d\eta_2\int_0^\infty t^{m_3}e^{-t^{2k\alpha_3}}dt$$

$$\le C\theta^{-\left(\left|\mu^3\right|+\left(m,\mu^3\right)\right)}.$$

Finally, by substituting $\xi = \theta^{-\mu^3}\eta$ in I_8, we get

$$I_8 \le C\theta^{-\left(\left|\mu^3\right|+\left(m,\mu^3\right)\right)}.$$

As a result, if $\frac{\alpha_1}{\alpha_3} < \frac{m_1+1}{m_3+1}$, $\frac{\alpha_2}{\alpha_3} = \frac{m_2+1}{m_3+1}$ then for some constants C_1, C_2 the following inequality holds:

$$I \le \theta^{-\max\limits_{i=1,2,3}\left(\left|\mu^i\right|+\left(m,\mu^i\right)\right)}(C_1\left|\ln\theta\right|+C_2). \tag{2.6}$$

Consider the third case, i.e. $\frac{\alpha_1}{\alpha_3} = \frac{m_1+1}{m_3+1}$, $\frac{\alpha_2}{\alpha_3} = \frac{m_1+1}{m_3+1}$, thus $\frac{\alpha_2}{\alpha_3} = \frac{m_2+1}{m_3+1}$.

Let

$$\mu_i^0 = \min_{j=1,2,3} \mu_i^j.$$

Then $\mu_1^0 = \mu_1^1$, $\mu_2^0 = \mu_2^2$, $\mu_3^0 = \mu_3^3$. As in the first case, we split I into 8 integrals by μ_1^0, μ_2^0 , μ_3^0. We consider each integral separately. By substituting $\xi = \theta^{-\mu^3}\eta$ in I_1, we get

$$I_1 \le C\theta^{-\left(\left|\mu^3\right|+\left(m,\mu^3\right)\right)}.$$

By substituting $\xi = \theta^{-\mu^2}\eta$ in I_2, we get

$$I_2 \le C\theta^{-\left(\left|\mu^2\right|+\left(m,\mu^2\right)\right)} \int_0^\infty \eta_1^{m_1} e^{-{\eta_1}^{2kl_1}} d\eta_1 \int_0^1 \eta_1^{m_2} e^{-{\eta_1}^{2k\alpha_1}{\eta_2}^{2k\alpha_2}{\eta_3}^{2k\alpha_3}} d\eta_2 \int_0^\infty \eta_3^{m_3} e^{-{\eta_3}^{2kl_3}} d\eta_3$$

$$\le C\theta^{-\left(\left|\mu^2\right|+\left(m,\mu^2\right)\right)}.$$

By substituting $\xi = \theta^{-\mu^3}\eta$ in I_3 and I_4 and estimating them similarly to I_2, an analogous estimate can be obtained.

By substituting $\xi = \theta^{-\mu^1}\eta$ in I_5, we get

$$I_5 \le C\theta^{-\left(\left|\mu^1\right|+\left(m,\mu^1\right)\right)} \int_0^1 d\eta_1 \int_0^\infty d\eta_2 \int_0^\infty e^{-{\eta_2}^{2kl_2}} e^{-{\eta_3}^{2kl_3}} e^{-{\eta_1}^{2k\alpha_1}{\eta_2}^{2k\alpha_2}{\eta_3}^{2k\alpha_3}} \eta_1^{m_1}\eta_2^{m_2}\eta_3^{m_3} d\eta_3$$

$$\le C\theta^{-\left(\left|\mu^1\right|+\left(m,\mu^1\right)\right)}.$$

Similarly, we make the substitutions $\xi = \theta^{-\mu^2}\eta$ and $\xi = \theta^{-\mu^3}\eta$ in I_6 and I_7, respectively.

Let us estimate I_8. By substituting $\xi = \theta^{-\mu^3}\eta$, we get

$$I_8 \le C\theta^{-\left(\left|\mu^3\right|+\left(m,\mu^3\right)\right)} \int_{\theta^{-\mu_1^0+\mu_1^3}}^\infty \frac{e^{-{\eta_1}^{2kl_1}}}{\eta_1} d\eta_1 \int_{\theta^{-\mu_2^0+\mu_2^3}}^\infty \frac{e^{-{\eta_2}^{2kl_2}}}{\eta_2} d\eta_2 \int_1^\infty t^{m_3} e^{2k\alpha_3} dt$$

$$\le \left(C_0(\ln\theta)^2 + C_1\left|\ln\theta\right| + C_2\right)\theta^{-\left(\left|\mu^3\right|+\left(m,\mu^3\right)\right)}.$$

Finally, we get that in the third case there exist constants C_0, C_1, C_2, such that

$$I \le \left(C_0(\ln\theta)^2 + C_1\left|\ln\theta\right| + C_2\right)\theta^{\max\limits_{i=1,2,3}\left|\mu^i\right|+\left(m,\mu^i\right)}. \tag{2.7}$$

Using inequalities (2.5), (2.6) and (2.7), we proceed with the proof of Lemma 2.1. Let us estimate $\hat{G}_{1,j}(t,\theta)$, $j = 1,2,3,4$. For any multi-index $m = (m_1, m_2, m_3)$

$$D^m \hat{G}_{1,j}(t,\theta) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int\limits_{\mathbb{R}^3} \xi_1^{m_1}\xi_2^{m_2}\xi_3^{m_3} e^{-i(t,\xi)} 2k\left(\theta\xi^{\alpha^j}\right)^{2k-1} e^{-P(\theta,\xi)} d\xi_1 d\xi_2 d\xi_3. \tag{2.8}$$

Consider the vertex $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ of $\mathfrak{N}$, and the multi-index $(m_1+\alpha_1^j, m_2+\alpha_2^j,\ m_3+\alpha_3^j)$. Suppose that:

$$\frac{\alpha_i}{\alpha_r} \ne \frac{m_i+\alpha_i^j+1}{m_r+\alpha_r^j+1},\ i,r = 1,2,3. \tag{2.9}$$

Let i_0 be the index, for which the following relation holds:

$$\max_{i=1,2,3} \frac{\alpha_i}{m_i+\alpha_i^j+1} = \frac{\alpha_{i_0}}{m_{i_0}+\alpha_{i_0}^j+1}.$$

Then let us substitute $\xi=\theta^{-\mu^{i_0}}\eta$ in integral (2.8).

According to (2.9), $\frac{\alpha_i}{m_i+\alpha_i^j+1}<\frac{\alpha_{i_0}}{m_{i_0}+\alpha_{i_0}^j+1}$ $i\neq i_0$. Considering $1-l_1\mu_1^{i_0}\geq 0$, $0<\theta<1$

$$|D^m\hat{G}_{1,j}\left(t,\theta\right)|\leq C\theta^{-(|\mu^{i_0}|+(m,\mu^{i_0}))}. \tag{2.10}$$

In the second case, i.e. when there is one equality in (2.9), (2.6) is obtained for $|D^m\hat{G}_{1,j}\left(t,\theta\right)|$. In the third case, i.e. when there are only equalities in (2.9), (2.7) is obtained for $|D^m\hat{G}_{1,j}\left(t,\theta\right)|$.

