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KORDAN NAURYZKHANOVICH OSPANOV

(to the 60th birthday)

On 25 September 2015 Kordan Nauryzhanovich Ospanov, professor of the Department "Fundamental Mathematics" of the L.N. Gumilyov Eurasian National University, Doctor of Physical and Mathematical Sciences (2000), a member of the Editorial Board of our journal, celebrated his 60th birthday.

He was born on September 25, 1955, in the village Zhanatalap of the Zhanaarka district of the Karaganda region. In 1976 he graduated from the Kazakh State University, and in 1981 he completed his postgraduate studies at the Abay Kazakh Pedagogical Institute.

Scientific works of K.N. Ospanov are devoted to application of methods of functional analysis to the theory of differential equations. On the basis of a local approach to the resolvent representation he has found weak conditions for the solvability of the singular generalized Cauchy-Riemann system and established coercive estimates for its solution. He has obtained a criterion of the spectrum discreteness for the resolvent of the system and the exact in order estimates of singular values and Kolmogorov widths. He has original research results on the coercive solvability of the quasilinear singular generalized Cauchy-Riemann system and degenerate Beltrami-type system. He has established important smoothness and approximation properties of non strongly elliptic systems. K.N. Ospanov has found separability conditions in Banach spaces for singular linear and quasi-linear second-order differential operators with growing intermediate coefficients and established a criterion for the compactness of its resolvent and finiteness of the resolvent type.

His results have contributed to a significant development of the theory of two-dimensional singular elliptic systems, degenerate differential equations and non strongly elliptic boundary value problems.

K.N. Ospanov has published more than 140 scientific papers. The list of his most important publications one may see on the

http://mmf.enu.kz/images/stories/photo/pasport/fm/ospanov

K.N. Ospanov is an Honoured Worker of Education of the Republic of Kazakhstan, and he was awarded the state grant "The best university teacher".

The Editorial Board of the Eurasian Mathematical Journal is happy to congratulate Kordan Nauryzkhanovich Ospanov on occasion of his 60th birthday, wishes him good health and further productive work in mathematics and mathematical education.

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INVERTIBILITY OF MULTIVALUED SUBLINEAR OPERATORS

I.V. Orlov, S.I. Smirnova

Communicated by T.V. Tararykova

Key words: sublinear multivalued operators, basis selectors, Hamel basis, extremal points.

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Abstract. We consider the representation of a compact-valued sublinear operator (K–operator) by means of the compact convex packet of single-valued so-called basis selectors. Such representation makes it possible to introduce the concept of an invertible K–operator via invertible selectors. The extremal points of direct and inverse selector representations are described, an analogue of the von Neumann theorem is obtained. A series of examples is considered.

1 Introduction

In the problems of modern nonsmooth analysis and nonsmooth optimization, the multivalued sublinear operators play, as is known, ever more important role (see, e.g., [2],[4],[6]–[7],[9],[10],[17]–[18]). In particular, the concepts of the subdifferential and the subsmoothness, which were researched in the series of our works (see [13]–[16]), are jointly connected to multivalued sublinear operators, that take convex compact values.

The questions on the nonsmooth form of the implicite function and on the inverse function theorems are very actual for any kind of subdifferential calculus and this question was researched long ago in the nonsmooth analysis (see [1],[8],[11]–[12]). However, in order to obtain developed tools of the nonsmooth invertibility, it seems that first an adequate invertibility theory for multivalued operators should be constructed.

The present work represents an outline of such theory. We describe the compact-valued sublinear operators by means of the packets of single-valued so-called basis selectors. This makes it possible to introduce a concept of the multivalued invertibility through the concept of the corresponding selectors. The construction and the properties of invertible multioperators are described explicitly. Special attention is given to the problem of extremal points of selector representation and corresponding application of the Krein–Milman theorem.

