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The Eurasian Mathematical Journal (EMJ)
The Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana
Kazakhstan

**KORDAN NAURYZKHANOVICH OSPANOV**

(to the 60th birthday)

On 25 September 2015 Kordan Nauryzhanovich Ospanov, professor of the Department "Fundamental Mathematics" of the L.N. Gumilyov Eurasian National University, Doctor of Physical and Mathematical Sciences (2000), a member of the Editorial Board of our journal, celebrated his 60th birthday.

He was born on September 25, 1955, in the village Zhanatalap of the Zhanaarka district of the Karaganda region. In 1976 he graduated from the Kazakh State University, and in 1981 he completed his postgraduate studies at the Abay Kazakh Pedagogical Institute.

Scientific works of K.N. Ospanov are devoted to application of methods of functional analysis to the theory of differential equations. On the basis of a local approach to the resolvent representation he has found weak conditions for the solvability of the singular generalized Cauchy-Riemann system and established coercive estimates for its solution. He has obtained a criterion of the spectrum discreteness for the resolvent of the system and the exact in order estimates of singular values and Kolmogorov widths. He has original research results on the coercive solvability of the quasilinear singular generalized Cauchy-Riemann system and degenerate Beltrami-type system. He has established important smoothness and approximation properties of non strongly elliptic systems. K.N. Ospanov has found separability conditions in Banach spaces for singular linear and quasi-linear second-order differential operators with growing intermediate coefficients and established a criterion for the compactness of its resolvent and finiteness of the resolvent type.

His results have contributed to a significant development of the theory of two-dimensional singular elliptic systems, degenerate differential equations and non strongly elliptic boundary value problems.

K.N. Ospanov has published more than 140 scientific papers. The list of his most important publications one may see on the

http://mmf.enu.kz/images/stories/photo/pasport/fm/ospanov

K.N. Ospanov is an Honoured Worker of Education of the Republic of Kazakhstan, and he was awarded the state grant "The best university teacher".

The Editorial Board of the Eurasian Mathematical Journal is happy to congratulate Kordan Nauryzkhanovich Ospanov on occasion of his 60th birthday, wishes him good health and further productive work in mathematics and mathematical education.

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# ON MONOTONICITY OF SOLUTIONS OF DIRICHLET PROBLEM FOR SOME QUASILINEAR ELLIPTIC EQUATIONS IN HALF-SPACES

**O.A. Salieva**

Communicated by V.I. Burenkov

**Key words:** quasilinear equations, monotonicity, nonexistence of solutions.

**AMS Mathematics Subject Classification:** 35J60, 35J70, 35J92.

**Abstract.** We prove the monotonicity of nonnegative bounded solutions to the Dirichlet problem for a quasilinear elliptic equation of the form $-\Delta_p u = f(u)g(x_n)$ with $p \geq 3$ in a half-space, where $x_n$ is the normal coordinate of the argument. This assertion implies new nonexistence results for the case $f(u) = u^q$ with the appropriate values of $q$.

## 1 Introduction

Let $n \geq 2$. Denote $\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) : x_n > 0\}$, $\partial\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) : x_n = 0\}$.

Consider the following Dirichlet problem:

$$\begin{cases} -\Delta_p u = f(u)g(x_n) & (x \in \mathbb{R}^n_+), \\ u = 0 & (x \in \partial\mathbb{R}^n_+), \end{cases} \tag{1.1}$$

where $\Delta_p u := \operatorname{div}(|Du|^{p-2}Du)$ is the $p$-Laplacian, and $f$ is a nonnegative function satisfying the following condition:

$f \in C^1(\mathbb{R}_+ \cup \{0\}) \cap C^2(\mathbb{R}_+)$ and for any $M > 0$ there exist $a = a(M) > 0$ and $A = A(M) > 0$ such that

$$as^q \leq f(s) \leq As^q \text{ and } |f'(s)| \leq As^{q-1} \text{ in } [0, M] \tag{$F_1$}$$

for some $q > p - 1$.

A typical example of a function $f$ satisfying ($F_1$) is $f(u) = u^q$ with $q > p - 1$.

As for $g(x_n)$, we suppose that it is a nonnegative non-decreasing continuous function such that

$$g(tx_n) \leq ct^\alpha g(x_n) \tag{$G_1$}$$

for all (at least sufficiently small) $t > 0$ and $x_n > 0$ with some constants $c > 0$ and $\alpha > -p$ independent of $t$. Further, in Theorem 1.2, we require in addition that there exists the limit

$$g_0 := \lim_{x_n \to 0} g(x_n) \in (0, +\infty). \tag{$G_2$}$$

As an example one can take $g(x_n) = \dfrac{x_n^\alpha}{1+x_n^\alpha}$ with $\alpha \geq 0$.

We study the problems of existence and monotonicity of nonnegative bounded solutions of problem (1.1) in the case $p \geq 3$.

For $1 < p < 3$ and $g(x_n) \equiv 1$ the problem was studied by a number of authors. In particular, the semilinear case (i.e., $p = 2$) was considered in the pioneering papers of H. Berestycki, L. Caffarelli, and L. Nirenberg [2], [3] for $n = 2$ and in those of E.N. Dancer (see [8]) for general $n$. For $p \neq 2$, the operator $\Delta_p$ becomes nonlinear and either singular (for $1 < p < 2$) or degenerate (for $p > 2$), which leads to various additional difficulties. The monotonicity of nonnegative bounded solutions to (1.1) with general $n$ and $g(x_n) \equiv 1$ was shown by A. Farina, L. Montoro, and B. Sciunzi in [11] for $\dfrac{2n+2}{n+2} < p < 2$ and in [12] for $2 < p < 3$ (see also the development of this approach in [10]). We also mention the paper of H. Zou [21] where a nonexistence result for problem (1.1) is obtained in the functional class $W^{1,p}(\mathbb{R}^n_+) \cap L^q_{\mathrm{loc}}(\mathbb{R}^n_+)$ under more restrictive assumptions on $f$. A detailed treatment of the case $q < p-1$ was given recently by E.N. Dancer, Y. Du, and M. Efendiev in [9]. For problems with singular coefficients, nonexistence results were obtained by M.F. Bidaut-Véron and S.I. Pohozaev in [4] and by E.I. Galakhov and the author in [14].

The case $p \geq 3$ and $g(x_n) \equiv 1$ was considered in another recent paper of E.I. Galakhov and the author [15]. Here we extend the methods of this paper to more general functions $g$.

The main statements of the present paper are as follows.

