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The Eurasian Mathematical Journal (EMJ)
The Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana
Kazakhstan

KORDAN NAURYZKHANOVICH OSPANOV

(to the 60th birthday)

On 25 September 2015 Kordan Nauryzhanovich Ospanov, professor of the Department "Fundamental Mathematics" of the L.N. Gumilyov Eurasian National University, Doctor of Physical and Mathematical Sciences (2000), a member of the Editorial Board of our journal, celebrated his 60th birthday.

He was born on September 25, 1955, in the village Zhanatalap of the Zhanaarka district of the Karaganda region. In 1976 he graduated from the Kazakh State University, and in 1981 he completed his postgraduate studies at the Abay Kazakh Pedagogical Institute.

Scientific works of K.N. Ospanov are devoted to application of methods of functional analysis to the theory of differential equations. On the basis of a local approach to the resolvent representation he has found weak conditions for the solvability of the singular generalized Cauchy-Riemann system and established coercive estimates for its solution. He has obtained a criterion of the spectrum discreteness for the resolvent of the system and the exact in order estimates of singular values and Kolmogorov widths. He has original research results on the coercive solvability of the quasilinear singular generalized Cauchy-Riemann system and degenerate Beltrami-type system. He has established important smoothness and approximation properties of non strongly elliptic systems. K.N. Ospanov has found separability conditions in Banach spaces for singular linear and quasi-linear second-order differential operators with growing intermediate coefficients and established a criterion for the compactness of its resolvent and finiteness of the resolvent type.

His results have contributed to a significant development of the theory of two-dimensional singular elliptic systems, degenerate differential equations and non strongly elliptic boundary value problems.

K.N. Ospanov has published more than 140 scientific papers. The list of his most important publications one may see on the

http://mmf.enu.kz/images/stories/photo/pasport/fm/ospanov

K.N. Ospanov is an Honoured Worker of Education of the Republic of Kazakhstan, and he was awarded the state grant "The best university teacher".

The Editorial Board of the Eurasian Mathematical Journal is happy to congratulate Kordan Nauryzkhanovich Ospanov on occasion of his 60th birthday, wishes him good health and further productive work in mathematics and mathematical education.

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OPEN NEIGHBOURHOOD COLOURING OF SOME PATH RELATED GRAPHS

N.N. Swamy, B. Sooryanarayana

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Key words: colouring, chromatic number, open neighbourhood, power graph, transformation graph.

AMS Mathematics Subject Classification: 05C15.

Abstract. An open neighbourhood k-colouring of a simple connected undirected graph $G(V, E)$ is a k-colouring $c : V \to \{1, 2, \cdots, k\}$, such that, for every $w \in V$ and for all $u, v \in N(w)$, $c(u) \neq c(v)$. The minimal value of k for which G admits an open neighbourhood k-colouring is called the open neighbourhood chromatic number of G and is denoted by $\chi_{onc}(G)$. In this paper, we obtain the open neighbourhood chromatic number of the line graph and total graph of a path P_n. We also obtain the open neighbourhood chromatic number of two families of graphs which are derived from a path P_n, namely k^{th} power of a path and transformation graph of a path.

1 Introduction

All the graphs considered in this paper are simple, non-trivial and undirected. For standard terminology, we refer to [5] and [11]. A *vertex colouring*, or simply *colouring*, of a graph $G = (V, E)$ is an assignment of colours to the vertices of G. A vertex colouring can be viewed as a function $c : V \to S$, S being a set of colours. A *k-vertex colouring*, or simply *k-colouring*, of G is a surjection $c : V \to \{1, 2, \cdots, k\}$. A *proper vertex colouring* or *proper colouring* of G is an assignment of colours to the vertices of G so that no two adjacent vertices are assigned the same colour. A *k-proper colouring* of G is a surjection $c : V \to \{1, 2, \cdots, k\}$ such that $c(u) \neq c(v)$ if u and v are adjacent in G. In such a case, G is said to be *k-proper colourable.* An *open neighbourhood colouring* [7, 8, 17] of a connected graph $G(V, E)$ is a colouring $c : V \to Z^+$, such that for each $w \in V$ and $\forall u, v \in N(w)$, $c(u) \neq c(v)$. An *open neighbourhood k-colouring* of a graph $G(V, E)$ is a k-colouring $c : V \to \{1, 2, \cdots, k\}$ which satisfies the conditions of an open neighbourhood colouring. The minimal value of k for which G admits an open neighbourhood k-colouring is called the *open neighbourhood chromatic number* of G and is denoted by $\chi_{onc}(G)$.

We extend the above definition to any graph, connected or disconnected, as follows.

Definition 6. If $G = (V, E)$ is any graph, then the *open neighbourhood chromatic number* of G is defined as $max\{\chi_{onc}(H)\}$ where maximum is taken over all components H of G.

We recall some of the definitions and results on the open neighbourhood chromatic number discussed in [7] for further reference.