2 Preliminaries. Sublinear K–operators and their simplest properties

Definition 1. Let F be a real normed space. Denote by F_K the convex cone consisting of all non-empty convex compacts in F, equipped with element-wise addition, non-negative scalar multiplication and the cone-norm:

$$\|C\| = \sup_{y \in C} \|y\| \quad (C \subset F).$$

This norm generates in F_K a locally convex cone-topology (see [14]) by means of the following ε–neighborhoods:

$$U_\varepsilon = \{C' \,|\, C \subset C' \subset C + D, \text{where}\, \|D\| < \varepsilon\}.$$

Remark 1. *Note that the cone-norm introduced above is coordinated with the inductive order in F_K by embedding and the corresponding cone-topology determines a unilateral cone-uniformity in F_K (see [14]). This makes it possible to introduce the concept of the quasi–completeness (or cone–completeness) in F_K. A quasi–complete normed cone F_K is called a Banach cone.*

Let us mention a useful result (see [14]).

Theorem 2.1. *If F is a real Banach space then the cone F_K is also a Banach one.*

Let us pass to description of sublinear multivalued operators that take compact convex values.

Definition 2. Let E and F be the real normed spaces, $A : E \to F_K$. We say that A is a *sublinear K–operator* if the following properties

(i) $A(h + k) \subset Ah + Ak$ (subadditivity);

(ii) $A(\lambda h) = \lambda \cdot Ah$ $(\lambda \geq 0)$ (positive homogeneity);

are satisfied. The cone–norm for a K–operator is introduced in the standard way:

$$\|A\| = \sup_{\|h\| \leq 1} \|Ah\|.$$

We say that the K–operator A is *bounded* if $\|A\| < \infty$. The normed cone of all bounded K–operators $A : E \to E_K$ is denoted by $L_K(E; F)$.

Remark 2. *Note, first of all, the standard properties of the K–operator norm:*

$$\|A_1 + A_2\| \leq \|A_1\| + \|A_2\|, \ \|\lambda \cdot A\| = \lambda \cdot \|A\| \,(\lambda \geq 0), \ \|Ah\| \leq \|A\| \cdot \|h\|.$$

Next, it is possible to introduce in F_K a multiplication for the negative scalars. In this case, $\|\lambda \cdot A\| = |\lambda| \cdot \|A\|$ $(\lambda \in \mathbb{R})$. Note also that any bounded K–operator is continuous at zero, but, in general, not at an arbitrary point of E.

Let us mention an important statement on quasi-completeness of $L_K(E;F)$ (see [14]).

Theorem 2.2. *If F is a real Banach space then the cone $L_K(E;F)$ is also a Banach one.*

For our purposes it is more appropriate to use *totally homogeneous* K–operators ($A(\lambda h) = \lambda \cdot Ah$ ($\forall\, \lambda \in \mathbb{R}$)). Therefore let us describe briefly a *symmetrization procedure*, which makes it possible to pass to the total homogeneity from the positive one (see [14]).

Definition 3. Let $A \in L_K(E;F)$. We introduce a *symmetrized* K–operator A^s in the following way:

$$A^s h = \overline{\mathrm{co}}((Ah) \cup (-A(-h))) \quad (h \in E),$$

here $\overline{\mathrm{co}}$ is the closed convex hull. Let us keep the previous notation $L_K(E;F)$ for the normed cone of symmetrized K–operators $A^s : E \to F_K$.

Theorem 2.3. *If $A \in L_K(E;F)$ then $A^s \in L_K(E;F)$, moreover*

$$A^s(\lambda h) = \lambda \cdot A^s h \quad (\forall\, \lambda \in \mathbb{R}) \quad \textit{and} \quad \|A^s\| = \|A\|.$$

3 Constructing of the packet of basis selectors for a given K–operator

In what follows, $H = \{h_i\}_{i \in I}$ is a fixed normed Hamel basis in a real Banach space E.

Definition 4. Let $A \in L_K(E;F)$. Choose an arbitrary element $a_i \in Ah_i$ for each $i \in I$ and set

$$A^s h_i = a_i \,(\forall\, i \in I), \quad A^s h = \sum_{k=1}^{n} \lambda_k a_{i_k} \quad \left(h = \sum_{k=1}^{n} \lambda_k h_{i_k} \in E \right).$$

Let us call the set $A_K = \{A^s\}$ the *packet of basis selectors* (or *s–representation*) of a sublinear K–operator A.