Let us estimate $t_1^{N\alpha_1}\ t_2^{N\alpha_2}\ t_3^{N\alpha_3}D^m\hat{G}_{1,j}\left(t,\theta\right)$. Using the properties of the Fourier transform we have

$$\theta^{-N}t_1^{N\alpha_1}\ t_2^{N\alpha_2}\ t_3^{N\alpha_3}D^m\hat{G}_{1,j}\left(t,\theta\right)$$

$$=\frac{\theta^{-N}}{(2\pi)^{\frac{3}{2}}}\int\limits_{\mathbb{R}^3}D_{\xi_1}^{N\alpha_1}D_{\xi_2}^{N\alpha_2}D_{\xi_3}^{N\alpha_3}e^{-i(t,\xi)}\xi_1^{m_1}\xi_2^{m_2}\xi_3^{m_3}2k(\theta\xi^{\alpha^j})^{2k-1}e^{-P(\theta,\xi)}d\xi_1d\xi_2d\xi_3.$$

Integration by parts makes it suffice to estimate the integral

$$\theta^{-N}\int_0^\infty\int_0^\infty\int_0^\infty D_{\xi_1}^{N\alpha_1}D_{\xi_2}^{N\alpha_2}D_{\xi_3}^{N\alpha_3}(\xi_1^{m_1}\xi_2^{m_2}\xi_3^{m_3}(\theta\xi^{\alpha^j})^{2k-1}e^{-P(\theta,\xi)})d\xi_1d\xi_2d\xi_3.$$

As in [4], we apply the following formula for the derivative of $\Phi(\xi)e^{-P(\theta,\xi)}$ of order $N\alpha$, where $\Phi(\xi)=\xi_1^{m_1}\xi_2^{m_2}\xi_3^{m_3}\left(\theta\xi^{\alpha^j}\right)^{2k-1}$

$$\theta^{-N}\sum_{\beta+\gamma=N\alpha}C_{|N\alpha|}^{|\beta|}\int_0^\infty\int_0^\infty\int_0^\infty D_\xi^\beta(\xi_1^{m_1}\xi_2^{m_2}\xi_3^{m_3}(\theta\xi^{\alpha^j})^{2k-1}\cdot$$
$$\cdot\sum_{\sigma^1+\dots+\sigma^{|\gamma|}=\gamma}e^{-P(\theta,\xi)}\prod_{j=1}^{|\gamma|}D_\xi^{\sigma^j}P(\theta,\xi)d\xi_1d\xi_2d\xi_3, \tag{2.11}$$

where the product is taken over all σ^j, satisfying $|\sigma^j|>0$.

Suppose that the substitution $\xi=\theta^{-\mu^i}\eta$ is made in (2.11) for some $i\,(i=1,2,3)$, then, considering the equalities $(\alpha N,\mu^i)=N\,(i=1,2,3)\,,D_\xi^\beta=\theta^{(\mu^i,\beta)}D_\eta^\beta$, we find that the "contribution" of θ to (2.11) is

$$\theta^{(2k-1)(1-(\alpha^j,\mu^i))}\prod_{j=1}^{|\gamma|}\theta^{(2k)(1-\left(\alpha^r,\mu^i\right))}\theta^{-(\left|\mu^i\right|+\left(m,\mu^i\right))}.$$

As $(\alpha N,\mu^i)=N,(\alpha^r,\mu^i)\leq 1\ ,r=1,2,3,4\ ,i=1,2,3$, the exponent of θ is at least $-(|\mu^i|+(m,\mu^i))$.

Let ρ_i be an exponent of ξ_i in (2.11) $(i=1,2,3)$. Consider $\rho=(\rho_1;\rho_2;\rho_3)$ and $\alpha=(\alpha_1;\alpha_2;\alpha_3)$. For the ratios $\frac{\alpha_i}{\rho_i+1}$ $(i=1,2,3)$ cases 1-3 may arise. Then (2.5) is true for $|t_1^{N\alpha_1}\ t_2^{N\alpha_2}\ t_3^{N\alpha_3}D^m\hat{G}_{1,j}\left(t,\theta\right)|$ in the first case, (2.6) in the second case, and (2.7) in the third case.

To estimate $\theta^{-N}|t_1^{N\beta_1}t_2^{N\beta_2}t_3^{N\beta_3}D^m\hat{G}_{1,j}\left(t,\theta\right)|$ a similar argument can be used. The exponent of θ will be at least $-\max\limits_{i=1,2,3}(|\mu^i|+(m,\mu^i))$, because the multi-index $(\beta_1,\beta_2,0)$ lies on the intersection of the plane passing through the points $\{\alpha_2, \alpha_3, \alpha_4\}$ and the plane passing through the points $\{\alpha_1, \alpha_3, \alpha_4\}$. Thus, a substitution by μ^1 or μ^2 will yield $(N\beta; \mu^1) = (N\beta; \mu^2) = N$, and a substitution by μ^3 will yield $(N\beta; \mu^3) > N$.

The estimate for $\theta^{-N} \left| t_1^{N\gamma_1} \, D^m \hat{G}_{1,j}(t, \theta) \right|$ is done similarly, taking into account that $\gamma = (\gamma_1, 0, 0)$ lies on the plane, passing through the points $\{\alpha_2, \alpha_3, \alpha_4\}$, and $(N\gamma; \mu^1) = N$, $(N\gamma; \mu^2) > N$, $(N\gamma; \mu^3) > N$.

In either case, one of (2.5), (2.6), (2.7) holds. □

Remark 1. *If (2.9) holds, then in (2.4) the constants C_0 and C_1 are 0. If there is at least one equality in (2.9), then $C_0 = 0$.*

Lemma 2.2. *Let $0 < \theta < 1$, $\alpha_1 < \alpha_2 < \alpha_3$. Then there is a constant PY̌, such that*

$$\int\limits_{\mathbb{R}^3} \frac{dt_1 dt_2 dt_3}{1 + \theta^{-N} \left(t_1^{N\alpha_1} t_2^{N\alpha_2} t_3^{N\alpha_3} + t_1^{N\beta_1} t_2^{N\beta_2} + t_1^{N\gamma_1} \right)} = C\theta^{|\mu^1|}. \tag{2.12}$$

Proof. Recall, that $\beta_1 = \frac{\alpha_1 l_3}{l_3 - \alpha_3}$, $\beta_2 = \frac{\alpha_2 l_3}{l_3 - \alpha_3}$, and thus $\frac{\beta_1}{\beta_2} = \frac{\alpha_1}{\alpha_2}$, $\beta_1 < \beta_2$.

By substituting $t = \theta^{\mu^1} \eta$ in (2.12) (note that it suffices to estimate the integral only on the closed upper half-space), we have

$$I = \int_0^\infty \int_0^\infty \int_0^\infty \frac{dt_1 dt_2 dt_3}{1 + \theta^{-N} \left(t_1^{N\alpha_1} t_2^{N\alpha_2} t_3^{N\alpha_3} + t_1^{N\beta_1} t_2^{N\beta_2} + t_1^{N\gamma_1} \right)}$$

$$= \theta^{|\mu^1|} \int_0^\infty \int_0^\infty \int_0^\infty \frac{d\eta_1 d\eta_2 d\eta_3}{1 + \left(\left(\eta_1^{\frac{\alpha_1}{\alpha_3}} \eta_2^{\frac{\alpha_2}{\alpha_3}} \eta_3 \right)^{N\alpha_3} + \left(\eta_1^{\frac{\beta_1}{\beta_2}} \eta_2 \right)^{N\beta_2} + \eta_1^{N\gamma_1} \right)}.$$

By applying the change of variables $\tau_1 = \eta_1$, $\tau_2 = \eta_1^{\frac{\alpha_1}{\alpha_2}} \eta_2$, $\tau_3 = \eta_1^{\frac{\alpha_1}{\alpha_3}} \eta_2^{\frac{\alpha_2}{\alpha_3}} \eta_3$, we get

$$I = C\theta^{|\mu^1|} \int_0^\infty \int_0^\infty \int_0^\infty \frac{d\tau_1 d\tau_2 d\tau_3}{\tau_1^{\frac{\alpha_1}{\alpha_2}} \tau_2^{\frac{\alpha_2}{\alpha_3}} \left(1 + \tau_3^{N\alpha_3} + \tau_2^{N\beta_2} + \tau_1^{N\gamma_1} \right)}. \tag{2.13}$$

Since $\frac{\alpha_1}{\alpha_2} < 1$, $\frac{\alpha_2}{\alpha_3} < 1$, the integral converges and $I = C\theta^{|\mu^1|}$. □

Lemma 2.3. *Let $0 < \theta < 1$, $\alpha_1 < \alpha_2 = \alpha_3$. Then for any multi-index $m = (m_1, m_2, m_3)$ and any N satisfying (A) there are constants $C_i (i = 0, 1, 2)$, such that*

$$\begin{aligned} \left| D^m \hat{G}_{1,j}(t, \theta) \right| \le \theta^{-\max\limits_{i=1,2,3} \left(|\mu^i| + (m, \mu^i) \right)} & \frac{\left(C_0 (\ln\theta)^2 + C_1 |\ln\theta| + C_2 \right)}{1 + \theta^{-N} \left(t_1^{N\alpha_1} t_2^{N\alpha_2} t_3^{N\alpha_3} + t_1^{N\beta_1} t_2^{N\beta_2} + t_1^{N\gamma_1} \right)} \\ & \cdot \frac{1}{1 + \theta^{-N} \left(t_1^{N\alpha_1} t_2^{N\alpha_2} t_3^{N\alpha_3} + t_1^{N\sigma_1} t_3^{N\sigma_3} + t_1^{N\gamma_1} \right)}, \end{aligned} \tag{2.14}$$

where $\gamma = (\gamma_1, \gamma_2, 0)$ is the point of intersection of the planes μ^2 and μ^3 and XOY, $\sigma = (\sigma_1, 0, \sigma_3)$ is the point of intersection of the planes μ^1 and μ^3 and XOZ, $\gamma = (\gamma_1, 0, 0)$ is the point of intersection of the μ^3 plane with the x-axis.