**Theorem 1.1.** *Let $p > 2$. Assume that $f$ satisfies $(F_1)$ and $g$ satisfies $(G_1)$. Then any nonnegative nontrivial solution $u(x', x_n) \in W^{1,p}(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$ of problem (1.1) is strictly monotonically increasing in $x_n$ on $\mathbb{R}_+$ for each fixed $x' \in \mathbb{R}^{n-1}$.*

**Theorem 1.2.** *Problem (1.1) with $f(u) = u^q$ and $g$ satisfying $(G_1)$ and $(G_2)$ has no nontrivial nonnegative bounded solutions if $p > 2$ and either*

$$n - 1 \leq \frac{p(p+3)}{p-1}, \quad q > p - 1$$

*or*

$$n - 1 > \frac{p(p+3)}{p-1}, \quad p - 1 < q < q_c(n-1, p),$$

*where*

$$q_c(n,p) = \frac{[(p-1)n - p]^2 + p^2(p-2) - p^2(p-1)n + 2p^2\sqrt{(p-1)(n-1)}}{(n-p)((p-1)n - p(p+3))}.$$

Note that for $g(x_n) \equiv 1$ these results coincide with those of [15].

The method of proof is based on certain rescaling techniques and on the weak comparison principle in unbounded narrow domains, similarly to [13], [11], [12]. This allows us to apply the moving plane method due to A.D. Alexandrov and J. Serrin [1], [18].

We note that for $p > 2$ the comparison principle can be derived from the weighted Poincaré inequality proven in [5], but one must cope with the possible blow-up of the constant in this inequality as the functions under comparison (i.e., the solution $u$ and its reflection with respect to a moving plane) approach zero. In [12], this blow-up is balanced by the power-like decay of $f$, but such an argument works only for $2 < p < 3$. Therefore we modify the approach used in [12] as follows:

1. First, we use a scaling argument to show that $u$ is monotonic in some layer near the boundary. In fact, the boundary maximum principle due to J.L. Vázquez [20] implies that for each $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$ there exists a constant $\delta(x') > 0$ such that the function $u(x', \cdot)$ is strictly increasing in $x_n$ for all $x_n \in (0, \delta(x'))$. We show that such $\delta > 0$ can be chosen independently of $x'$. Hence the moving plane process can get started.

2. In order to be able to continue this process for any $x_n$, we formulate the comparison principle for the solution $u$ and its reflection $u_\lambda$ with respect to a moving plane $x_n = \lambda$ rather than for generic $u$ and $v$. This allows us to use the specific properties of $u_\lambda$.

3. In the proof of the comparison principle, we estimate $(u - u_\lambda)_+$ on a narrow unbounded set $A_{\lambda,\delta}$ such that $u(x) \le u_\lambda(x)$ on its boundary, as in [11] and [12]. By another scaling argument, we show that $u$ and $u_\lambda$ are uniformly bounded away from zero on the set $\mathrm{supp}(u-u_\lambda)_+ \cap A_{\lambda,\delta}$. Therefore we can apply to $u-u_\lambda$ the weighted Poincaré inequality with a uniformly bounded constant in any set from a sufficiently fine covering of $A_{\lambda,\delta}$, which implies that $(u - u_\lambda)_+ \equiv 0$, that is, $u(x) \le u_\lambda(x)$ in $A_{\lambda,\delta}$, and hence the moving plane process can be continued. Thus we obtain the monotonicity of $u(x', x_n)$ in $x_n \in \mathbb{R}_+$ for any $x' \in \mathbb{R}^{n-1}$.

Since monotonic solutions of (1.1) are necessarily stable, this implies nonexistence of positive bounded solutions to (1.1) with $f(u) = u^q$ for any $q \in (p-1, q_{\mathrm{cr}}(n,p))$, where $q_{\mathrm{cr}}(n,p)$ is the critical exponent for stable solutions of (1.1) obtained in [6]. By passing to the limit as $x_n \to \infty$, we can extend this result to the case $q_{\mathrm{cr}}(n,p) \le q < q_{\mathrm{cr}}(n-1,p)$.

The rest of the paper consists of three sections. Section 2 contains some known results, which we will use in the sequel. In Section 3, we formulate our preliminary results, namely, lower estimates on $\delta$ and $u$ based on scaling arguments. In Section 4, we prove Theorems 1.1 and 1.2.

## 2 Some known results

In the sequel we will make use of the following results in [12]. Let $\Omega$ be an open set in $\mathbb{R}^n$. For a nonnegative weight function $\rho \in L^1(\Omega)$, we denote

$$L^2_\rho(\Omega) = \{u : \Omega \to \mathbb{R}, \int\limits_\Omega \rho u^2 \, dx < \infty\},$$

$$H^{1,2}_\rho(\Omega) = \{u : \Omega \to \mathbb{R}, \|u\|^2_{H^{1,2}_\rho(\Omega)} = \int\limits_\Omega \rho(u^2 + |\nabla u|^2) \, dx < \infty\},$$

$H^{1,2}_{0,\rho}(\Omega)$ - the closure of $C^\infty_0(\Omega)$ with respect to the norm of $H^{1,2}_\rho(\Omega)$.

**Theorem 2.1.** *Let $C^* > 0$. Assume that $\rho$ is a weight function such that*

$$\int\limits_{\Omega} \frac{dy}{\rho^t |x-y|^\gamma} \leq C^*, \tag{2.1}$$

*with $t = \frac{p-1}{p-2} r$, $\frac{p-1}{p-2} < r < 1$, and $n - 2t < \gamma < n-2$ if $n \geq 3$ (or $\gamma = 0$ if $n = 2$).*

*Then, for any $w \in H^{1,2}_{0,\rho}(\Omega) \cap W^{1,1}_0(\Omega)$,*

$$\|w\|_{L^q(\Omega)} \leq C_\rho \|\nabla w\|_{L^2_\rho(\Omega)} \tag{2.2}$$

*for any $1 \leq q < 2^*(t)$, where*

$$\frac{1}{2^*(t)} = \frac{1}{2} - \frac{1}{n} + \frac{1}{t}\left(\frac{1}{2} - \frac{\gamma}{2n}\right), \tag{2.3}$$

$$C_\rho = \hat{C}(C^*)^{\frac{1}{2t}} (C_\eta)^{1-\frac{1}{2t}}, \tag{2.4}$$

*$\hat{C} = \frac{1}{n\omega_n}$, where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$, and*

$$C_\eta = \left(\frac{1-\eta}{\frac{\alpha}{n} - \eta}\right)^{1-\eta} \omega_n^{1-\frac{\alpha}{n}} |\Omega|^{\frac{\alpha}{n} - \eta}$$