Note that s–representation depends on the choice of the Hamel basis H in E. First, we explain that all basis selectors are linear continuous operators.

Theorem 3.1. *Let E and F be the real Banach spaces, H be a Hamel basis in E and $A \in L_K(E;F)$. Then for every selector $A^s \in A_K$ the following properties:*

$$A^s \in L(E;F); \quad \|A\| \le \sup_{A^s \in A_K} \|A^s\| \le C \cdot \|A\|; \tag{3.1}$$

are valid. Here the constant $C = C(H)$ in the right-hand side of (3.1) depends only on the choice of a Hamel basis H in E.

Proof. 1) Let us check first the homogeneity of A^s. Let $h = \sum\limits_{k=1}^{n} \lambda_k h_{i_k} \in E$, $\mu \in \mathbb{R}$. Then:

$$A^s(\mu h) = A^s \left(\sum_{k=1}^{n} (\mu\lambda_k) h_{i_k} \right) = \sum_{k=1}^{n} (\mu\lambda_k) \cdot Ah_{i_k} = \mu \cdot \sum_{k=1}^{n} \lambda_k \cdot Ah_{i_k} = \mu \cdot A^s h.$$

2) Next, check the additivity of A^s. Let $h^1 = \sum\limits_{k=1}^{n} \lambda_k h_{i_k} \in E$ and $h^2 = \sum\limits_{l=1}^{m} \mu_l h_{j_l} \in E$. Denote by $\{\widetilde{h}_q\}_{q=1}^{p}$ the set of all mutual vectors in the expansions of h^1 and h^2 above. Without loss of generality they can be considered as h_{i_k} $(k = \overline{n-p+1, n})$ and h_{j_l} $(l = \overline{m-p+1, m})$, respectively. The corresponding scalar multipliers are denoted by $\widetilde{\lambda}_q$ and $\widetilde{\mu}_q$ $(q = \overline{1, p})$. Then we obtain

$$\begin{aligned} A^s(h^1 + h^2) &= A^s \left(\sum_{k=1}^{n} \lambda_k h_{i_k} + \sum_{l=1}^{m} \mu_l h_{j_l} \right) \\ &= A^s \left(\sum_{k=1}^{n-p+1} \lambda_k h_{i_k} + \sum_{l=1}^{m-p+1} \mu_l h_{j_l} + \sum_{q=1}^{p} (\widetilde{\lambda}_q + \widetilde{\mu}_q) \widetilde{h}_q \right) \\ &= \sum_{k=1}^{n-p+1} \lambda_k A h_{i_k} + \sum_{l=1}^{m-p+1} \mu_l A h_{j_l} + \left(\sum_{q=1}^{p} \widetilde{\lambda}_q A \widetilde{h}_q + \sum_{q=1}^{p} \widetilde{\mu}_q A \widetilde{h}_q \right) \\ &= \sum_{k=1}^{n} \lambda_k A h_k + \sum_{l=1}^{m} \mu_l A h_{j_l} = Ah^1 + Ah^2. \end{aligned}$$

3) Now, let us estimate of the selector norms. First, because $Ah_i = \{A^s h_i |\, A^s \in A_K\}$ for each $h_i \in H$, then

$$\left(h = \sum_{k=1}^{n} \lambda_k h_{i_k} \in E \right) \Rightarrow \left(Ah \subset \sum_{k=1}^{n} \lambda_k A h_{i_k} = \sum_{k=1}^{n} \lambda_k \cdot \{A^{s_k} h_{i_k} |\, A^{s_k} \in A_K\} \right).$$

From here, setting $A^s h_{i_k} = A^{s_k} h_{i_k}$ $(k = \overline{1, n})$, we obtain:

$$Ah \subset \{ \sum_{k=1}^{n} \lambda_k \cdot A^s h_{i_k} |\, A^s \in A_K \} = \{A^s h |\, A^s \in A_K\}.$$

Hence,

$$\left(\|Ah\| \le \sup_{A^s \in A_K} \|A^s h\| \ (\forall\, h \in E) \right) \Rightarrow \left(\|A\| \le \sup_{A^s \in A_K} \|A^s\| \right).$$