Proof. Since $\gamma = (\gamma_1, \gamma_2, 0)$ is the point of intersection of the μ_2- and μ_3-planes, we have $(\gamma;\mu^1) > 1$, $(\gamma;\mu^2) = 1$, $(\gamma;\mu^3) = 1$. Analogously, $\sigma = (\sigma_1, 0, \sigma_3)$ is the point of intersection of planes μ_1 and μ_3, thus $(\sigma;\mu^1) = 1$, $(\sigma;\mu^3) = 1$, $(\sigma;\mu^2) > 1$. Similarly, if $\gamma = (\gamma_1, 0, 0)$, then $\gamma_1\mu_1^1 = 1$, $\gamma_1\mu_1^2 > 1$, $\gamma_1\mu_1^3 > 1$.

To prove the lemma, let us estimate:

$$\begin{aligned} &\left(1+\theta^{-N}\left(t_1^{N\alpha_1}t_2^{N\alpha_2}t_3^{N\alpha_3}+t_1^{N\beta_1}t_2^{N\beta_2}+t_1^{N\gamma_1}\right)\right) \\ &\cdot\left(1+\theta^{-N}\left(t_1^{N\alpha_1}t_2^{N\alpha_2}t_3^{N\alpha_3}+t_1^{N\sigma_1}t_3^{N\sigma_3}+t_1^{N\gamma_1}\right)\right) D^m\hat{G}_{1,j}(t,\theta) \\ = \frac{1}{(2\pi)^{\frac{3}{2}}}\int\limits_{\mathbb{R}^3} &e^{-i(t,\xi)}\left(1+\theta^{-N}\left(D_{\xi_1}^{N\alpha_1}D_{\xi_2}^{N\alpha_2}D_{\xi_3}^{N\alpha_3}+D_{\xi_1}^{N\beta_1}D_{\xi_2}^{N\beta_2}+D_{\xi_1}^{N\gamma_1}\right)\right) \\ &\cdot\left(1+\theta^{-N}\left(D_{\xi_1}^{N\alpha_1}D_{\xi_2}^{N\alpha_2}D_{\xi_3}^{N\alpha_3}+D_{\xi_1}^{N\sigma_1}D_{\xi_2}^{N\sigma_2}+D_{\xi_1}^{N\gamma_1}\right)\right) \\ &\cdot\left(2k\left(\theta\xi^{\alpha^j}\right)^{2k-1}\xi_1^{m_1}\xi_2^{m_2}\xi_3^{m_3}e^{-P(\theta,\xi)}\right)d\xi_1 d\xi_2 d\xi_3. \end{aligned} \tag{2.15}$$

We expand (2.15) and estimate each summand separately. First, we estimate the exponent of θ. Considering the choice of points $\beta = (\beta_1, \beta_2, 0)$; $\sigma = (\sigma_1, 0, \sigma_3)$, $\gamma = (\gamma_1, 0, 0)$, we apply the change of variables $\xi = \theta^{-\mu^i}\eta$. It follows that the exponent of θ is at least $-\left(|\mu^i| + (m, \mu^i)\right)$.

After applying (2.11) to each summand of (2.15) and comparing the exponents of $\xi_i (i = 1, 2, 3)$ with the multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, cases 1-3 may arise. Finally, for each summand one of inequalities (2.5), (2.6), (2.7) holds. □

Lemma 2.4. *Let $0 < \theta < 1$, $\alpha_1 < \alpha_2 = \alpha_3$. Then there is a constant $C > 0$, such that*

$$\begin{aligned} &\int\limits_{\mathbb{R}^3}\frac{1}{1+\theta^{-N}\left(t_1^{N\alpha_1}t_2^{N\alpha_2}t_3^{N\alpha_3}+t_1^{N\beta_1}t_2^{N\beta_2}+t_1^{N\gamma_1}\right)} \\ &\cdot\frac{dt_1 dt_2 dt_3}{1+\theta^{-N}\left(t_1^{N\alpha_1}t_2^{N\alpha_2}t_3^{N\alpha_3}+t_1^{N\sigma_1}t_3^{N\sigma_3}+t_1^{N\gamma_1}\right)} \le C\theta^{|\mu^1|}. \end{aligned} \tag{2.16}$$

Proof. As in Lemma 2.2, it suffices to consider the integral only on the closed upper half-space. We split the integral into the following parts:

$$\begin{aligned} I = \int_0^\infty\int_0^\infty\int_0^\infty &\frac{1}{1+\theta^{-N}\left(t_1^{N\alpha_1}t_2^{N\alpha_2}t_3^{N\alpha_3}+t_1^{N\beta_1}t_2^{N\beta_2}+t_1^{N\gamma_1}\right)} \\ &\cdot\frac{dt_1 dt_2 dt_3}{1+\theta^{-N}\left(t_1^{N\alpha_1}t_2^{N\alpha_2}t_3^{N\alpha_3}+t_1^{N\sigma_1}t_3^{N\sigma_3}+t_1^{N\gamma_1}\right)} \\ = \int_0^\infty dt_1\int_0^{\theta^{\mu_2^1}}dt_2\int_0^{\theta^{\mu_3^1}}dt_3 &+ \int_0^\infty dt_1\int_{\theta^{\mu_2^1}}^\infty dt_2\int_0^{\theta^{\mu_3^1}}dt_3 + \int_0^\infty dt_1\int_0^{\theta^{\mu_2^1}}dt_2\int_{\theta^{\mu_3^1}}^\infty dt_3 \\ &+ \int_0^\infty dt_1\int_{\theta^{\mu_2^1}}^\infty dt_2\int_{\theta^{\mu_3^1}}^\infty dt_3 = I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Let us estimate each integral. Considering the points $\alpha,\beta,\gamma,\sigma$, $(\alpha,\mu^1)=(\beta,\mu^1)=(\sigma,\mu^1)=(\gamma,\mu^1)=1$

By substituting $t=\theta^{\mu^1}\eta$ in I_1, we get

$$I_1\le\theta^{|\mu^1|}\int_0^\infty d\eta_1\int_0^1 d\eta_2\int_0^1 d\eta_3\le C\theta^{|\mu^1|}\int_0^\infty\frac{d\eta_1}{1+\eta_1^{N\gamma_1}}\le C\theta^{|\mu^1|}.$$

By substituting $t=\theta^{\mu^1\eta}$ in I_2, we get

$$I_2\le C\theta^{|\mu^1|}\int_0^\infty d\eta_1\int_0^1 d\eta_2\int_1^\infty\frac{d\eta_3}{1+\eta_1^{N\alpha_1}\eta_2^{N\alpha_2}\eta_3^{N\alpha_3}+\eta_1^{N\sigma_1}\eta_3^{N\sigma_3}+\eta_1^{N\gamma_1}}$$

$$\le C\theta^{|\mu^1|}\int_0^\infty d\eta_1\int_1^\infty\frac{d\eta_3}{1+\eta_1^{N\sigma_1}\eta_3^{N\sigma_3}+\eta_1^{N\gamma_1}}$$

$$\le C\theta^{|\mu^1|}\int_0^\infty\frac{d\eta_1}{\eta_1^{\sigma_1/\sigma_3}}\int_1^\infty\frac{d({\eta_1}^{\sigma_1/\sigma_3}\eta_3)}{1+({\eta_1}^{\sigma_1/\sigma_3}\eta_3)^{N\sigma_3}+\eta_1^{N\gamma_1}}$$

$$\le C\theta^{|\mu^1|}\int_0^\infty\frac{d\eta_1}{\eta_1^{\sigma_1/\sigma_3}}\int_0^\infty\frac{d\tau}{1+\tau^{N\sigma_3}+\eta_1^{N\gamma_1}}\le C\theta^{|\mu^1|},$$

since $\sigma_1/\sigma_3<1$.