*with*

$$\eta = 1 - \frac{1}{2t} - \frac{1}{q}$$

*and*

$$\alpha = n - \left(n - 1 - \frac{\gamma}{2t}\right)\frac{2t}{2t-1}.$$

*Proof.* See Theorem 5.1, [12], with $\Omega_1 = \Omega$ and $\Omega_2 = \varnothing$. □

**Corollary 2.1.** *Assume that $w \in H^{1,2}_{0,\rho}(\Omega)$. If the weight $\rho$ fulfils (2.1), then*

$$\|w\|_{L^2(\Omega)} \leq K_p C_\rho |\nabla w\|_{L^2_\rho(\Omega)},$$

*and there exists $0 < \theta < 1$ such that*

$$K_p = C|\Omega|^{\frac{\theta}{(p-1)n}} \tag{2.5}$$

*and $C > 0$ depends only on the numerical parameters.*

*Proof.* See Corollary 5.3, [12], with $\Omega_1 = \Omega$ and $\Omega_2 = \varnothing$. □

**Proposition 2.1.** *Let $u \in C^1(\mathbb{R}^n_+)$ be a solution to problem (1.1) with $p > 1$. Assume that $f$ satisfies $(F_1)$ and that $|\nabla u|$ is uniformly bounded on $\mathbb{R}^n_+$. Let $\Omega' \subset\subset \mathbb{R}^n_+$ and $0 < \delta < \operatorname{dist}(\Omega', \partial\mathbb{R}^n_+)$ be such that $f(u(x)) > 0$ in $\overline{\Omega'_\delta}$, where*

$$\Omega'_\delta = \{x \in \mathbb{R}^n_+ : d(x, \Omega') < \delta\} \subset\subset \mathbb{R}^n_+.$$

*Consider a finite covering $\Omega' \subset \bigcup_{i=1}^{S} B_\delta(x^i)$ with $x^i \in \Omega'$ and $S = S(\delta)$. Then there exists $C(n) > 0$ such that*

$$\int\limits_{\Omega'} \frac{dy}{|\nabla u(y)|^\tau |x-y|^\gamma} \leq C(n) \frac{S(\delta)}{a^2\delta^2 (\inf\limits_{x\in\Omega'_\delta} u(x))^{2q}} \tag{2.6}$$

*where $\gamma$ is as in Theorem 2.1, $a = a(M)$ with $M = \sup\limits_{x\in\Omega'_\delta} u(x)$ as in $(F_1)$, and*

$$\max\{p-2, 0\} \leq \tau \leq p-1.$$

*Proof.* See Proposition 4.2 and Corollary 4.3, [12], and references therein. □

**Remark 6.** In particular, if in Proposition 2.1 $\Omega'$ is a cube of diameter $d$, then (2.6) holds with

$$S(\delta) = \left(\left[\frac{d}{\delta}\right] + 1\right)^n$$

(see Remark 4.4, [12]).

## 3 Preliminary results

For our auxiliary results, we will need the condition

$$(F) \qquad \sup_{x\in\mathbb{R}^n_+} \frac{f(u(x))}{u^{p-1}(x)} < \infty$$

or a stronger one

$$(F') \qquad \lim_{u\to 0^+} \frac{f(u)}{u^{p-1}} = 0.$$

For $x' \in \mathbb{R}^{n-1}$, we denote

$$z(x') = \inf\left\{x_n > 0 : \frac{\partial u(x', x_n)}{\partial x_n} = 0\right\}. \tag{3.1}$$

**Remark 7.** One necessarily has $z(x') > 0$ for any $x' \in \mathbb{R}^{n-1}$ due to the boundary maximum principle for quasi-linear operators [20].

**Lemma 3.1.** *Let $p > 1$. Suppose that $f$ satisfies $(F_1)$, $g$ satisfies $(G_1)$, $u \in W^{1,p}(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$ is a nonnegative nontrivial solution of problem (1.1), and the pair $(f, u)$ satisfies $(F)$.*

*Then there exists $\varkappa > 0$ such that $u(x', x_n) \geq u(x', y_n)$ for all $x_n, y_n$ such that*

$$0 \leq y_n \leq x_n < \varkappa.$$

*Proof.* If $\inf\limits_{x'\in\mathbb{R}^{n-1}} z(x') > 0$, the assertion obviously holds, since $u = 0$ on $\partial\mathbb{R}^n_+$ and $u \geq 0$ in $\mathbb{R}^n_+$. Hence we suppose that

$$\inf_{x'\in\mathbb{R}^{n-1}} z(x') = 0. \tag{3.2}$$

By Remark 7, $z(0) > 0$. Define

$$\begin{aligned} &Z_0 = \{x' \in \mathbb{R}^{n-1} : \, z(x') \geq z(0)\}, \\ &Z_k = \{x' \in \mathbb{R}^{n-1} : \, 2^{-k}z(0) \leq z(x') \leq 2^{1-k}z(0)\} \quad (k \in \mathbb{N}). \end{aligned} \tag{3.3}$$

Evidently, $\mathbb{R}^{n-1} = \bigcup\limits_{k=0}^{\infty} Z_k$. Some of the sets $Z_k$ may be empty.

We claim that there exists a sequence $\{x'_m\}$ of points $x'_m \in \mathbb{R}^{n-1}$ with the following properties.

1. For each $m \in \mathbb{N}$, there exists $k_m \in \mathbb{N}$ such that $x'_m \in Z_{k_m}$ and $k_m \to \infty$ as $m \to \infty$.

2. Define $r_m = \operatorname{dist}\left(x'_m, \bigcup\limits_{j=k_m+1}^{\infty} Z_j\right)$. Then

$$\lim_{m\to\infty} \sup 2^{k_m} r_m = \infty. \tag{3.4}$$

In fact, a sequence with property 1 does exist, since otherwise the number of non-empty sets $Z_k$ would be finite, say $K$, and hence

$$\inf_{x'\in\mathbb{R}^{n-1}} z(x') \geq 2^{-K} z(0),$$

which would contradict (3.2).