Theorem 1.1. *For any graph G, $\chi_{onc}(G) \geq \Delta(G)$.*

Theorem 1.2. *If H is a connected subgraph of G, then $\chi_{onc}(H) \leq \chi_{onc}(G)$.*

Theorem 1.3. *Let $G(V,E)$ be a connected graph on $n \geq 3$ vertices. Then $\chi_{onc}(G) = n$ if and only if $N(u) \bigcap N(v) \neq \emptyset$ holds for every pair of vertices $u, v \in V(G)$.*

Theorem 1.4. *For a path P_n, $n \geq 2$,*

$$\chi_{onc}(P_n) = \begin{cases} 1, & \text{if } n = 2, \\ 2, & \text{otherwise.} \end{cases}$$

Theorem 1.5. *For a cycle C_n, $n \geq 3$,*

$$\chi_{onc}(C_n) = \begin{cases} 2, & \text{if } n \equiv 0 \ (mod\ 4), \\ 3, & \text{otherwise.} \end{cases}$$

Definition 7. Given a graph $G(V,E)$ and an integer k with $1 \leq k \leq diam(G)$, the k^{th} power of G is a graph, denoted by G^k, having the vertex set $V(G^k) = V(G)$ with the property that two vertices u and v are adjacent in $V(G^k)$ if and only if $d_G(u,v) \leq k$. In particular, if $k = 1$, then $G^1 = G$ and if $k = diam(G)$, then $G \cong K_{V(G)}$.

As discussed in [19], the transformation graph G^{xyz} with $x, y, z \in \{-,+\}$ is a generalization of the concept of total graph of a graph G and is defined as follows.

Definition 8. Given a graph $G(V,E)$, the *transformation graph* of G is a graph, denoted $G^{xyz} = (V_1, E_1)$, with $V_1 = V \bigcup E$ such that two vertices $u, v \in V_1$ are adjacent in G^{xyz} if one of the following conditions holds:

i) $x = +$ and the vertices $u, v \in V$ are adjacent in G or
$x = -$ and the vertices $u, v \in V$ are non-adjacent in G.

ii) $y = +$ and the edges $u, v \in E$ are adjacent in G or
$y = -$ and the edges $u, v \in E$ are non-adjacent in G.

iii) $z = +$ and the vertex $u \in V$ and the edge $v \in E$ are incident in G or
$z = -$ and the vertex $u \in V$ and the edge $v \in E$ are non-incident in G.

By definition, it follows that for every graph G, there correspond eight transformation graphs namely $G^{+++}, G^{++-}, G^{+-+}, G^{+--}, G^{-++}, G^{-+-}, G^{--+}$ and G^{---}. Each of these eight transformation graphs appear to possess nice properties, some of which are discussed in [20, 16, 4, 1]. In particular, the graph G^{+++} is nothing but the total graph of G and the graph G^{---} is the complement of the graph G^{+++}.

In this paper, we obtain the open neighbourhood chromatic number of the line graph, total graph & k^{th} power of a path and transformation graphs of a path.

2 Open neighbourhood colouring of the line graph, total graph and k^{th} power of a path

In this section, we obtain the open neighbourhood chromatic number of the line graph and total graph of a path. We also obtain the open neighbourhood chromatic number of the graph P_n^k, the k^{th} power of the path $P_n, n \geq 2, 1 \leq k \leq diam(P_n) = n-1$.

Theorem 2.1. *For the line graph $L(P_n)$ of the path P_n on n vertices, $n \geq 3$,*

$$\chi_{onc}(L(P_n)) = \begin{cases} 1, & \text{if } n = 3, \\ 2, & \text{otherwise.} \end{cases}$$

Proof. Since the line graph $L(P_n)$ is isomorphic to the path P_{n-1}, $\chi_{onc}(L(P_n)) = \chi_{onc}(P_{n-1})$.
The result then follows by Theorem 1.4. □

Theorem 2.2. *For the total graph $T(P_n)$ of the path P_n on n vertices, $n \geq 3$, $\chi_{onc}(T(P_n)) = 5$.*

Proof. For $n = 3$, it is easy to observe that every two vertices in $T(P_3)$ are connected by a path of length two. Hence, in any open neighbourhood colouring of $T(P_3)$, every vertex must be given a different colour so that $\chi_{onc}(T(P_3)) = 5$.

We now consider the case $n \geq 4$.

Consider the path P_n on $n \geq 4$ vertices. Without loss of generality, let $V(P_n) = \{v_1, v_2, \cdots, v_n\}$ & $E(P_n) = \{e_1 = v_1v_2, e_2 = v_2v_3, \cdots, e_{n-1} = v_{n-1}v_n\}$.

Then, the total graph $T(P_n)$ has $V(T(P_n)) = \{v_1, v_2, \cdots, v_n, e_1, e_2, \cdots, e_{n-1}\}$.

Since $T(P_3)$ is a subgraph of $T(P_n)$, we have $\chi_{onc}(T(P_n)) \geq \chi_{onc}(T(P_3)) = 5$.

For each i, $1 \leq i \leq 5$, we define the set $S_i = \{v_j | \ j \equiv i \pmod 5\} \bigcup \{e_k | \ \ k+3 \equiv i \pmod 5\}$. In each S_i, every vertex is at a distance of at least 3 from every other vertex. Thus, each $S_i, 1 \leq i \leq 5$, is a P_3-independent set of $T(P_n)$.