I_3 is estimated similarly, as $\beta_1/\beta_2<1$.

By substituting $t=\theta^{\mu^1\eta}$ in I_4, we get

$$I_4\le C\theta^{|\mu^1|}\int_0^\infty\int_1^\infty\int_1^\infty\frac{d\eta_1 d\eta_2 d\eta_3}{1+(\eta_1^{\alpha_1/\alpha_2}\eta_2\eta_3)^{N\alpha_3}+({\eta_1}^{\frac{\beta_1}{\beta_2}}\eta_2)^{N\beta_2}+\eta_1^{N\gamma_1}}.$$

In the last integral the change of variables $\tau_1=\eta_1,\tau_2={\eta_1}^{\alpha_1/\alpha_2}\eta_2,\ \tau_3=\eta_1^{\alpha_1/\alpha_2}\eta_2\eta_3$ yields

$$I_4\le C\theta^{|\mu^1|}\int_0^\infty\frac{d\tau_1}{\tau_1^{\alpha_1/\alpha_2}}\int_{{\tau_1}^{\alpha_1/\alpha_2}}^\infty d\tau_2\int_0^\infty\frac{d\tau_3}{\tau_2(1+\tau_1^{N\gamma_1}+\tau_2^{N\beta_2}+\tau_3^{N\alpha_3})}$$

$$=C\theta^{|\mu^1|}\int_0^\infty\frac{d\tau_1}{\tau_1^{\alpha_1/\alpha_2}}\int_{{\tau_1}^{\alpha_1/\alpha_2}}^1 d\tau_2\int_0^\infty\frac{d\tau_3}{\tau_2\left(1+\tau_1^{N\gamma_1}+\tau_2^{N\beta_2}+\tau_3^{N\alpha_3}\right)}$$

$$+C\theta^{|\mu^1|}\int_0^\infty\frac{d\tau_1}{\tau_1^{\alpha_1/\alpha_2}}\int_1^\infty d\tau_2\int_0^\infty\frac{d\tau_3}{1+\tau_1^{N\gamma_1}+\tau_2^{N\beta_2}+\tau_3^{N\alpha_3}}=I_4^1+I_4^2.$$

The integral I_4^2 is convergent. Let us estimate I_4^1.

$$I_4^1\le C\theta^{|\mu^1|}\int_0^\infty\frac{d\tau_1}{\tau_1^{\alpha_1/\alpha_2}}\int_{{\tau_1}^{\alpha_1/\alpha_2}}^1\frac{d\tau_2}{\tau_2}\int_0^\infty\frac{d\tau_3}{1+\tau_1^{N\gamma_1}+\tau_3^{N\alpha_3}}$$

$$\le C\theta^{|\mu^1|}\int_0^\infty\frac{\left|\ln\frac{1}{\tau_1}\right|}{\tau_1^{\alpha_1/\alpha_2}}d\tau_1\int_0^\infty\frac{d\tau_3}{1+\tau_1^{N\gamma_1}+\tau_3^{N\alpha_3}}+C_2\theta^{|\mu^1|}\le C\theta^{|\mu^1|},$$

since $\int\limits_0^1\frac{\ln\frac{1}{\tau}}{\tau^{\alpha_1/\alpha_2}}d\tau$ converges when $\alpha_1/\alpha_2<1$. □

Now we consider the case $\alpha_1 = \alpha_2 = \alpha_3$.

Lemma 2.5. *Let $0 < \theta < 1$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$. Then for any multi-index $m = (m_1, m_2, m_3)$ and any N satisfying (A) there are constants C_i, $(i = 0, 1, 2)$, such that*

$$|D^m \hat{G}_{1,j}(t, \theta)| \leq \theta^{-\max\limits_{i=1,2,3}\left(|\mu^i| + (m, \mu^i)\right)} \left(C_0 (\ln\theta)^2 + C_1 |\ln\theta| + C_2\right) \cdot$$

$$\cdot \frac{1}{1 + \theta^{-N}\left(t_1^{N\alpha} t_2^{N\alpha} t_3^{N\alpha} + t_1^{N\beta} t_2^{N\beta} + t_1^{N\gamma}\right)} \cdot \frac{1}{1 + \theta^{-N}\left(t_1^{N\alpha} t_2^{N\alpha} t_3^{N\alpha} + t_1^{N\sigma} t_3^{N\sigma} + t_3^{Nr}\right)} \cdot$$

$$\cdot \frac{1}{1 + \theta^{-N}\left(t_1^{N\alpha} t_2^{N\alpha} t_3^{N\alpha} + t_2^{Nq} t_3^{Nq} + t_2^{N\delta}\right)},$$

where $(\beta, \beta, 0)$ is the point of intersection the planes μ^1, μ^2 and $(x, 0, y)$; $(\sigma, 0, \sigma)$ is the point of intersection of the planes μ^1, μ^3 and $(x, 0, z)$; $(0, q, q)$ is the point of intersection of the planes μ^2, μ^3 and $(y, 0, z)$; $\gamma\mu_1^1 = 1; \delta\mu_2^2 = 1; r\mu_3^3 = 1$.

The proof is analogous to the one of Lemma 2.3, if to take into account that

$$\beta\mu_1^2 + \beta\mu_2^2 = 1;\ \beta\mu_1^3 + \beta\mu_2^3 = 1; \beta\mu_1^1 + \beta\mu_2^1 > 1,$$

$$\sigma\mu_1^1 + \sigma\mu_3^1 = 1;\ \sigma\mu_1^3 + \sigma\mu_3^3 = 1; \sigma\mu_1^2 + \sigma\mu_3^2 > 1,$$

$$q\mu_2^1 + q\mu_3^1 = 1;\ q\mu_2^2 + q\mu_3^2 = 1; q\mu_2^3 + q\mu_3^3 > 1,$$

$$\delta\mu_1^1 = 1;\ \delta\mu_1^2 > 1; \delta\mu_1^3 > 1,$$

$$r\mu_3^3 = 1;\ r\mu_3^2 > 1; r\mu_3^1 > 1,$$

$$\gamma\mu_2^2 = 1;\ \gamma\mu_2^1 > 1; \gamma\mu_2^3 > 1,$$

Lemma 2.6. *Let $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$. Then there are constants C_0, C_1, C_2, such that*

$$\int\limits_{\mathbb{R}^3} \frac{dt_1 dt_2 dt_3}{\left(1 + \theta^{-N}\left((t_1 t_2 t_3)^{N\alpha} + (t_1 t_2)^{N\beta} + t_1^{N\gamma}\right)\right)\left(1 + \theta^{-N}\left((t_1 t_2 t_3)^{N\alpha} + (t_1 t_3)^{N\sigma} + t_3^{Nr}\right)\right)}$$

$$\cdot \frac{1}{1 + \theta^{-N}\left((t_1 t_2 t_3)^{N\alpha} + (t_2 t_3)^{Nq} + t_2^{N\delta}\right)} \leq \theta^{|\mu^1|}\left(C_0 (\ln\theta)^2 + C_1 |\ln\theta| + C_2\right). \tag{2.17}$$

Proof. Since $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$, we have $(\mu^1, \alpha) = (\mu^2, \alpha) = (\mu^3, \alpha) = 1$, i.e. $|\mu^1| = |\mu^2| = |\mu^3| = \frac{1}{\alpha}$.

Let $\max\limits_{i=1,2,3} \mu_j^i = \mu_j^0$ $j = 1, 2, 3$. Then

$$\mu_1^1 = \frac{1}{\alpha}\left(1 - \frac{\alpha}{l_2} - \frac{\alpha}{l_3}\right),\ \mu_2^1 = \frac{1}{l_2},$$

$$\mu_3^1 = \frac{1}{l_3}, \mu_1^2 = \frac{1}{l_1}, \mu_2^2 = \frac{1}{\alpha}\left(1 - \frac{\alpha}{l_1} - \frac{\alpha}{l_3}\right),$$
$$\mu_3^2 = \frac{1}{l_3},\ \mu_1^3 = \frac{1}{l_1},$$
$$\mu_2^3 = \frac{1}{l_2},\ \mu_3^3 = \frac{1}{\alpha}\left(1 - \frac{\alpha}{l_1} - \frac{\alpha}{l_2}\right)$$

.