Now assume that each sequence with property 1 does not satisfy 2, that is, for each such sequence one has

$$\lim_{m\to\infty} \sup 2^{k_m} r_m < \infty. \tag{3.5}$$

We construct a sequence $\{x'_m\}$ such that $x'_1 = 0$ and $x'_{m+1} \in \bigcup\limits_{j=k_m+1}^{\infty} Z_j$ is chosen so that

$$|x'_{m+1} - x'_m| \leq 2\operatorname{dist}\left(x'_m, \bigcup_{j=k_m+1}^{\infty} Z_j\right) = 2r_m.$$

We show that the series $\sum\limits_{m=0}^{\infty} |x'_{m+1} - x'_m|$ necessarily diverges. In fact, otherwise one would have

$$\begin{aligned} |x'_m| \leq |x'_0| + \sum_{j=0}^{m-1} |x'_{j+1} - x'_j| \leq 0 + \sum_{j=0}^{\infty} |x'_{j+1} - x'_j| \leq \\ \leq 2\sum_{j=0}^{\infty} r_j \leq 2c \sum_{j=0}^{\infty} 2^{-k_j} < \infty \quad (m \in \mathbb{N}), \end{aligned}$$

where $c := \limsup_{j\to\infty} 2^{k_j} r_j < \infty$ due to (3.5).

Thus the sequence $\{x'_m\}$ is bounded and therefore converges to a point $x'_* \in \mathbb{R}^{n-1}$ up to a subsequence. Denote $z_m = z(x'_m)$. Since $u \in C^{1,\beta}(\mathbb{R}^n_+)$ and $\dfrac{\partial u(x'_m, z_m)}{\partial x_n} = 0$ by (3.2), for some constant $c > 0$ we have

$$\frac{\partial u(x'_m, 0)}{\partial x_n} = \frac{\partial u(x'_m, 0)}{\partial x_n} - \frac{\partial u(x'_m, z_m)}{\partial x_n} \leq c z_m^\beta \to 0 \text{ as } m \to \infty$$

and by continuity

$$\frac{\partial u(x'_*, 0)}{\partial x_n} = 0,$$

which is impossible due to the boundary maximum principle.

Hence claim (3.4) holds, i. e. there exists a sequence $\{x'_m\}$ with elements in $\mathbb{R}^{n-1}$ such that

$$x'_m \in Z_{k_m}, \quad B_{r_m/2}(x'_m) \subset \bigcup_{j=0}^{k_m} Z_j,$$

that is,

$$z(x') \geq 2^{-k_m} z(0) \geq z(x'_m)/2 = z_m/2 \text{ for all } x' \in B_{r_m/2}(x'_m). \tag{3.6}$$

Now for each $m \in \mathbb{N}$ we introduce a rescaled function

$$u_m(x) = \frac{1}{u(x'_m, z_m)} \cdot u(\tilde{x}_m), \text{ where } \tilde{x}_m = (z_m x_1 + x_1^m, \ldots, z_m x_{n-1} + x_{n-1}^m, z_m x_n).$$

By (1.1) and $(F)$, one has

$$-\Delta_p u_m(x) = -\frac{z_m^p}{u^{p-1}(x'_m, z_m)} \Delta_p u(\tilde{x}_m) = d_m u_m^{p-1}(x) g(x_n), \tag{3.7}$$

where

$$0 \leq d_m = z_m^p \frac{f(u(\tilde{x}_m)) g(z_m x_n)}{u^{p-1}(\tilde{x}_m) g(x_n)} \leq z_m^{p+\alpha} \cdot \sup_{x\in\mathbb{R}^n_+} \frac{f(u(x))}{u^{p-1}(x)}. \tag{3.8}$$

Evidently, one has $u_m(0, \ldots, 0, 1) = 1$ and $\dfrac{\partial u_m(0, \ldots, 0, 1)}{\partial x_n} = 0$. Due to (3.1) and (3.6), $u$ is monotonic in the cylinder

$$Z_m := \left\{ (x', x_n) : \, 0 \leq |x' - x'_m| \leq \frac{r_m}{2}, \, 0 \leq x_n \leq \frac{z_m}{2} \right\},$$

and therefore the rescaled functions $u_m$ are monotonic in the respective cylinders

$$Z'_m := \left\{ (x', x_n) : \, 0 \leq |x'| \leq \frac{r_m}{2 z_m}, \, 0 \leq x_n \leq \frac{1}{2} \right\}.$$

Note that by (3.3) and the definition $z_m = z(x'_m)$ with $x'_m \in Z_{k_m}$ we have

$$2^{-k_m} z(0) \leq z_m \leq 2^{1-k_m} z(0),$$

which by (3.4) implies

$$\lim_{m\to\infty} \frac{r_m}{2z_m} \geq \lim_{m\to\infty} \frac{2^{k_m-1} r_m}{z(0)} = \infty. \tag{3.9}$$

Being a solution of the equation

$$-\Delta_p u_m(x) = d_m(x) u_m^{p-1}(x) g(x_n) \tag{3.10}$$

obtained in (3.7), $u_m$ satisfies the Harnack inequality (see, e.g., Theorem 7.2, [17]), i.e., for some constant $c_1 > 0$, each $\delta > 0$, and each compact set $X \subset \overline{\mathbb{R}^n_+}$ containing $(0, \dots, 0, 1)$ we get

$$\sup_{x\in X_\delta} u_m(x) \leq c_1 \inf_{x\in X_\delta} u_m(x) \leq c_1,$$

where $X_\delta = \{x = (x_1, \dots, x_n) \in X : x_n \geq \delta\}$. Due to the monotonicity of $u_m(x', x_n)$ in $x_n \in [0, 1/2]$ in $Z'_m$, for $X \subset Z'_m$ this implies

$$\sup_{x\in X} u_m(x) \leq c_1.$$

By standard estimates (see [19]) for the solution $u_m$ of (3.10) with $d_m$ satisfying (3.8) we have

$$\|u_m\|_{C^{1,\beta}(X)} \leq c_2(X)$$

with some $\beta > 0$ and $c_2(X) > 0$.

Due to (3.9), the sequence of cylinders $Z'_m$ covers the whole layer $\mathbb{R}^{n-1} \times [0, 1/2]$. Hence, by the Arzela–Ascoli Theorem, there exists a subsequence of $\{u_m\}$ converging to some function

$$u_0 \in C^{1,\beta'}_{\mathrm{loc}}(\mathbb{R}^n_+)$$

with $\beta' > 0$. Moreover, by the fact that $\lim\limits_{m\to\infty} z_m = 0$, (3.7)–(3.8), and the boundedness of $u$ it follows that $u_0$ satisfies

$$\begin{cases} -\Delta_p u_0 = 0 & (x \in \mathbb{R}^n_+), \\ u_0 = 0 & (x \in \partial\mathbb{R}^n_+). \end{cases} \tag{3.11}$$

Hence $u_0(x', x_n) = c_2 x_n$ with some $c_2 \in \mathbb{R}$ by Theorem 3.1 [16]. Furthermore, one has $c_2 = 1$, since $u_0(0, \dots, 0, 1) = \lim\limits_{k\to\infty} u_m(0, \dots, 0, 1) = 1$. On the other hand,

$$\frac{\partial u_0(0, \dots, 0, 1)}{\partial x_n} = \lim_{m\to\infty} \frac{\partial u_m(0, \dots, 0, 1)}{\partial x_n} = 0,$$

which contradicts $u_0(x', x_n) = x_n$. Thus the assumption (3.2) cannot hold. This completes the proof. □

Now define $u_\lambda(x', x_n) = u(x', 2\lambda - x_n)$ for $\lambda > 0$, $x' \in \mathbb{R}^{n-1}$, and $x_n \in [0, \lambda]$.