Now, define a colouring $c : V(T(P_n)) \to \{1, 2, 3, 4, 5\}$ as $c(S_i) = i$ for each $1 \leq i \leq 5$. Clearly, c is an open neighbourhood colouring of $T(P_n)$ so that $\chi_{onc}(T(P_n)) \leq 5$.

Therefore, $\chi_{onc}(T(P_n)) = 5$. □

Lemma 2.1. *For any $3 \leq n \leq 2k+1, \chi_{onc}(P_n^k) = n$ where $2 \leq k \leq n-1$.*

Proof. Consider the graph P_n^k with $3 \leq n \leq 2k+1$. It is easy to observe that every two vertices in P_n^k are connected by a path of length two. Thus, in any open neighbourhood colouring of P_n^k, every vertex has to be given a different colour.

Hence, $\chi_{onc}(P_n^k) = n$ if $3 \leq n \leq 2k+1$. □

Theorem 2.3. *For the k^{th} power of a path on n vertices $P_n^k, n \geq 3, 2 \leq k \leq n-1$,*

$$\chi_{onc}(P_n^k) = \begin{cases} n, & \text{if } n \leq 2k+1, \\ 2k+1, & \text{otherwise.} \end{cases}$$

Proof. Consider P_n^k, the k^{th} power of a path P_n, $n \geq 3, k \geq 2$. Let the vertices of P_n^k be $v_1, v_2, \cdots, v_n$.
We prove the result in the two cases as follows.

Case (1) : Suppose $n \leq 2k+1$. By Lemma 2.1, we have $\chi_{onc}(P_n^k) = n$ if $n \leq 2k+1$.

Case (2) : Suppose $n > 2k+1$. Then, the graph P_n^k contains P_{2k+1}^k as its subgraph so that $\chi_{onc}(P_n^k) \geq \chi_{onc}(P_{2k+1}^k) = 2k+1$.

For each $i, 1 \leq i \leq 2k+1$, we define the set $S_i = \{v_j | \ j \equiv i(mod(2k+1))\}$.

As seen from the graph, the distance between any two vertices in each S_i is at least 3. Thus, each S_i is a P_3-independent set of P_n^k.

Now, define a colouring $c : V(P_n^k) \rightarrow \{1, 2, \cdots, 2k+1\}$ as $c(S_i) = i$ for each $1 \leq i \leq 2k+1$. Clearly, c is an open neighbourhood colouring of C_n^k so that $\chi_{onc}(P_n^k) \leq 2k+1$. Hence, $\chi_{onc}(P_n^k) = 2k+1$ if $n > 2k+1$. □

3 Open neighbourhood colouring of transformation graphs of a path

In this section, we obtain the open neighbourhood chromatic number of the eight transformation graphs of the path P_n on n vertices. Let $V(P_n) = \{v_1, v_2, \cdots, v_n\}$ and $E(P_n) = \{e_1 = v_1v_2, e_2 = v_2v_3, \cdots, e_{n-1} = v_{n-1}v_n\}$. Then, the transformation graph $P_n^{xyz}, n \geq 3$ has $V(P_n^{xyz}) = \{v_1, v_2, \cdots, v_n, e_1, e_2, \cdots, e_{n-1}\}$.

Theorem 3.1. *For the transformation graph* $P_n^{+++}, n \geq 3, \chi_{onc}(P_n^{+++}) = 5$.

Proof. As observed earlier, the graph G^{+++} is nothing but the total graph $T(G)$ of G.
Therefore, $\chi_{onc}(P_n^{+++}) = \chi_{onc}(T(P_n)) = 5$. □

Theorem 3.2. *For the transformation graph* $P_n^{++-}, n \geq 3$,

$$\chi_{onc}(P_n^{++-}) = \begin{cases} 3, & \text{if } n = 3, \\ 6, & \text{if } n = 4, \\ 2n-1, & \text{otherwise.} \end{cases}$$

Proof. The result is proved in various cases as follows.

Case (1) : For $n = 3$, the transformation graph $P_3^{++-} \cong C_5$. Hence, it follows by Theorem 1.5 that $\chi_{onc}(P_3^{++-}) = \chi_{onc}(C_5) = 3$.

Case (2) : Suppose $n = 4$. It is easy to observe that in P_4^{++-}, every two vertices except v_2 and v_3 are connected by a path of length two. Thus, it follows that the colouring $c : V(P_4^{++-}) \rightarrow \{1, 2, 3, 4, 5, 6\}$ defined as $c(e_1) = 1, c(e_2) = 2, c(e_3) = 3, c(v_1) = 4, c(v_5) = c(v_6) = 5, c(v_7) = 6$ is an open neighbourhood colouring of P_4^{++-} with minimum colours so that $\chi_{onc}(P_4^{++-}) = 6$.