The polyhedron $\mathfrak{N}$ is convex, thus

$$\frac{\alpha}{l_1} + \frac{\alpha}{l_2} + \frac{\alpha}{l_3} > 1,$$

and hence $\max\limits_{i=1,2,3} \mu_j^i = \frac{1}{l_i}$.

Let us split the integral (2.17) into 8 integrals (note that it is sufficient to consider the integrals only on the closed upper half-space)

$$I = \int_0^\infty dt_1 \int_0^\infty dt_2 \int_0^\infty dt_3 = I_1 + \cdots + I_8 =$$

$$= \int_0^{\theta^{\mu_1^0}} dt_1 \int_0^{\theta^{\mu_2^0}} dt_2 \int_0^{\theta^{\mu_3^0}} \ldots\ dt_3 + \int_{\theta^{\mu_1^0}}^\infty dt_1 \int_0^{\theta^{\mu_2^0}} dt_2 \int_0^{\theta^{\mu_3^0}} \ldots\ dt_3 + \int_0^{\theta^{\mu_1^0}} dt_1 \int_{\theta^{\mu_2^0}}^\infty dt_2 \int_0^{\theta^{\mu_3^0}} \ldots\ dt_3$$
$$+ \int_0^{\theta^{\mu_1^0}} dt_1 \int_0^{\theta^{\mu_2^0}} dt_2 \int_{\theta^{\mu_3^0}}^\infty \ldots\ dt_3 + \int_{\theta^{\mu_1^0}}^\infty dt_1 \int_{\theta^{\mu_2^0}}^\infty dt_2 \int_0^{\theta^{\mu_3^0}} \ldots\ dt_3 + \int_0^{\theta^{\mu_1^0}} dt_1 \int_{\theta^{\mu_2^0}}^\infty dt_2 \int_{\theta^{\mu_3^0}}^\infty \ldots\ dt_3$$
$$+ \int_{\theta^{\mu_1^0}}^\infty dt_1 \int_0^{\theta^{\mu_2^0}} dt_2 \int_{\theta^{\mu_3^0}}^\infty \ldots\ dt_3 + \int_{\theta^{\mu_1^0}}^\infty dt_1 \int_{\theta^{\mu_2^0}}^\infty dt_2 \int_{\theta^{\mu_3^0}}^\infty \ldots\ dt_3.$$

By substituting $t = \theta^{\mu^1}\eta$ in I_1 and taking into account that $\mu_i^0 - \mu_i^1 \geq 0$, $i = 1, 2, 3$ and $0 < \theta < 1$, we have

$$I_1 \leq C\theta^{|\mu^1|} \int_0^1 \int_0^1 \int_0^1 \frac{d\eta_1 d\eta_2 d\eta_3}{\left(1 + (\eta_1\eta_2\eta_3)^{N\alpha} + (\eta_1\eta_2)^{N\beta} + \eta_1{}^{N\gamma}\right)} \leq C\theta^{|\mu^1|}.$$

By substituting $t = \theta^{\mu^1}\eta$ in I_2, we get

$$I_2 \leq C\theta^{|\mu^1|} \int_0^\infty \frac{d\eta_1}{1 + \eta_1^{N\gamma}} \int_0^1 d\eta_2 \int_0^1 d\eta_3 \leq\ C\theta^{|\mu^1|}.$$

I_3 and I_4 can be estimated similarly to I_2 by the substitutions $t = \theta^{\mu^2}\eta$ and $t = \theta^{\mu^3}\eta$. By substituting $t = \theta^{\mu^1}\eta$ in I_5, we get

$$I_5 \leq C\theta^{|\mu^1|} \int_{\theta^{\mu_1^0 - \mu_1^1}}^\infty d\eta_1 \int_{\theta^{\mu_2^0 - \mu_2^1}}^\infty d\eta_2 \int_0^{\theta^{\mu_3^0 - \mu_3^1}} \frac{d\eta_3}{1 + (\eta_1\eta_2\eta_3)^{N\alpha} + (\eta_1\eta_2)^{N\beta} + \eta_1{}^{N\gamma}}$$
$$\leq C\theta^{|\mu^1|} \int_{\theta^{\mu_1^0 - \mu_1^1}}^\infty \frac{d\tau_1}{\tau_1} \int_0^\infty \frac{d\tau_2}{1 + (\tau_2)^{N\beta} + \tau_1{}^{N\gamma}} \leq \theta^{|\mu^1|} \left(C_1 \left|\ln\theta\right| + C_2\right).$$

I_6 and I_7 are estimated similarly. By substituting $t=\theta^{\mu^1}\eta$ in I_8, we get

$$I_8 \le C\theta^{|\mu^1|}\int_{\theta^{\mu_1^0-\mu_1^1}}^{\infty}\frac{d\eta_1}{\eta_1}\int_1^{\infty}\frac{d(\eta_1\eta_2)}{\eta_1\eta_2}\int_1^{\infty}\frac{d(\eta_1\eta_2\eta_3)}{1+(\eta_1\eta_2\eta_3)^{N\alpha}+(\eta_1\eta_2)^{N\beta}+\eta_1{}^{N\gamma}}.$$

Let us apply the change of variables $\tau_1=\eta_1,\ \tau_2=\eta_1\eta_2,\ \tau_3=\eta_1\eta_2\eta_3$ in I_8, then

$$I_8 \le\ C\theta^{|\mu^1|}(\int_{\theta^{\mu_1^0-\mu_1^1}}^{1}\frac{d\tau_1}{\tau_1}\int_{\tau_1}^{1}\frac{d\tau_2}{\tau_2}\int_0^{\infty}\frac{d\tau_3}{1+(\tau_3)^{N\alpha}+(\tau_2)^{N\beta}+(\tau_1)^{N\gamma}}$$
$$+\int_{\theta^{\mu_1^0-\mu_1^1}}^{1}\frac{d\tau_1}{\tau_1}\int_1^{\infty}\frac{d\tau_2}{\tau_2}\int_0^{\infty}\frac{d\tau_3}{1+(\tau_3)^{N\alpha}+(\tau_2)^{N\beta}+(\tau_1)^{N\gamma}}$$
$$+\int_1^{\infty}\frac{d\tau_1}{\tau_1}\int_1^{\infty}\frac{d\tau_2}{\tau_2}\int_0^{\infty}\frac{d\tau_3}{1+(\tau_3)^{N\alpha}+(\tau_2)^{N\beta}+(\tau_1)^{N\gamma}})=C\theta^{|\mu^1|}(J_1+J_2+J_3).$$

Let us estimate J_i $(i=1,2,3)$

$$J_1\le\int_{\theta^{\mu_1^0-\mu_1^1}}^{1}\frac{d\tau_1}{\tau_1}|\ln\tau_1\ |\int_0^{\infty}\frac{d\tau_3}{1+(\tau_3)^{N\alpha}}\le C(\ln\theta)^2.$$

$$J_2\le C|\ln\theta|.$$

$$J_3\le C\int_1^{\infty}d\tau_1\int_1^{\infty}d\tau_2\int_0^{\infty}\frac{d\eta_3}{1+(\tau_3)^{N\alpha}+(\tau_2)^{N\beta}+\tau_1{}^{N\gamma}}\le C.$$

As a result, for some constants C_0,C_1,C_2

$$I_8\le\ \theta^{|\mu^1|}\left(C_0(\ln\theta)^2+C_1|\ln\theta|+C_2\right).$$

□

3 Regularization of a function by a set of multi-indices and its properties

For any measurable function U consider the regularization with the kernel $\hat{G}_0\,(t,\theta)$:

$$U_\theta\,(x)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int\limits_{\mathbb{R}^3}U(t)\hat{G}_0\,(t-x,\theta)dt,\quad x\in\mathbb{R}^3. \tag{3.1}$$

As in [4], the regularization U_θ satisfies the following properties.