**Lemma 3.2.** *Let $p > 1$. Suppose that $f$ satisfies $(F')$, $g$ satisfies $(G_1)$ and $(G_2)$, and $u \in W^{1,p}(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$ is a nonnegative nontrivial solution of problem (1.1). Denote*

$$A_\lambda = \{(x', x_n) : \ u_\lambda(x', x_n) < u(x', x_n), \, 0 \le x_n \le \lambda\}.$$

*Then for each $u$ such that $u(x', x_n)$ monotonically increases in $x_n \in [0, y]$ for each fixed $x' \in \mathbb{R}^{n-1}$ and some $y \in (0, \lambda)$ one has*

$$c_\lambda := \inf_{x \in A_\lambda} u(x) > 0. \tag{3.12}$$

*Proof.* Suppose, to the contrary, that $\inf\limits_{x \in A_\lambda} u(x) = 0$. Then there exists a sequence $\{(x'_k, y_k)\}$ such that

$$\lim_{k \to \infty} u(x'_k, y_k) = 0 \tag{3.13}$$

and $(x'_k, y_k) \in A_\lambda$, that is, for some $z(x'_k) \in [0, \lambda)$ one has

$$u_\lambda(x'_k, z(x'_k)) = u(2\lambda - x'_k, z(x'_k)) < u(x'_k, z(x'_k)).$$

Since $u(x'_k, x_n)$ monotonically increases in $x_n \in [0, y]$, for some $t(x'_k) \in [y, z(x'_k))$ one has $\dfrac{\partial u(x'_k, x_n)}{\partial x_n}\Big|_{x_n = t(x'_k)} = 0$, and the sequence $\{t(x'_k)\}$ converges to some $t_0 \in [y, \lambda]$ up to a subsequence. Now introduce the family of functions

$$u_k(x', x_n) = \frac{u(x' + x'_k, x_n)}{u(x'_k, x_n)}.$$

Note that

$$\frac{\partial u_k(0, x_n)}{\partial x_n}\Big|_{x_n = t(x'_k)} = 0. \tag{3.14}$$

Similarly to (3.7), due to (1.1) one has

$$\begin{aligned} -\Delta_p u_k(x) &= -\frac{1}{u^{p-1}(x'_k, x_n)} \Delta_p u(x' + x'_k, x_n) = \\ = \frac{1}{u^{p-1}(x'_k, x_n)} u^{p-1}(x' + x'_k, x_n) &\cdot \frac{f(u(x' + x'_k, x_n)) g(x_n)}{u^{p-1}(x' + x'_k, x_n)} = d_k u_k^{p-1}(x) g(x_n), \end{aligned} \tag{3.15}$$

where $d_k := \dfrac{f(u(x' + x'_k, x_n))}{u^{p-1}(x' + x'_k, x_n)}$ tends to zero as $k \to \infty$ due to (3.13) and $(F')$. Repeating the argument of the proof of Lemma 3.1, we obtain that there exists $u_0(x) := \lim\limits_{k \to \infty} u_k(x) = c x_n$ with $c > 0$. But (3.14) implies

$$\frac{\partial u_0(0, \ldots, 0, x_n)}{\partial x_n}\Big|_{x_n = t_0} = 0.$$

This contradiction proves the claim. $\square$

To prove the central results of the present paper, we will also need the following statement on passing to the limit.

**Lemma 3.3.** *Let a function $u \in C^{1,\beta}_{\mathrm{loc}}(\mathbb{R}^n_+;\mathbb{R}_+)$ with $\beta > 0$ be a weak solution of the equation*

$$-\Delta_p u(x) = u^q(x)g(x_n) \quad (x = (x', x_n) \in \mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \mathbb{R}_+), \tag{3.16}$$

*where $p, q > 1$ and $g$ satisfies $(G_1)$, $(G_2)$. Suppose also that for any $x' \in \mathbb{R}^{n-1}$ there exists a positive bounded limit*

$$U(x') := \lim_{x_n\to\infty} u(x', x_n), \tag{3.17}$$

*where the convergence in (3.17) is uniform in $C^1(G)$ for any compact subset $G \subset \mathbb{R}^{n-1}$. Then $U$ is a weak solution of the equation*

$$-\Delta_p U(x') = g_0 U^q(x') \quad (x' \in \mathbb{R}^{n-1}), \tag{3.18}$$

*where $g_0$ is the limit in hypothesis $(G_2)$.*

*Proof.* The locally uniform convergence of (3.17) in $C^1$ means, in particular, that

$$\begin{aligned} &\frac{\partial u(x', x_n)}{\partial x_i} \to \frac{\partial U(x')}{\partial x_i} \quad (i = 1, \dots, n-1),\\ &\frac{\partial u(x', x_n)}{\partial x_n} \to 0 \end{aligned} \tag{3.19}$$

uniformly on any compact subset $G \subset \mathbb{R}^{n-1}$. Introduce test functions $\varphi \in \mathcal{D}(\mathbb{R}^{n-1})$ and $\psi \in \mathcal{D}(\mathbb{R})$ with $\operatorname{supp}\psi \subset (0,1)$ and $\int\limits_0^1 \psi(x_n)\,dx_n = 1$. Multiplying equation (3.16) by $\psi(x_n - k)\varphi(x')$ $(k \in \mathbb{N})$ and integrating by parts, we get

$$\begin{aligned} &\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^{n-1}} u^q(x', x_n)g(x_n)\psi(x_n - k)\varphi(x')\,dx_n\,dx' =\\ &= \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^{n-1}} |Du(x', x_n)|^{p-2}\frac{\partial u(x', x_n)}{\partial x_n}\cdot\frac{\partial \psi(x_n - k)}{\partial x_n}\,\varphi(x')\,dx_n\,dx' +\\ &+ \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^{n-1}} |Du(x', x_n)|^{p-2}\sum_{i=2}^{n}\frac{\partial u(x', x_n)}{\partial x_i}\cdot\frac{\partial \varphi(x')}{\partial x_i}\,\psi(x_n - k)\,dx_n\,dx'. \end{aligned} \tag{3.20}$$