Case (3) : Suppose $n \geq 5$. It is seen that every two vertices in P_n^{++-} are connected by a path of length two so that each vertex has to be given a different colour in any open neighbourhood colouring of P_n^{++-}. Hence, $\chi_{onc}(P_n^{++-}) = |V(P_n^{++-})| = 2n-1$. □

Theorem 3.3. *For the transformation graph $P_n^{+-+}, n \geq 3$,*

$$\chi_{onc}(P_n^{+-+}) = \begin{cases} 5, & \text{if } n \leq 5, \\ n+1, & \text{if } n = 6,7, \\ n+2, & \text{otherwise.} \end{cases}$$

Proof. **Case (1) :** Suppose $n \leq 5$. Here, we consider the following subcases.

`Subcase (1)`: Suppose $n = 3$. Since every two vertices in P_3^{+-+} are connected by a path of length two, every vertex receives a different colour in any open neighbourhood colouring of P_3^{+-+}. Thus, $\chi_{onc}(P_3^{+-+}) = 5$.

`Subcase (2)`: Suppose $n = 4, 5$. It is easily seen that P_n^{+-+} has P_3^{+-+} as its subgraph so that $\chi_{onc}(P_n^{+-+}) \geq \chi_{onc}(P_3^{+-+}) = 5$. Thus, $\chi_{onc}(P_n^{+-+}) \geq 5$.

The reverse inequality can be easily established by means of an open neighbourhood colouring of P_n^{+-+} using five colours so that $\chi_{onc}(P_n^{+-+}) \leq 5$.

Therefore, $\chi_{onc}(P_n^{+-+}) = 5$ if $n = 4, 5$.

Case (2) : Suppose $n = 6, 7$.

`Subcase (1)`: Suppose $n = 6$. The vertices e_1, e_3 and e_5 in P_6^{+-+} cannot have any other vertex in any P_3-independent set of P_6^{+-+} as there is a path of length two between each of them and every other vertex in P_6^{+-+}. Further, if v_1 is taken in a P_3-independent set, then the same set can contain only one of the vertices v_4, v_5 or v_6 and nothing else. In the same way, a P_3-independent set containing v_2 cannot have any vertex other than one of v_5 or v_6. Likewise, if a P_3-independent set contains v_3, it cannot contain any vertex other than v_6. Proceeding this way, we see that the colouring shown in Fig. 1 is an open neighbourhood colouring of P_6^{+-+} using minimum colours. Thus, we have $\chi_{onc}(P_6^{+-+}) = 7$.

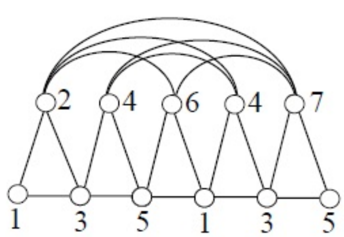

Figure 1: An open neighbourhood colouring of P_6^{+-+}.

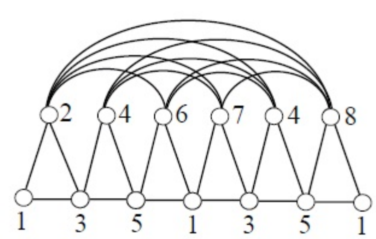

Figure 2: An open neighbourhood colouring of P_7^{+-+}.

`Subcase (2)`: Suppose $n = 7$. The vertices e_1, e_3, e_4 and e_6 in P_7^{+-+} cannot have any other vertex in any P_3-independent set of P_6^{+-+}. Further, if v_1 is taken in a P_3-independent set, then the same set can contain only one of the vertices v_4, v_5 or v_6 and nothing else. In particular, if v_4 is taken in the set, then the set may contain no other vertex other than v_7. In the same way, a P_3-independent set containing v_2 cannot have any vertex other than one of v_4, v_5, v_6 or v_7. Likewise, if a P_3-independent set contains v_3, it cannot contain any vertex other than v_6. Proceeding this way, we see that the colouring shown in Fig. 2 is an open neighbourhood colouring of P_7^{+-+} using minimum colours.

Thus, we have $\chi_{onc}(P_7^{+-+}) = 8$.

Case (3) : Suppose $n \geq 8$. In the transformation graph P_n^{+-+}, not more than one vertex of the form $e_i, 1 \leq i \leq n-1$, can be in one P_3-independent set as each such vertex is connected to every vertex in the graph by a path of length two. Further, it is easy to observe that not more than one vertex in a set of vertices v_i, v_{i+1}, v_{i+2} can be in a P_3-independent set of P_n^{+-+}. Thus, the minimum number of P_3-independent sets in P_n^{+-+} is given by $n+2$, consisting of $n-1$ P_3-independent sets, one each for the vertices $e_i, 1 \leq i \leq n-1$, and three P_3-independent sets S_0, S_1, S_2 given by $S_k = \{v_j |\ \mathrm{j} \equiv \mathrm{k} (\mathrm{mod}\ 3)\}, k = 0, 1, 2$.
Thus, $\chi_{onc}(P_n^{+-+}) \geq n+2$.