Lemma 3.1. *Let $p>1,\ f\in L_p\left(\mathbb{R}^3\right)$. Then $f_\theta\in L_p\left(\mathbb{R}^3\right)$ and $\lim\limits_{\theta\to\infty}\|f_\theta\|_{L_p(\mathbb{R}^3)}=0$.*

Proof. Let $\theta>1,\ \lambda=\left(\frac{1}{l_1},\frac{1}{l_2},\frac{1}{l_3}\right)=(\lambda_1,\lambda_2,\lambda_3)$. As in [4], we substitute $t=\theta^{-\lambda}\eta$ in $\hat{G}_0\,(t,\theta)$. It can be shown, that for some constant C and a number N satisfying (A) the following inequality holds:

$$\left|\hat{G}_0\,(t,\theta)\right|\le C\theta^{|\lambda|}\frac{1}{1+\theta^{-N}(t_1^{Nl_1}+t_2^{Nl_2}+t_3^{Nl_3})}. \tag{3.2}$$

As $C_0^\infty(\mathbb{R}^n)$ is dense in $L_p(\mathbb{R}^3)$, it follows that for any $\varepsilon > 0$ there is an $\Phi_\varepsilon \in C_0^\infty(\mathbb{R}^3)$, such that $\|f - \Phi_\varepsilon\|_{L_p(\mathbb{R}^3)} < \varepsilon$. Applying Young's inequality to $\|f_\theta\|_{L_p(\mathbb{R}^3)}$ we have

$$\|f_\theta\|_{L_p(\mathbb{R}^3)} \leq \frac{1}{(2\pi)^{\frac{3}{2}}} \left\| \int\limits_{\mathbb{R}^3} \hat{G}_0(t-x,\theta)(f-\Phi_\varepsilon)dt \right\|_{L_p(\mathbb{R}^3)} + \frac{1}{(2\pi)^{\frac{3}{2}}} \left\| \int\limits_{\mathbb{R}^3} \hat{G}_0(t-x,\theta)\,\Phi_\varepsilon(t)dt \right\|_{L_p(\mathbb{R}^3)}$$

$$\leq \frac{1}{(2\pi)^{\frac{3}{2}}} \left\|\hat{G}_0\right\|_{L_1} \|(f-\mathit{\Phi}_\varepsilon)\|_{L_p} + \frac{1}{(2\pi)^{\frac{3}{2}}} \left\|\hat{G}_0\right\|_{L_p} \|\Phi_\varepsilon\|_{L_1} = I_1 + I_2.$$

Using (3.2) for $\left\|\hat{G}_0\right\|_{L_1}$

$$\left\|\hat{G}_0\right\|_{L_1(\mathbb{R}^3)} \leq C\theta^{-|\lambda|} \int\limits_{\mathbb{R}^3} \frac{dt_1dt_2dt_3}{1+\theta^{-N}\left(t_1^{Nl_1}+t_2^{Nl_2}+t_3^{Nl_3}\right)} \leq C_1.$$

Thus $I_1 \leq \varepsilon C_1$. Consider I_2:

$$I_2 \leq C\theta^{-|\lambda|} \left(\int\limits_{\mathbb{R}^3} \frac{dt_1dt_2dt_3}{\left(1+\theta^{-N}(t_1^{Nl_1}+t_2^{Nl_2}+t_3^{Nl_3})\right)^p} \right)^{\frac{1}{p}} \leq C_\varepsilon \theta^{-|\lambda|+\frac{|\lambda|}{p}}.$$

As $p > 1$, $\lim\limits_{\theta\to\infty} I_2 = 0$. □

Lemma 3.2. *If $f \in L_p(\mathbb{R}^3)\,(1 \leq p < \infty)$, then $f_\theta \in L_p(\mathbb{R}^3)$ and $\lim\limits_{\theta\to 0} \|f_\theta - f\|_{L_p(\mathbb{R}^3)} = 0$.*

Proof. Since

$$\frac{1}{(2\pi)^{\frac{3}{2}}} \int\limits_{\mathbb{R}^3} \hat{G}_0(t,\theta)dt = G_0(0,\theta) = 1,$$

we can estimate the difference $f_\theta - f$ as follows:

$$\|f_\theta - f\|_{L_p} \leq \frac{1}{(2\pi)^{\frac{3}{2}}} \int\limits_{\mathbb{R}^3} \|f(\cdot+\tau) - f(\cdot)\|_{L_p} \left|\hat{G}_0(\tau,\theta)\right| d\tau.$$

Applying the generalized Minkowski inequality we have

$$\|f_\theta - f\|_{L_p} \leq \frac{1}{(2\pi)^{\frac{3}{2}}} \int\limits_{\mathbb{R}^3} \|f(\cdot+\tau) - f(\cdot)\|_{L_p} \left|\hat{G}_0(\tau,\theta)\right| d\tau. \tag{3.3}$$

Suppose that the vertex $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ of $\mathfrak{N}$ satisfies $\alpha_1 < \alpha_2 < \alpha_3$ (other cases are proven similarly with the use of Lemmas 2.3 and 2.5)

It can be shown, that for any non-negative integer N there are constants C_1, C_2, C_3, such that as in Lemma 3.1 for any $\theta \in (0,1)$ and $\lambda = (\lambda_1, \lambda_2, \lambda_3) = \left(\frac{1}{l_1}, \frac{1}{l_2}, \frac{1}{l_3}\right)$ the following inequality holds:

$$\left|\hat{G}_0(t,\theta)\right| \le \theta^{-|\lambda|+(1-(\alpha;\lambda))2k}\left(C_1(\ln\theta)^2+C_2\left|\ln\theta\right|+C_3\right)\frac{1}{1+\theta^{-N}\left(t_1^{Nl_1}+t_2^{Nl_2}+t_3^{Nl_3}\right)}. \quad (3.4)$$

The proof is similar to the one in Lemma 2.1.

Let $\rho_\lambda(x)=\left(x_1^{2l_1}+x_2^{2l_2}+x_3^{2l_3}\right)^{\frac{1}{2}}$ (see [2]). Let $\gamma\in(0;1)$. Integral (3.3) can be expressed the following way:

$$\|f_\theta-f\|_{L_p}\le C\int\limits_{\rho_\lambda(\tau)\le\theta^\gamma}\|f(\cdot+\tau)-f(\cdot)\|\left|\hat{G}_0(\tau,\theta)\right|d\tau$$

$$+C\int\limits_{\rho_\lambda(\tau)\ge\theta^\gamma}\|f(\cdot+\tau)-f(\cdot)\|\left|\hat{G}_0(\tau,\theta)\right|d\tau=A_1(\theta)+A_2(\theta).$$

By (3.4) for $A_2(\theta)$, we have

$$A_2(\theta)\le 2\|f\|_{L_p(\mathbb{R}^3)}\theta^{-|\lambda|+(1-(\alpha;\lambda))2k}\left(C_1(\ln\theta)^2+C_2\left|\ln\theta\right|+C_3\right)$$

$$\cdot\int\limits_{\rho_\lambda(\tau)\ge\theta^\gamma}\frac{d\tau_1 d\tau_2}{1+\theta^{-N}\left(\tau_1^{Nl_1}+\tau_2^{Nl_2}+\tau_3^{Nl_3}\right)}.$$

By substituting $\tau=\theta^\lambda\eta$, we get

$$A_2(\theta)\le 2\|f\|_{L_p(\mathbb{R}^3)}\theta^{(1-(\alpha;\lambda))2k}\left(C_1(\ln\theta)^2+C_2\left|\ln\theta\right|+C_3\right)$$

$$\cdot\int\limits_{\rho_\lambda(\eta)\ge\theta^{\gamma-1}}\frac{d\eta_1 d\eta_2}{1+\eta_1^{Nl_1}+\eta_2^{Nl_2}+\eta_3^{Nl_3}}.$$

The latter integral is estimated using the λ-spherical transformation (see [2]), i.e. we apply the change of variables $\eta_1=r^{\lambda_1}w_1,\ \eta_2=r^{\lambda_2}w_2,\ \eta_3=r^{\lambda_3}w_3$, where ${w_1}^{2l_1}+{w_2}^{2l_2}+{w_3}^{2l_3}=1$.

As a result,

$$A_2(\theta)\le 2\|f\|_{L_p(\mathbb{R}^3)}\theta^{(1-(\alpha;\lambda))2k}\left(C_1(\ln\theta)^2+C_2\left|\ln\theta\right|+C_3\right)$$

$$\cdot\int_{\theta^{\gamma-1}}^{\infty}\int\limits_{\rho_\lambda(\omega)=1}\frac{r^{|\lambda|-1}dr}{1+r^N({w_1}^{2Nl_1}+{w_2}^{2Nl_2}+{w_3}^{2Nl_3})}\sum_{i=1}^{3}\lambda_i^2{\omega_i}^2 d\omega$$

$$\le 2|f\|_{L_p(\mathbb{R}^3)}\theta^{(1-(\alpha;\lambda))2k}\left(C_1(\ln\theta)^2+C_2\left|\ln\theta\right|+C_3\right)\int_{\theta^{\gamma-1}}^{\infty}r^{|\lambda|-1-N}dr$$

$$=2\|f\|_{L_p(\mathbb{R}^3)}\theta^{(N-|\lambda|)(1-\gamma)+(1-(\alpha;\lambda))2k}\left(C_1(\ln\theta)^2+C_2\left|\ln\theta\right|+C_3\right).$$

Let N be such that the exponent of θ is positive. Then $\lim\limits_{\theta\to 0}A_2(\theta)=0$.