Change of variables and passing to the limit as $k \to \infty$ result in

$$\begin{aligned} &\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^{n-1}} u^q(x', x_n)g(x_n)\psi(x_n - k)\varphi(x')\,dx_n\,dx' = \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^{n-1}} u^q(x', s + k)g(s + k)\psi(s)\varphi(x')\,ds\,dx' \to\\ &\to g_0\int\limits_{\mathbb{R}}\psi(s)\,ds\int\limits_{\mathbb{R}^{n-1}} U^q(x')\varphi(x')\,dx' = g_0\int\limits_{\mathbb{R}^{n-1}} U^q(x')\varphi(x')\,dx', \end{aligned} \tag{3.21}$$

$$\begin{aligned}
&\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^{n-1}} |Du(x',x_n)|^{p-2}\frac{\partial u(x',x_n)}{\partial x_n}\cdot\frac{\partial\psi(x_n-k)}{\partial x_n}\,\varphi(x')\,dx_n\,dx' = \\
&= \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^{n-1}} |Du(s+k,x')|^{p-2}\frac{\partial u(s+k,x')}{\partial s}\cdot\frac{\partial\psi(s)}{\partial s}\,\varphi(x')\,ds\,dx' \to \\
&\to \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^{n-1}} |DU(x')|^{p-2}\frac{\partial U(x')}{\partial s}\cdot\frac{\partial\psi(s)}{\partial s}\varphi(x')\,ds\,dx' = 0.
\end{aligned}\tag{3.22}$$

Similarly

$$\begin{aligned}
&\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^{n-1}} |Du(x',x_n)|^{p-2}\sum_{i=1}^{n-1}\frac{\partial u(x',x_n)}{\partial x_i}\cdot\frac{\partial\varphi(x')}{\partial x_i}\,\psi(x_n-k)\,dx_n\,dx' = \\
&= \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^{n-1}} |Du(s+k,x')|^{p-2}\sum_{i=1}^{n-1}\frac{\partial u(s+k,x')}{\partial x_i}\cdot\frac{\partial\varphi(x')}{\partial x_i}\,\psi(s)\,ds\,dx' \to \\
&\to \int\limits_{\mathbb{R}}\psi(s)\,ds\int\limits_{\mathbb{R}^{n-1}} |DU(x')|^{p-2}\sum_{i=1}^{n-1}\frac{\partial U(x')}{\partial x_i}\cdot\frac{\partial\varphi(x')}{\partial x_i}\,dx' = \\
&= \int\limits_{\mathbb{R}^{n-1}} |DU(x')|^{p-2}\sum_{i=1}^{n-1}\frac{\partial U(x')}{\partial x_i}\cdot\frac{\partial\varphi(x')}{\partial x_i}\,dx'.
\end{aligned}\tag{3.23}$$

Combining (3.20)–(3.23), we get

$$g_0\int\limits_{\mathbb{R}^{n-1}} U^q(x')\varphi(x')\,dx' = \int\limits_{\mathbb{R}^{n-1}} |DU(x')|^{p-2}(DU,D\varphi)\,dx' \tag{3.24}$$

for any $\varphi\in C_0^1(\mathbb{R}^{n-1})$. Thus $U$ is a positive weak solution of equation (3.18). □

## 4 Proof of Theorems 1.1 and 1.2

*Proof of Theorem 1.1.* Now denote

$$\Lambda = \{\lambda\geq 0:\ u(x)\leq u_\lambda(x) \text{ for all } x'\in\mathbb{R}^{n-1},\ x_n\in[0,\lambda]\}$$

and $\lambda_0=\sup\Lambda$. By Lemma 3.1, $(0,\varkappa/2)\subset\Lambda$ and hence $\lambda_0\geq\varkappa/2>0$. To prove Theorem 1.1, we must show that $\lambda_0=\infty$, that is, $\Lambda=\overline{\mathbb{R}_+}$. Since $\Lambda$ is closed by the continuity of $u$ and non-empty, it suffices to show that it is relatively open in the topology of $\overline{\mathbb{R}_+}$.

Suppose the converse. Similarly to the proof of Theorem 1.1 in [12], we multiply (1.1) and the corresponding equation for $u_\lambda$ by the test function

$$\Psi=(u-u_\lambda)_+^\alpha\varphi_R^2\chi_{\mathbb{R}^{n-1}\times[\delta;+\infty)},$$

where $\alpha>0$ and $\delta>0$ will be specified later. Here $\varphi_R\in C_0^\infty(\mathbb{R}^{n-1})$ is a nonnegative function such that

$$\begin{cases}\varphi_R\equiv 1 & \text{in } B_R(0)\subset\mathbb{R}^{n-1},\\ \varphi_R\equiv 0 & \text{in } \mathbb{R}^{n-1}\setminus B_{2R}(0),\\ |D\varphi_R|\leq cR^{-1} & \text{in } B_{2R}(0)\setminus B_R(0) \text{ with some } c>0.\end{cases}\tag{4.1}$$

After subtraction, for each $\lambda > 0$ we obtain in a standard way (see [12])

$$\begin{aligned}
&\alpha c_1 \int\limits_{\mathcal{C}(2R)} (|Du|^{p-2} + |Du_\lambda|^{p-2})|D(u-u_\lambda)_+|^2 (u-u_\lambda)_+^{\alpha+1}\varphi_R^2\,dx \le \\
&\le c_2 \int\limits_{\mathcal{C}(2R)} (|Du|^{p-2} + |Du_\lambda|^{p-2})|D(u-u_\lambda)_+| \cdot |D\varphi_R^2|(u-u_\lambda)_+^{\alpha}\,dx + \\
&+ \int\limits_{\mathcal{C}(2R)} (f(u)g(x_n) - f(u_\lambda)g(x_{n,\lambda}))(u-u_\lambda)_+^{\alpha}\varphi_R^2\,dx := I_1 + I_2,
\end{aligned} \tag{4.2}$$

where $\mathcal{C}(2R) = \{(x', x_n) : |x'| \le 2R,\ \delta \le x_n \le \lambda\}$, $x_{n,\lambda} = 2\lambda - x_n \ge x_n$ for $0 < x_n \le \lambda$. Setting

$$I_1 := c_2 \int\limits_{\mathcal{C}(2R)} (|Du|^{p-2} + |Du_\lambda|^{p-2})|D(u-u_\lambda)_+| \cdot |D\varphi_R^2|(u-u_\lambda)_+^{\alpha}\,dx \tag{4.3}$$

and

$$I_2 := \int\limits_{\mathcal{C}(2R)} (f(u) - f(u_\lambda))(u-u_\lambda)_+^{\alpha}\varphi_R^2\,dx, \tag{4.4}$$

we can rewrite inequality (4.2) as

$$\alpha c_1 \int\limits_{\mathcal{C}(2R)} (|Du|^{p-2} + |Du_\lambda|^{p-2})|D(u-u_\lambda)_+|^2 (u-u_\lambda)_+^{\alpha+1}\varphi_R^2\,dx \le I_1 + I_2. \tag{4.5}$$