Secondly, since

$$E = \operatorname{span} H = \bigcup_{\lambda > 0} (\lambda \cdot \operatorname{co} H),$$

it easily follows by the Baire theorem on categories that each set $\lambda \cdot \operatorname{co} H$ has the second Baire category everywhere in its closure. Hence, the set $\operatorname{co} H$ is dense in some ball $B_r(0) \subset E$. From here it follows $\forall A^s \in A_K$:

$$\|A^s\| \le \frac{1}{r} \sup_{h \in \operatorname{co} H} \|A^s h\| = \frac{1}{r} \sup\{\|A^s(\sum_{k=1}^{n} \lambda_k h_{i_k})\| \,|\, 0 \le \lambda_k \le 1,\, \sum_{k=1}^{n} \lambda_k = 1,\, h_{i_k} \in H\}$$

$$\le \frac{1}{r} \sup\{\sum_{k=1}^{n} \lambda_k \cdot \|A^s h_{i_k})\| \,|\, 0 \le \lambda_k \le 1,\, \sum_{k=1}^{n} \lambda_k = 1,\, h_{i_k} \in H\}$$

$$\le \frac{1}{r} \sup\{\|A\| \cdot \sum_{k=1}^{n} \lambda_k \,|\, 0 \le \lambda_k \le 1,\, \sum_{k=1}^{n} \lambda_k = 1\} = \frac{1}{r} \cdot \|A\|.$$

So, $A^s \in L(E;F)$ and the bilateral estimate (3.1) with $C = 1/r$, $r = r(H)$ holds. □

Remark 3. *It is possible to identify the packet of basis selectors $A_K = \{A^s\}$ and the K–operator $A_K h = \{A^s h |\, A^s \in A_K\}$. Then estimate (3.1) can be rewritten in the form of norm equivalence for the K–operators A and A_K:*

$$\|A\| \le \|A_K\| \le C_H \cdot \|A\| \quad (\forall\, A \in L_K(E;F)), \tag{3.2}$$

where the constant C_H depends only on the choice of a Hamel basis H. In addition, $Ah \subset A_K h$ ($\forall\, h \in E$) and the correspondence $A \mapsto A_K$ is sublinear. Note that, under such representation, $A_K \in L_K(E;F)$.

Theorem 3.2. *For every K–operator $A \in L_K(E;F)$ its s–representation A_K is also a sublinear bounded K–operator.*

Proof. First, let us check the homogeneity of A_K:

$$A_K(\lambda h) = \{A^s(\lambda h)|\, A^s \in A_K\} = \{\lambda \cdot A^s h|\, A^s \in A_K\} = \lambda \cdot \{A^s h|\, A^s \in A_K\} = \lambda \cdot A_K h.$$

Next, we check the subadditivity of A_K:

$$A_K(h+k) = \{A^s(h+k)|\, A^s \in A_K\} = \{A^s h + A^s k|\, A^s \in A_K\} \subset$$

$$\subset \{A^s h|\, A^s \in A_K\} + \{A^{s'} k|\, A^{s'} \in A_K\} = A_K h + A_K k.$$

Finally, as is shown above, $\|A_K\| \le C_H \cdot \|A\|$. □

Note also that the packet of basis selectors A_K forms a convex compact.

Theorem 3.3. *For every K–operator $A_K \in L_K(E;F)$ its packet of basis selectors A_K is a convex compact in $L(E;F)$.*

Proof. First, let us check the convexity of A_K. Let A^s, $B^s \in A_K$, $0 \le \lambda \le 1$, $h_i \in H$. Then:

$$[(1-\lambda)A^s + \lambda B^s]h_i = (1-\lambda)\cdot A^s h_i + \lambda \cdot B^s h_i \in Ah_i,$$

in view of convexity of the set Ah_i. Hence, by the definition of A_K, it follows that $(1-\lambda)A^s + \lambda B^s \in A_K$.

Next, we check the compactness of A_k. Let $\{A^s_n\}_{n=1}^\infty \subset A_K$. Note that the property of sequential compactness can be formulated as follows: in any subsequence $\{A^s_{n_k}\}_{k=1}^\infty$ a convergent (to an element of A_K) subsubsequence $\{A^s_{n_{k_j}}\}_{j=1}^\infty$ can be chosen. In addition, by virtue of the Banach–Steinhaus theorem, we can consider only the pointwise convergence.