Let us estimate $A_1(\theta)$. By applying Lemma 2.1 for the case $m=0$, $N=0$, we have

$$A_1(\theta) \le C \sup_{\rho_\lambda(\eta)\le\theta^\gamma} \|f(\cdot+\tau)-f(\cdot)\|_{L_p(R^3)} \theta^{-\max\limits_{i=1,2,3}|\mu^i|} \int\limits_{\rho_\lambda(\tau)\le\theta^\gamma} d\eta_1 d\eta_2.$$

As $|\lambda| > \max\limits_{i=1,2,3}|\mu^i|$, there is $\gamma\in(0;1)$, such that $\gamma|\lambda| > \max\limits_{i=1,2,3}|\mu^i|$.
By applying the λ-spherical transformation

$$A_1(\theta) \le C\theta^{\gamma|\lambda|-\max\limits_{i=1,2,3}|\mu^i|} \sup_{\rho_\lambda(\eta)\le\theta^\gamma} \|f(\cdot+\eta)-f(\cdot)\|_{L_p(\mathbb{R}^3)}.$$

Since the exponent of θ is positive and the function $f\in L_p$ $(1\le p<\infty)$ is continuous with respect to translation in L_p (see [2]), it follows that $\lim\limits_{\theta\to0} A_1(\theta)=0$. □

Corollary 3.1. *Let* $f\in L_p(\mathbb{R}^3)$ $(1\le p<\infty)$. *Then there is a sequence* θ_k, *such that* $\lim\limits_{k\to\infty}\theta_k=0$ *and* $\lim\limits_{k\to\infty} f_{\theta_k}(x)=f(x)$ *almost everywhere.*

Proof follows by the properties of convergence in L_p.

4 Integral representation of functions

Theorem 4.1. *Let a function* f *have the weak derivatives* $D^{\alpha^i}f, i=1,2,3,4,$ *where* α^i *are the vertices of a completely regular polyhedron* $\mathfrak{N}$, *and* $D^\alpha f\in L_p(\mathbb{R}^3)\,(1\le p<\infty), i=1,2,3,4.$ *Then for almost all* $x\in\mathbb{R}^3$

$$f(x) = f_h(x) + \lim_{\varepsilon\to0}\sum_{i=1}^{4}\int_\varepsilon^h d\theta \int\limits_{\mathbb{R}^3} D^{\alpha^i} f(t)\,\hat{G}_{1,i}(t-x,\theta)\,dt. \tag{4.1}$$

Proof. By using (3.1) and the Fundamental Theorem of Calculus

$$\begin{aligned} f_h(x)-f_\varepsilon(x) &= \frac{1}{(2\pi)^{\frac{3}{2}}}\int_\varepsilon^h \frac{d}{d\theta}\int\limits_{\mathbb{R}^3} f(x+t)\,\hat{G}_0(t,\theta)\,dt d\theta \\ &= \frac{1}{(2\pi)^{\frac{3}{2}}}\int_\varepsilon^h \int\limits_{\mathbb{R}^3} f(x+t)\,\frac{d}{d\theta}\hat{G}_0(t,\theta)\,dt. \end{aligned} \tag{4.2}$$

Let us calculate $\frac{d}{d\theta}\hat{G}_0(t,\theta)$.

$$\frac{d}{d\theta}\hat{G}_0(t,\theta) = \frac{1}{(2\pi)^{\frac{3}{2}}}\sum_{j=1}^{4}\int\limits_{\mathbb{R}^3} e^{-i(t,\xi)}e^{-P(\theta,\xi)}(-2k)\,\theta^{2k-1}\xi^{2k\alpha^j}\,d\xi$$

$$= \frac{1}{(2\pi)^{\frac{3}{2}}}\sum_{j=1}^{4} D_t^{\alpha^j}\int\limits_{\mathbb{R}^3} e^{-i(t,\xi)}e^{-P(\theta,\xi)}(-2k)\left(\theta\,\xi^{\alpha^j}\right)^{2k-1} d\xi$$

$$= \sum_{j=1}^{4} D_t^{\alpha^j} \hat{G}_{1j}(t,\theta).$$

Hence

$$f_h(x) - f_\varepsilon(x) = \sum_{j=1}^{4} \int_\varepsilon^h d\theta \int_{\mathbb{R}^3} D_t^{\alpha^j} f(x+t)\, \hat{G}_{1,j}(t,\theta)\, dt. \tag{4.3}$$

By applying Corollary 3.1, we complete the proof. □

5 Embedding theorems for multianisotropic spaces

Let $\mathfrak{N}$ be a completely regular polyhedron described in Section 2, then

$$W_p^{\mathfrak{N}}(\mathbb{R}^3) = \{f : f \in L_p(\mathbb{R}^3);\, D^{\alpha^i} f \in L_p(\mathbb{R}^3),\quad i = 1,2,3,4\}$$

is called the multianisotropic Sobolev space. If $\mathfrak{N} = \{\alpha; |\alpha| \le m\}$, it coincides with the Sobolev space $W_p^m(\mathbb{R}^3)$, if $\mathfrak{N} = \{\alpha; (\alpha,\mu) \le 1\}$, it coincides with the anisotropic Sobolev space $W_p^{m_1,m_2,m_3}(\mathbb{R}^3)$.

Let us prove embedding theorems for $W_p^{\mathfrak{N}}$.

Theorem 5.1. *Let $1 \le p \le q < \infty$ or $1 \le p < \infty$ and $q = \infty$, $m = (m_1, m_2, m_3)$ be a multi-index, and let, for the vertex $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ of $\mathfrak{N}$, $\alpha_1 \le \alpha_2 < \alpha_3$ or $\alpha_1 < \alpha_2 = \alpha_3$. Denote $\chi = \max\limits_{i=1,2,3} (|\mu^i| + (m,\mu^i)) - |\mu^1| \left(1 - \frac{1}{p} + \frac{1}{q}\right)$.*

If $\chi < 1$, then $D^m W_p^{\mathfrak{N}}(\mathbb{R}^3) \hookrightarrow L_q(\mathbb{R}^3)$, i.e. any function $f \in W_p^{\mathfrak{N}}(\mathbb{R}^3)$ has the weak derivative $D^m f \in L_q(\mathbb{R}^3)$, and the following inequality holds

$$\|D^m f\|_{L_q(\mathbb{R}^3)} \le h^{1-\chi}\left(a_1(\ln h)^2 + a_2|\ln h|) + a_3\right) \cdot \sum_{i=1}^{4} \left\| D^{\alpha^i} f \right\|_{L_p(\mathbb{R}^3)}$$
$$+ h^{-\chi}(b_1(\ln h)^2 + b_2|\ln h| + b_3)\|f\|_{L_p(\mathbb{R}^3)}, \tag{5.1}$$

where h is an arbitrary positive parameter and $a_1, a_2, a_3, b_1, b_2, b_3$ are constants, that are independent of f and h.

Proof. Since $\alpha_1 \le \alpha_2 \le \alpha_3$, by (4.3)

$$D^m f_h(x) - D^m f_\varepsilon(x) = \sum_{i=1}^{4} \int_\varepsilon^h d\theta \int_{\mathbb{R}^3} D^{\alpha^j} f(t)\, D^m \hat{G}_{1,i}(t-x,\theta)\, dt.$$

Applying Young's inequality to the right-hand side of this equality we have

$$\|D^m f_h(x) - D^m f_\varepsilon(x)\|_{L_q(\mathbb{R}^3)} \le \sum_{i=1}^{4} \int_\varepsilon^h d\theta \left\|D^{\alpha^j} f\right\|_{L_p(\mathbb{R}^3)} \left\|D^m \hat{G}_{1,j}(\cdot,\theta)\right\|_{L_r(\mathbb{R}^3)}, \tag{5.2}$$

where $\frac{1}{r} = 1 - \frac{1}{p} + \frac{1}{q}$.