We proceed in two steps:

*Step 1: Estimation of $I_1$.*

We use the argument from [12] (Section 6, step 1), simplifying it by means of Lemma 3.1. Namely, by (4.3), using the parametric Young inequality, we obtain

$$I_1 \le I_1^a + I_1^b, \tag{4.6}$$

where

$$I_1^a := \delta'\hat{C} \int\limits_{\mathcal{C}(2R)} (|Du|^{p-2} + |Du_\lambda|^{p-2})|D(u-u_\lambda)_+|^2 \varphi_R^2 (u-u_\lambda)_+^{\alpha-1}\,dx \tag{4.7}$$

and

$$I_1^b := \hat{C}\delta' \int\limits_{\mathcal{C}(2R)} (|Du|^{p-2} + |Du_\lambda|^{p-2})|D\varphi_R|^2 (u-u_\lambda)_+^{\alpha+1}\,dx. \tag{4.8}$$

For sufficiently small $\delta' > 0$, $I_1^a$ can be estimated from above by the left-hand side of (4.5). It remains to estimate $I_2^b$.

Note that by Proposition 4.2 [11], for any $\varepsilon > 0$, if $\lambda - y < \varepsilon$, one can find $\delta = \delta(\varepsilon) > 0$ such that $\operatorname{supp}(u-u_\lambda)_+ \subset A_{\lambda,\delta}$, where $A_{\lambda,\delta} = \mathbb{R}^{n-1} \times [y-\delta, \lambda]$ and $\delta(\varepsilon) \to 0^+$ as $\varepsilon \to 0^+$. Here the case $\operatorname{supp}(u-u_\lambda)_+ \cap (\mathbb{R}^{n-1} \times [0,\delta]) \ne \varnothing$ is excluded due to the factor $\chi_{\mathbb{R}^{n-1}\times[\delta;+\infty)}$ in the test function, where $\delta = \delta(\varepsilon)$ and $\varepsilon > 0$ is still to be fixed. Thus, if we denote the width of $\operatorname{supp}(u - u_\lambda)_+$ in the direction $x_n$ by $d_\lambda$:

$$d_\lambda := \sup_{(x',z_1)\in\operatorname{supp}(u-u_\lambda)_+} \sup_{(x',z_2)\in\operatorname{supp}(u-u_\lambda)_+} |z_1 - z_2|,$$

we have $d_\lambda \le d(\varepsilon) := \varepsilon + \delta(\varepsilon)$, where $d(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Now consider a family $\{Q_i\}_{i=1}^N$ of $N = N(R)$ disjoint cubes $Q_i$ with edge $d(\varepsilon)$ and with the $x_n$ coordinate of the center, say $y_C$, such that $y_C = \frac{y-\delta(\varepsilon)+\lambda}{2}$ and

$$\mathcal{C}(2R) \subset \bigcup_{i=1} \overline{Q_i}. \tag{4.9}$$

It follows that the diameter of each cube $Q_i$ is

$$\operatorname{diam} Q_i = \sqrt{n}d(\varepsilon), \quad i = 1, \dots, N. \tag{4.10}$$

By (4.1) and (4.8), since $\nabla u, \nabla u_\lambda \in L^\infty(\mathbb{R}^n_+)$, we get

$$I_b^1 \le \sum_{i=1}^N \frac{C}{\delta' R^2} \int\limits_{\mathcal{C}(2R)\cap Q_i} \left([(u-u_\lambda)_+]^{\frac{\alpha+1}{2}}\right)^2 dx. \tag{4.11}$$

Similarly to [12], due to Proposition 2.1 we can use in each cube $Q_i$ the weighted Poincaré inequality of Corollary 2.1 with $\rho \equiv |Du|^{p-2}$ and take advantage of the constant $\hat{C}$ that turns to be independent of the index $i$ in (4.9). Thus we obtain

$$I_b^1 \le \sum_{i=1}^N C_i^* \frac{C}{\delta' R^2} \int\limits_{Q_i} |Du|^{p-2}[(u-u_\lambda)_+]^{\alpha-1}|D(u-v)_+|^2\, dx. \tag{4.12}$$

We estimate the constant $C_i^*$ by Proposition 2.3 and Remark 2.4. Denote

$$C := \sup_{x\in\mathbb{R}^n_+} \frac{|u(x)-u(z)|}{|x-z|},$$

which is finite due to the fact that $u \in L^\infty(\mathbb{R}^n_+)$ and standard elliptic estimates. By Lemma 3.2, we have

$$\inf_{x\in\operatorname{supp}(u-u_\lambda)_+} u(x) = c_\lambda > 0, \tag{4.13}$$

where $c_\lambda$ is the constant defined in (3.12), whence one has

$$\inf_{x\in(Q_i)_\delta} u(x) \ge c_\lambda/2 > 0 \quad (i = 1, \dots, N). \tag{4.14}$$

Moreover, by Lemma 3.1 we have $y \ge \varkappa > 0$ and thus $x_n > \varkappa - \delta > 0$ in $\operatorname{supp}(u-u_\lambda)_+$, whence

$$\operatorname{dist}(Q_i, \{u = 0\}) \ge \varkappa - \delta > 0 \quad (i = 1, \dots, N). \tag{4.15}$$