Suppose, to the contrary, that there exists such subsequence $\{A^s_{n_k}\}_{k=1}^\infty$ that for any subsubsequence $\{A^s_{n_{k_j}}\}_{j=1}^\infty$ there exists such $h \in E$ so that $A^s_{n_{k_j}}h$ does not converge. On the other hand, by virtue of compactness of the set Ah, in the sequence $\{A^s_{n_k}h\}_{k=1}^\infty \subset Ah$ a convergent subsequence $\{A^s_{n_{k_j}}h\}_{j=1}^\infty$ can be chosen, that leads to a contradiction.

It remains to check the belonging of the limit operator to A_K. Let $A^s h = \lim\limits_{j\to\infty} A^s_{n_{k_j}}h$. Then, for some $h_i \in H$, in view of compactness of Ah_i we obtain:

$$(A^s_{n_{k_j}}h_i \in Ah_i,\ j \in \mathbb{N}) \Rightarrow (A^s h_i \in Ah_i).$$

It follows, in view of the obvious linearity of A^s, that $A^s \in A_K$. □

Corollary 3.1. *The following bounded sublinear (but not injective) embedding:*

$$L_K(E;F) \hookrightarrow (L(E;F))_K \quad (A \mapsto A_K)$$

takes place.

Finally, if the Banach space E possesses a topological basis then, it is possible to describe the K–operator A_K by means of its values on the basis in E.

Corollary 3.2. *Let a real Banach space have a topological basis $\{h_n\}_{n=1}^\infty$, $A \in L_K(E;F)$. Then, under a suitable choice of a Hamel basis E:*

$$(h = \sum_{n=1}^{\infty} \lambda_n h_n \in E) \Rightarrow (A_K h = \sum_{n=1}^{\infty} \lambda_n \cdot Ah_n |\ dist_H).$$

Here we denote by $dist_H$ the Hausdorff metric in the set of all compacts contained in E.

Proof. As is known [19], the set $\{h_n\}_{n=1}^\infty$ can be included as a part in some Hamel basis in E. By constructing an s–representation A_K with respect to such a Hamel basis we obtain $Ah_n = A_K h_n$ $(n = 1, 2, \ldots)$.

Next, let $h = \sum\limits_{n=1}^{\infty} \lambda_n h_n \in E$. For very $N \in \mathbb{N}$ denote $h^N = \sum\limits_{n=1}^{N} h_n$; then $h^N \to_E h$ as $N \to \infty$. By the definition of A_K it follows:

$$A_K h^N = \left\{ \sum_{n=1}^{N} \lambda_n A^s h_n |\ A^s \in A_K \right\} = \left\{ \sum_{n=1}^{N} \lambda_n a^s_n |\ a^s_n \in Ah_n \right\} = \sum_{n=1}^{N} \lambda_n \cdot Ah_n. \quad (3.3)$$

The set $A_K = \{A^s\}$, in view of its compactness in $L(E;F)$, is equicontinuous. Hence,

$$\forall\, \varepsilon > 0\, \exists\, N(\varepsilon) \quad (N > N(\varepsilon)) \Rightarrow (\forall\, A^s \in A_K : \; \|A^s h^N - A^s h\| < \varepsilon). \tag{3.4}$$

So, using (3.3) and (3.4) implies for $N > N(\varepsilon)$ (in terms of the Hausdorff metric):

$$(A_K h^N \subset U_\varepsilon(A_K h),\ A_K h \in U_\varepsilon(A_K h^N)) \Rightarrow (\mathrm{dist}_H(A_K h^N, A_K h) < \varepsilon).$$

Hence,

$$A_K h = \lim_{N\to\infty} A_K h^N = \lim_{N\to\infty} \sum_{n=1}^{N} \lambda_n A h_n = \sum_{n=1}^{\infty} \lambda_n A h_n \ (\mathrm{dist}_H).$$

□

4 K–invertibility of K–operators

In what follows, E and F are real Banach spaces, $E \cong F$, $\mathrm{Isom}(E;F)$ is the set of all isomorphisms between E and F, $A \in L_K(E;F)$, H is a fixed Hamel basis in E and all s–representations of the K–operators $E \to F_K$ are considered with respect to H.