To prove the reverse inequality, define a colouring $c : V(P_n^{+-+}) \to \{1, 2, \cdots, n+2\})$ as follows;

$$c(v) = \begin{cases} i, & \text{if } v = e_i \text{ for some } i, \\ n, & \text{if } v = v_i \text{ with } i \equiv 0 (\mathrm{mod}\ 3), \\ n+1, & \text{if } v = v_i \text{ with } i \equiv 1 (\mathrm{mod}\ 3), \\ n+2, & \text{otherwise.} \end{cases}$$

Clearly, c is an open neighbourhood colouring of P_n^{+-+} using $n+2$ colours so that $\chi_{onc}(P_n^{+-+}) \leq n+2$.

Hence, we conclude that $\chi_{onc}(P_n^{+-+}) = n+2$. □

Theorem 3.4. *For the transformation graph $P_n^{+--}, n \geq 3$,*

$$\chi_{onc}(P_n^{+--}) = \begin{cases} 2, & \text{if } n = 3, \\ 4, & \text{if } n = 4, \\ 2n-1, & \text{otherwise.} \end{cases}$$

Proof. We prove the result in various cases as follows.

Case (1) : Suppose $n = 3$. Since the transformation graph $P_3^{+--} \cong P_5$, from Theorem 1.4, we have $\chi_{onc}(P_3^{+--}) = \chi_{onc}(P_5) = 2$.

Case (2) : Suppose $n = 4$. We first prove that in any open neighbourhood colouring of P_4^{+--}, not more than two vertices can receive the same colour. To prove this, we take various subcases as follows.

`Subcase (1)`: Consider any three vertices of the form v_i, v_j, v_k. Since these are the vertices in a path P_4, we cannot give the same colour to all the three vertices.

`Subcase (2)`: Consider the three vertices e_1, e_2, e_3. Since e_2 is connected by a path of length two with e_1 and e_3, it cannot be given the same colour as that of the other two vertices.

`Subcase (3)`: Consider three vertices of the form v_i, v_j, e_k. Suppose v_i and v_j are adjacent, then either of them or both form a path of length two with the vertex e_k. Otherwise, v_i and v_j are connected by a path of length two. In either case, all the three vertices cannot be given the same colour.

`Subcase (4)`: Consider the vertices v_i, e_j, e_k. Suppose e_j and e_k are the consecutive vertices, one of them is connected by a path of length two with v_i. Otherwise, all the three vertices form a path of length two. In either case, all the three vertices cannot be given the same colour.

To conclude, we see that not more than two vertices can be given the same colour in any open neighbourhood colouring of P_4^{+--}. Thus, $\chi_{onc}(P_4^{+--}) \geq 4$.

To prove the reverse inequality, define a colouring c of P_4^{+--} as $c(v_1) = c(e_2) = 1, c(v_2) = c(v_3) = 2, c(e_1) = c(e_3) = 3, c(v_4) = 4$. It is easy to verify that c is an open neighbourhood colouring of P_4^{+--} so that $\chi_{onc}(P_4^{+--}) \leq 4$.

Therefore, $\chi_{onc}(P_4^{+--}) = 4$.

Case (3) : Suppose $n \geq 5$. Since every two vertices in P_n^{+--} are connected by a path of length two, each vertex has to be given a different colour in any open neighbourhood colouring of P_n^{+--}. Hence, $\chi_{onc}(P_n^{+--}) = |V(P_n^{+--})| = 2n - 1$. □

Theorem 3.5. *For the transformation graph $P_n^{-++}, n \geq 3$,*

$$\chi_{onc}(P_n^{-++}) = \begin{cases} 3, & \text{if } n = 3, \\ n+1, & \text{if } n = 4, 5, \\ n+2, & \text{if } n = 6, \\ n+3, & \text{otherwise.} \end{cases}$$

Proof. We prove the result in various cases as follows.

Case (1) : Suppose $n = 3$. Since P_3^{-++} contains a C_3 as its subgraph, we have $\chi_{onc}(P_3^{-++}) \geq \chi_{onc}(C_3) = 3$.

For the reverse inequality, define $c : V(P_3^{-++}) \to \{1, 2, 3\}$ as $c(v_i) = i, i = 1, 2, 3$ and $c(e_1) = 1, c(e_2) = 3$. c can be easily verified to be an open neighbourhood colouring of P_3^{-++} so that $\chi_{onc}(P_3^{-++}) \leq 3$.

Combining the two, we have $\chi_{onc}(P_3^{-++}) = 3$.

Case (2) : Suppose $n = 4$. The graph P_4^{-++} contains P_3^{+-+} as its subgraph so that $\chi_{onc}(P_4^{-++}) \geq \chi_{onc}(P_3^{+-+}) = 5$ by Theorem 3.3.

Consider a colouring of P_4^{-++} as in the Fig. 3. It is easy to verify that this colouring is an open neighbourhood colouring of P_4^{-++} using five colours so that $\chi_{onc}(P_4^{-++}) \leq 5$.

Therefore, $\chi_{onc}(P_4^{-++}) = 5$.

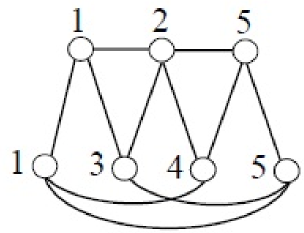

Figure 3: An open neighbourhood colouring of P_4^{-++}.