Combining Proposition 2.1 and Remark 6 with (4.14) and (4.15), we deduce that the constants $C_i^*$ in (4.12) are in fact independent of $i$ and (unlike the argument in [12]) of $R$. Hence,

$$\begin{aligned} I_1 \le \delta' \hat{C} \int\limits_{\mathcal{C}(2R)} (|Du|^{p-2} + |Du_\lambda|^{p-2})|D(u-u_\lambda)_+|^2 \varphi_R^2 (u-u_\lambda)_+^{\alpha-1}\, dx + \\ + \frac{C(d_Q,\delta')}{R^2} \int\limits_{\mathcal{C}(2R)} (|Du|^{p-2} + |Du_\lambda|^{p-2})|D(u-u_\lambda)_+|^2 (u-u_\lambda)_+^{\alpha-1}\, dx, \end{aligned} \tag{4.16}$$

where $\delta' > 0$ can be chosen arbitrarily small and

$$C_1(d_Q,\delta') \to 0 \text{ as } d_Q \to 0.$$

*Step 2. Estimation of $I_2$ in (4.5).*

We can assume that $p \ge 3$ and therefore $q > p-1 \ge 2$, since the case $2 < p < 3$ was considered in [12]. Note that

$$\begin{aligned} f(u)g(x_n) - f(u_\lambda)g(x_{n,\lambda}) &= (f(u) - f(u_\lambda))g(x_n) + f(u_\lambda(g(x_n) - g(x_{n,\lambda})) \\ &\le c(\lambda)(f(u) - f(u_\lambda)) \end{aligned}$$

with some constant $c(\lambda) > 0$ by the assumptions on the continuity and monotonicity of $g(x_n)$. By the Taylor expansion of $f(\cdot)$, we obtain

$$f(u) = f(u_\lambda) + f'(\xi)(u - u_\lambda)$$

with $u_\lambda(x) \le \xi(x) \le u(x)$ for $x \in \operatorname{supp}(u-u_\lambda)_+$.

For $x = (x', y)$ such that $x' \in \mathbb{R}^{n-1}$ and

$$0 < \delta < \min\left\{\varkappa, \frac{c_\lambda}{2C} - (\lambda - y)\right\}$$

with $\varkappa$ defined in Lemma 3.1, the Lipschitz continuity of $u$ and (4.13) imply

$$\inf_{x\in\operatorname{supp}(u-u_\lambda)_+} u_\lambda(x) \ge \frac{c_\lambda}{2} > 0$$

and consequently

$$\xi_0 := \inf_{x\in\operatorname{supp}(u-u_\lambda)_+} \min\{u(x), u_\lambda(x)\} > \frac{c_\lambda}{2} > 0. \tag{4.17}$$

On the other hand,

$$\xi_1 := \sup_{x\in\operatorname{supp}(u-u_\lambda)_+} \max\{u(x), u_\lambda(x)\} \le \|u\|_{L^\infty(\mathbb{R}^n)}^{q-1} < \infty. \tag{4.18}$$

Therefore, similarly to the previous step (see (4.11) and (4.12), we can apply to $I_2$ the weighted Poincaré inequality given in Corollary 2.1:

$$\begin{aligned} I_2 &\le C\|\xi_1\|^{q-1}\sum_{i=1}^{N}\int\limits_{\mathcal{C}(2R)\cap Q_i}[(u-u_\lambda)_+^{\frac{\alpha+1}{2}}]^2dx \le \\ &\le C_3(d_Q)C_4\sum_{i=1}^{N}\int\limits_{\mathcal{C}(2R)\cap Q_i}|Du|^{p-2}|D(u-u_\lambda)_+|^2(u-u_\lambda)_+^{\alpha-1}dx \le \\ &\le C_3(d_Q)C_4(y-\delta,\xi_0)\sum_{i=1}^{N}\int\limits_{\mathcal{C}(2R)\cap Q_i}(|Du|^{p-2}+|Du_\lambda|^{p-2})|D(u-u_\lambda)_+|^2(u-u_\lambda)_+^{\alpha-1}dx, \end{aligned} \tag{4.19}$$

where $C_3(d_Q)\to 0$ as $d_Q\to 0$, and

$$C_4(y-\delta,\xi)\le C(y-\delta)^{[n-2p+(p-1)r-\gamma]\frac{p-2}{(p-1)r}}\sup_{x\in Q_i}\xi^s(x),$$

with $\frac{p-2}{p-1}<r<1$, $s=s(p,q,r)=[2p-2-(p-1)r-2q]\frac{p-2}{(p-1)r}$, and $\gamma=0$ if $n=2$ or any $\gamma>n-2$ if $n\ge 3$. If $s\ge 0$, we have $\sup\limits_{x\in Q_i}\xi^s(x)\le\xi_1^s$ with $\xi_1$ defined in (4.18), while for $s<0$, we obtain $\sup\limits_{x\in Q_i}\xi^s(x)\le\xi_0^s$, where $\xi_0$ is defined in (4.17). In both cases the quantity $C_4(y-\delta,\xi)$ is uniformly bounded by some $C_5(y)$ for small $\delta>0$.

Hence, combining (4.2)–(4.19), we get

$$\begin{aligned} &\alpha c_1\int\limits_{\mathcal{C}(2R)}(|Du|^{p-2}+|Du_\lambda|^{p-2})|D(u-u_\lambda)_+|^2(u-u_\lambda)_+^{\alpha+1}\varphi^2\,dx\le \\ &\le\delta' c_3\int\limits_{\mathcal{C}(2R)}(|Du|^{p-2}+|Du_\lambda|^{p-2})|D(u-u_\lambda)_+|^2\varphi^2(u-u_\lambda)_+^{\alpha-1}\,dx+ \\ &+\frac{C_1(d_Q,\delta')}{R}\int\limits_{\mathcal{C}(2R)}(|Du|^{p-2}+|Du_\lambda|^{p-2})|D(u-u_\lambda)_+|^2(u-u_\lambda)_+^{\alpha-1}\,dx+\frac{C_2(d_Q,\delta')}{R}+ \\ &+C_3(d_Q)C_5(y)\int\limits_{\mathcal{C}(2R)}(|Du|^{p-2}+|Du_\lambda|^{p-2})|D(u-u_\lambda)_+|^2(u-u_\lambda)_+^{\alpha-1}dx. \end{aligned} \tag{4.20}$$

Choosing $\delta'>0$ and $\varepsilon>0$ to be sufficiently small (recall that $0\le d_Q\le\varepsilon+\delta_\varepsilon$, and hence $d_Q\to 0$ as $\varepsilon\to 0$), we can pass to the limit as $R\to\infty$ and complete the proof similarly to that of Theorem 1.1 in [12]. □

*Proof of Theorem 1.2.* Since monotonic solutions of (1.1) are stable (see [12], Theorem 1.4), Theorem 1.1 and Proposition 2.3 [6] imply the assertion of Theorem 1.2 for $p-1<q<q_c(n,p)$. For $q_c(n,p)\le q<q_c(n-1,p)$, by Lemma 3.3 one can reduce the dimension of the problem by one, passing to the limit as $x_n\to+\infty$ and using once more Proposition 2.3 [6], which leads to the same result. □

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Olga Alekseevna Salieva
Department of Applied Mathematics
Moscow State Technological University "Stankin"
3a Vadkovsky pereulok
127993 Moscow, Russia
E-mail: olga.a.salieva@gmail.com

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