Let us estimate $\left\|D^m\hat{G}_{1,j}\left(\mathrm{B}\cdot,\theta\right)\right\|_{L_r(R^3)}$ by applying Lemma 2.1 for the case $\alpha_1<\alpha_2<\alpha_3$, Lemma 2.3 for the case $\alpha_1<\alpha_2=\alpha_3$ and Lemma 2.5 for the case $\alpha_1=\alpha_2=\alpha_3$. Assume that $\alpha_1<\alpha_2<\alpha_3$, the other cases being similar.

$$\left\|D^m\hat{G}_{1,j}\left(\cdot,\theta\right)\right\|_{L_r(\mathbb{R}^3)}\leq\left(a_1(\ln\theta)^2+a_2|\ln\theta|+a_3\right)\theta^{-\max\limits_{i=1,2,3}\left(\left|\mu^i\right|+\left(m,\mu^i\right)\right)}$$

$$\cdot\left(\int\limits_{\mathbb{R}^3}\frac{dt_1dt_2dt_3}{\left(1+\theta^{-N}\left(t_1^{N\alpha_1}t_2^{N\alpha_2}t_3^{N\alpha_3}+t_1^{N\beta_1}t_2^{N\beta_2}+t_3^{N\gamma_1}\right)\right)^r}\right)^{\frac{1}{r}}.$$

By substituting $t=\theta^{\mu_1}\tau$ and applying Lemma 2.2 we conclude that

$$\left\|D^m\hat{G}_{1,j}\left(\mathrm{B}\cdot,\theta\right)\right\|_{L_r(\mathbb{R}^3)}\leq\theta^{-\chi}\left(a_1(\ln\theta)^2+a_2|\ln\theta|+a_3\right).$$

Thus,

$$\begin{aligned}\|D^mf_h\left(x\right)-D^mf_\varepsilon\left(x\right)\|_{L_q(\mathbb{R}^3)}\leq\int_\varepsilon^h\theta^{-\chi}\left(a_1(\ln\theta)^2+a_2|\ln\theta|+a_3\right)d\theta\cdot\\ \cdot\sum_{j=1}^4\left\|D^{\alpha^j}f\right\|_{L_p(\mathbb{R}^3)}\leq h^{1-\chi}\left(a_1(\ln h)^2+a_2|\ln h|+a_3\right)\sum_{j=1}^4\left\|D^{\alpha^j}f\right\|_{L_p(\mathbb{R}^3)}.\end{aligned}\tag{5.3}$$

We carried out integration with respect to θ and denoted the constants again by a_i ($i=1,2,3$).

Inequality (5.3) implies that D^mf_h is a Cauchy family in $L_q\left(\mathbb{R}^3\right)$ as $h\to0$. f_ε converges to f in the norm of $L_p(1\leq p<\infty)$ as $\varepsilon{\to}0$ (by Lemma 3.2). Taking into account the properties of the weak derivatives (see Lemma 6.2 of [2]), it follows that the weak derivative D^mf exists, $D^mf{\in}L_q\left(\mathbb{R}^3\right)$ and $\|D^mf-D^mf_\varepsilon\|_{L_q(\mathbb{R}^3)}\to0$ as $\varepsilon{\to}0$.

Thus, applying inequality (5.3) to $\|D^mf\|_{L_q(R^3)}$ we have

$$\begin{aligned}\|D^mf\|_{L_q(\mathbb{R}^3)}\leq\|D^mf_h\|_{L_q(\mathbb{R}^3)}+\|D^mf-D^mf_h\|_{L_q(\mathbb{R}^3)}\leq\\ \leq\|D^mf_h\|_{L_q(\mathbb{R}^3)}+h^{1-\chi}\left(a_1(\ln h)^2+a_2\left(\ln|h|\right)+a_3\right)\cdot\sum_{i=1}^4\left\|D^{\alpha^j}f\right\|_{L_p(\mathbb{R}^3)}.\end{aligned}\tag{5.4}$$

Now we only need to estimate $\|D^mf_h\|_{L_q(R^3)}$. Putting together the integral representation (3.1), the properties of $\hat{G}_0\left(t,\theta\right)$ (see Lemma 2.1), and Young's inequality we conclude, that

$$\|D^mf_h\|_{L_q(\mathbb{R}^3)}\leq C\|f\|_{L_p(\mathbb{R}^3)}\left\|\hat{G}_0\left(\cdot,\theta\right)\right\|_{L_r(\mathbb{R}^3)},$$

where $C>0$ is independent of f and h.

By substituting $t=\theta^{\mu_1}\eta$ and by applying the inequality (2.4), we get

$$\|D^mf_h\|_{L_q(\mathbb{R}^3)}$$

$$\leq h^{-\max\limits_{i=1,2,3}\left(|\mu^i|+(m,\mu^i)\right)+\frac{|\mu^1|}{r}}\left(b_1(\ln h)^2+b_2\left(|\ln h|\right)+b_3\right)$$

$$\cdot\left(\int\limits_{\mathbb{R}^3}\frac{d\eta_1 d\eta_2 d\eta_3}{\left(1+\eta_1^{N\alpha_1}\eta_2^{N\alpha_2}\eta_3^{N\alpha_3}+\eta_1^{N\beta_1}\eta_2^{N\beta_2}+\eta_3^{N\gamma_1}\right)^r}\right)^{\frac{1}{r}}\|f\|_{L_p(\mathbb{R}^3)}$$

$$\leq h^{-\chi}(b_1(\ln h)^2+b_2\left(|\ln h|\right)+b_3)\|f\|_{L_p(\mathbb{R}^3)}.$$

□

In particular, if $q=+\infty$ we obtain the embedding $D^\alpha W_p^{\mathfrak{N}}(\mathbb{R}^3)\hookrightarrow \mathrm{C}(\mathbb{R}^3)$.

Theorem 5.2. *Let* $1\leq p<\infty$, $m=(m_1,m_2,m_3)$ *be a multi-index, and* $\alpha_1\leq\alpha_2<\alpha_3$ *or* $\alpha_1<\alpha_2=\alpha_3$. *Let* $\chi=\max\limits_{i=1,2,3}\left(|\mu^i|+(m,\mu^i)\right)-|\mu^1|\left(1-\frac{1}{p}\right)$.

If $\chi<1$, *then* $D^m W_p^{\mathfrak{N}}(\mathbb{R}^3)\hookrightarrow C(\mathbb{R}^3)$, *i.e. for any function* $f\in W_p^{\mathfrak{N}}(\mathbb{R}^3)$ *the weak derivative* $D^m f$ *is equal almost everywhere in* $\mathbb{R}^3$ *to a continuous function which we again denote by* $D^m f$, *and the following inequality holds:*

$$\sup_{x\in\mathbb{R}^3}|D^m f(x)|\leq h^{1-\chi}\left(a_1(\ln h)^2+a_2\left(|\ln h|+a_3\right)\right)\sum_{i=1}^{4}\left\|D^{\alpha^i}f\right\|_{L_p(\mathbb{R}^3)}$$

$$+h^{-\chi}\left(b_1(\ln h)^2+b_2\left(|\ln h|\right)+b_3\right)\|f\|_{L_p(\mathbb{R}^3)}.$$

Remark 2. *If* $\alpha_1=\alpha_2=\alpha_3$ *and if there is a summand in (2.11) for which* $\rho_1=\rho_2=\rho_3$, *where* ρ_i *be the exponent of* $\xi_i,(i=1,2,3)$, *then a fourth-degree polynomial in* $|\ln h|$ *appears in inequality (5.1).*

Remark 3. *In (5.1) the logarithm appears only if there is a summand in (2.11) for which* $\frac{\rho_1+1}{\rho_2+1}=\frac{\alpha_1}{\alpha_2}$ *or* $\frac{\rho_1+1}{\rho_3+1}=\frac{\alpha_1}{\alpha_3}$. *In other cases* a_i,b_i $(i=1,2)$ *are zero.*

Remark 4. *As in [4] the presence of the logarithm in (5.1) is natural in the case* $\frac{\rho_1+1}{\rho_2+1}=\frac{\alpha_1}{\alpha_2}$, $\frac{\rho_1+1}{\rho_3+1}=\frac{\alpha_1}{\alpha_3}$.

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