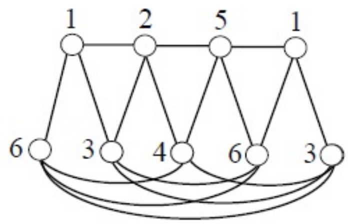

Figure 4: An open neighbourhood colouring of P_5^{-++}.

Case (3) : Suppose $n = 5$.
In P_5^{-++}, each of the vertices v_2, v_3, e_1, e_2 and e_3 have to be given different colours as there is a path of length two connecting every two of them. Further, the vertex v_4 cannot be assigned the colour given to any of the vertices v_3, e_1, e_2 and e_3. Suppose v_4 is assigned a new colour, we have $\chi_{onc}(P_5^{-++}) \geq 6$. Otherwise, v_4 is assigned the colour given to v_2, in which case, the vertex v_5 has to be given a new colour.

Either way, we need a minimum of six colours to have an open neighbourhood colouring of P_5^{-++}.

To prove the reverse inequality, consider a colouring of P_5^{-++} as in Fig. 4. It is evident from the figure that the colouring satisfies the conditions of open neighbourhood colouring.

Therefore, $\chi_{onc}(P_5^{-++}) = 6$.

Case (4) : Suppose $n = 6$. It can be seen that in any open neighbourhood colouring of P_6^{-++}, none of the vertices $v_1, v_2, v_3, v_4, v_6, e_2, e_3$ and e_4 can be given the same colour as each of them is connected with the others by a path of length two. Thus, $\chi_{onc}(P_6^{-++}) \geq 8$.

The reverse inequality can be proved by means of an open neighbourhood colouring $c : V(P_6^{-++}) \to \{1, 2, \cdots, 8\}$ defined by $c(v_i) = i$ for $i = 1, 2, 3, 4, 6$ and $c(v_5) = 2, c(e_1) = c(e_4) = 8, c(e_2) = c(e_5) = 5, c(e_3) = 7$.

We thus conclude that $\chi_{onc}(P_6^{-++}) = 8$.

Case (5) : Suppose $n \geq 7$. In the transformation graph P_n^{-++}, not more than one vertex of the form $v_i, 1 \leq i \leq n$, can be in one P_3-independent set as each such vertex is connected to every vertex in the graph by a path of length two. Further, not more than one vertex in a set of vertices e_i, e_{i+1}, e_{i+2} can be in a P_3-independent set of P_n^{-++}. Thus, the minimum number of P_3-independent sets in P_n^{-++} is given by $n + 3$, consisting of n P_3-independent sets, one each for the vertices $v_i, 1 \leq i \leq n$, and three P_3-independent sets S_0, S_1, S_2 given by $S_k = \{e_j | \text{ j} \equiv \text{k(mod 3)}\}, k = 0, 1, 2$.

Thus, $\chi_{onc}(P_n^{-++}) \geq n+3$.

To prove the reverse inequality, define a colouring $c : V(P_n^{-++}) \rightarrow \{1, 2, \cdots, n+3\}$ as follows;

$$c(v) = \begin{cases} i, & \text{if } v = v_i \text{ for some } i, \\ n+1, & \text{if } v = e_i \text{ with } i \equiv 0(\text{mod } 3), \\ n+2, & \text{if } v = e_i \text{ with } i \equiv 1(\text{mod } 3), \\ n+3, & \text{otherwise.} \end{cases}$$

Clearly, c is an open neighbourhood colouring of P_n^{-++} using $n+3$ colours so that $\chi_{onc}(P_n^{-++}) \leq n+3$.

Therefore, we conclude that $\chi_{onc}(P_n^{-++}) = n+3$. □

Theorem 3.6. *For the transformation graph $P_n^{-+-}, n \geq 3$,*

$$\chi_{onc}(P_n^{-+-}) = \begin{cases} 2, & \text{if } n = 3, \\ 5, & \text{if } n = 4, \\ 8, & \text{if } n = 5, \\ 2n-1, & \text{otherwise.} \end{cases}$$

Proof. We prove the result in various cases as follows.

Case (1) : Suppose $n = 3$. The transformation graph P_3^{-+-} is disconnected. Hence, by Definition 6, $\chi_{onc}(P_3^{-+-}) = \chi_{onc}(C_4) = 2$.

Case (2) : Suppose $n = 4$. The graph P_4^{-+-} contains P_3^{+-+} as its subgraph so that $\chi_{onc}(P_4^{-+-}) \geq \chi_{onc}(P_3^{+-+}) = 5$ from Theorem 3.3.

The reverse inequality can be proved by observing the colouring of P_4^{-+-} as given in Fig. 5. From the figure, it follows that the colouring is an open neighbourhood colouring of P_4^{-+-} so that $\chi_{onc}(P_4^{-+-}) \leq 5$. To conclude, $\chi_{onc}(P_4^{-+-}) = 5$.

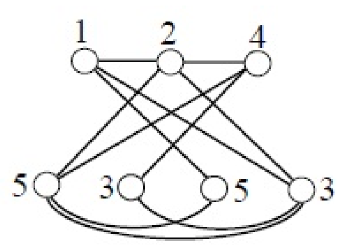

Figure 5: An open neighbourhood colouring of P_4^{-+-}.

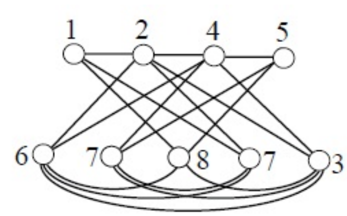

Figure 6: An open neighbourhood colouring of P_5^{-+-}.

Case (3) : Suppose $n = 5$. All the vertices in P_5^{-+-} except v_4 have to be given different colours as each such vertex is connected to every other vertex by a path of length two. This implies that $\chi_{onc}(P_5^{-+-}) \geq 8$.

The reverse inequality follows from the colouring of P_5^{-+-} shown in Fig. 6.

Therefore $\chi_{onc}(P_5^{-+-}) = 8$.

Case (4) : Suppose $n \geq 6$. In the transformation graph P_n^{-+-}, every two vertices are connected by a path of length two so that each vertex has to be given a different colour in any open neighbourhood colouring of P_n^{-+-}.

Hence, $\chi_{onc}(P_n^{-+-}) = |V(P_n^{-+-})| = 2n - 1$. □

Theorem 3.7. *For the transformation graph* $P_n^{--+}, n \geq 3$, $\chi_{onc}(P_n^{--+}) = n$.

Proof. We prove the result in various cases as follows.

Case (1) : Suppose $n = 3$. It is easy to observe that the transformation graph $P_3^{--+} \cong C_5$. Thus, from Theorem 1.5, $\chi_{onc}(P_3^{--+}) = \chi_{onc}(C_5) = 3$.

Case (2) : Suppose $n = 4$. In the transformation graph P_4^{--+}, it can be seen that each of the vertices v_1, v_2 and e_3 has to be given a different colour in any open neighbourhood colouring of P_4^{--+}. Further, out of the remaining four vertices, not more than one vertex can be assigned a colour given to any of these three vertices. Thus, we need at least four colours for an open neighbourhood colouring of P_4^{--+}.

To prove the reverse inequality, consider a colouring of P_4^{--+} as in Fig. 7. It follows from the figure that the colouring satisfies the conditions of open neighbourhood colouring so that $\chi_{onc}(P_4^{--+}) \leq 4$.

Therefore, $\chi_{onc}(P_4^{--+}) = 4$.

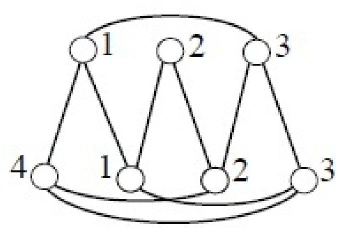

Figure 7: An open neighbourhood colouring of P_4^{--+}.

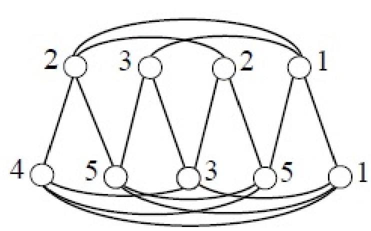

Figure 8: An open neighbourhood colouring of P_5^{--+}.

Case (3) : Suppose $n = 5$. We first prove that in any open neighbourhood colouring of P_5^{--+}, not more than two vertices can receive the same colour. To prove this, we take various subcases as follows.

`Subcase (1)`: Consider any three vertices of the form v_i, v_j, v_k. Since these are the vertices in a path P_4, we cannot give the same colour to all the three vertices.

`Subcase (2)`: Consider any three vertices of the form e_i, e_j, e_k. Once again, these vertices lie in a path P_4 so that the same colour cannot be given to all three of them.

`Subcase (3)`: Consider any three vertices of the form e_i, e_j, v_k. Then, either e_i and e_j are connected by a path of length two or the three vertices are connected by a path P_4. In either case, the same colour cannot be given to all the three vertices.

`Subcase (4)`: Consider any three vertices of the form v_i, v_j, e_k. Here again, either v_i and v_j are connected by a path of length two or the three vertices are connected by a path P_4 so that all the three vertices cannot be given the same colour.

From the above discussion, we conclude that at least five colours are needed for an open neighbourhood colouring of P_5^{--+}.
The colouring of P_5^{--+} shown in Fig. 8 can be easily verified to be an open neighbourhood colouring so that $\chi_{onc}(P_5^{--+}) \leq 5$.

Therefore, $\chi_{onc}(P_5^{--+}) = 5$.

Case (4) : Suppose $n = 6$. In the graph P_5^{--+}, we see that none of the vertices v_1, v_2, v_3, v_4 and v_6 can be given the same colour in any open neighbourhood colouring as every two such vertices are connected by a path of length two. Further, we cannot give the vertex v_5 any of the colours assigned to the vertices v_1, v_3, v_4 or v_6. However, v_5 may or may not be assigned the same colour as that of v_2. Based on this, we consider two subcases as follows.

`Subcase (1)`: Suppose v_5 is given the same colour as v_2. Then, it is evident from the figure that a sixth colour has to be assigned to at least one of the vertices e_1, e_2, e_3, e_4 and e_5 to have an open neighbourhood colouring of P_6^{--+}.

`Subcase (2)`: Suppose v_5 is given a new colour. Then, we need at least six colours to have an open neighbourhood colouring of P_6^{--+}.

In either case, we have $\chi_{onc}(P_6^{--+}) \geq 6$.

To establish the reverse inequality, consider a colouring of P_6^{--+} as $c : V(P_6^{--+}) \to \{1, 2, \cdots, 6\}$ as $c(e_i) = i$ for each $i = 1, 2, 3, 4, 5$ and $c(v_i) = i$ for each $i = 1, 2, \cdots, 6$. Clearly, c is an open neighbourhood colouring of P_6^{--+} so that $\chi_{onc}(P_6^{--+}) \leq 6$.

Therefore, $\chi_{onc}(P_6^{--+}) = 6$.

Case (5) : Suppose $n \geq 7$. Each of the vertices $v_1, v_2, \cdots, v_n$ in P_n^{--+} has to be assigned a different colour in every open neighbourhood colouring of P_n^{--+} as every two of them are connected by a path of length two. Thus, we have $\chi_{onc}(P_n^{--+}) \geq n$.

For the reverse inequality, define a colouring $c : V(P_n^{--+}) \to \{1, 2, \cdots, n\}$ as $c(v) = i$ where $v = v_i$ or e_i for some i.

It is easy to verify that the colouring is an open neighbourhood colouring of P_n^{--+} using n colours so that $\chi_{onc}(P_n^{--+}) \leq n$.

Therefore, $\chi_{onc}(P_n^{--+}) = n$ if $n \geq 7$. □

Theorem 3.8. *For the transformation graph $P_n^{---}, n \geq 3$,*

$$\chi_{onc}(P_n^{---}) = \begin{cases} 2, & \text{if } n = 3, \\ 4, & \text{if } n = 4, \\ 7, & \text{if } n = 5, \\ 2n-1, & \text{otherwise.} \end{cases}$$

Proof. We prove the result in various cases as follows.

Case (1) : Suppose $n = 3$. Since P_3^{---} is disconnected, from Definition 6, it follows that $\chi_{onc}(P_3^{---}) = \chi_{onc}(P_4) = 2$.

Case (2) : Suppose $n = 4$. It is easy to verify that $\Delta(P_4^{---}) = 4$. Hence, by Theorem 1.1, it follows that $\chi_{onc}(P_4^{---}) \geq \Delta(P_4^{---}) = 4$.

To prove the reverse inequality, consider the colouring of P_4^{---} as in Fig. 9.

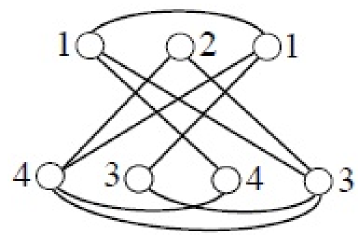

Figure 9: An open neighbourhood colouring of P_4^{---}.

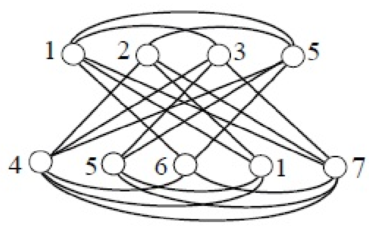

Figure 10: An open neighbourhood colouring of P_5^{---}.

Easily, the above colouring can be verified to be an open neighbourhood colouring of P_4^{---} so that $\chi_{onc}(P_4^{---}) \leq 4$.

Therefore, $\chi_{onc}(P_4^{---}) = 4$.

Case (3) : Suppose $n = 5$. In the transformation graph P_5^{---}, each of the seven vertices $e_1, e_2, e_3, v_1, v_2, v_3$ and v_5 has to be given a different colour in any open neighbourhood colouring P_5^{---} as every two of these vertices are connected by a path of length two so that $\chi_{onc}(P_5^{---}) \geq 7$.

The reverse inequality can be easily established by observing that the colouring in Fig. 10 is an open neighbourhood colouring of P_5^{---}.

Therefore, $\chi_{onc}(P_5^{---}) = 7$.

Case (4) : Suppose $n \geq 6$. Since every two vertices of P_n^{---} are connected by a path of length two, each vertex has to be given a different colour in any open neighbourhood colouring of P_n^{---}. Hence, we conclude that $\chi_{onc}(P_n^{---}) = |V(P_n^{---})| = 2n - 1$ if $n \geq 6$. □

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Narahari Narasimha Swamy
Department of Mathematics
University College of Science
Tumkur University
Tumakuru, Karnataka 572103, India
E-mail: narahari_nittur@yahoo.com

Badekara Sooryanarayana
Department of Mathematical and Computational Studies
Dr. Ambedkar Institute of Technology
Bengaluru, Karnataka 560056, India
E-mail: dr_bsnrao@dr-ait.org

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