Definition 5. We say that the K–operator A is K–*invertible* if $A_K \subset \mathrm{Isom}(E;F)$. In this case, we introduce K–*inverse* K–operator A_K^{-1} as follows:

$$A_K^{-1} = \overline{\mathrm{co}}\left\{(A^s)^{-1} |\, A^s \in A_K\right\} \quad \left(A_K^{-1} k = \left\{B^\sigma k |\, B^\sigma \in A_K^{-1}\right\}\right).$$

The set of the all K–invertible K–operators $A : E \to F_K$ is denoted by $\mathrm{Isom}_K(E;F)$.

Consider some properties of K–invertible K–operators.

Theorem 4.1. *If a K–operator A is K–invertible, then A_K^{-1} forms a convex compact in $L(E;F)$.*

Proof. The set $\{(A^s)^{-1} |\, A^s \in A_K\}$ is compact by virtue of compactness of A_K and continuity of the mapping $A^s \mapsto (A^s)^{-1}$ (see [3]). From here, in view of the Krein theorem (see [19]), the compactness follows of its closed convex hull A_K^{-1}. □

Theorem 4.2. *If a K–operator A is K–invertible, then*

$$I_E h \in [A_K^{-1} \cdot A_K] h \ (h \in E); \quad I_F k \in [A_K \cdot A_K^{-1}] k \ (k \in F). \tag{4.1}$$

Proof. Let us remind that by the definition of the K–*composition* of K–operators $A \in L_K(E;F)$ and $B \in L_K(F;G)$ (see [14]),

$$[B \cdot A] h = \overline{\mathrm{co}} \left(\bigcup_{k \in Ah} Bk \right).$$

Hence,

$$[A_K^{-1} \cdot A_K] h = \overline{\mathrm{co}} \left(\bigcup_{k \in A_K h} A_K^{-1} k \right). \tag{4.2}$$

Next, since $A_K^{-1} \supset \{(A^s)^{-1} | \, A^s \in A_K\}$, it follows from (4.2) that:

$$[A_K^{-1} \cdot A_K]h \supset \{(A^s)^{-1} \cdot (A^s h) | \, A^s \in A_K\} = \{h\} = \{I_E h\}.$$

So, the first of estimates (4.1) is satisfied. The second estimate can be checked in an analogous way. □

Now, let us consider a simple example.

Example 4.1. *Let $E = F = \mathbb{R}$. As is shown in ([14]), the cone $L_K(\mathbb{R}, \mathbb{R}) = \mathbb{R}^*_K$ consisting of real K–functionals can be identified with the half-plane*

$$\{(k_1, k_2) | \, k_1 \le k_2\}, \quad A = [k_1; k_2].$$

Moreover, a K–functional A is K–invertible if and only if $0 \notin [k_1; k_2]$. In this case, $A_K = A = [k_1; k_2]$,

$$1 \in [A_K^{-1} \cdot A_K] = [A_K \cdot A_K^{-1}] = \begin{cases} \left[\frac{k_1}{k_2}; \frac{k_2}{k_1}\right], & 0 < k_1 \le k_2; \\ \left[\frac{k_2}{k_1}; \frac{k_1}{k_2}\right], & k_1 \le k_2 < 0. \end{cases}$$

Next, let us explain the structure of the operator which is K–inverse to a K–composition.

Theorem 4.3. *If $A \in Isom_K(E; F)$, $B \in Isom_K(F; G)$, then*

$$[B \cdot A]_K^{-1}(l) \subset [B_K \cdot A_K]_K^{-1}(l) \subset [A_K^{-1} \cdot B_K^{-1}](l) \quad (\forall \, l \in G).$$

Proof. Let $A_K = \{A^s\}$, $B_K = \{B^s\}$. As $Ah \subset A_K h$, $Bh \subset B_K h$, then

$$[B \cdot A]_K h \subset [B_K \cdot A_K]h = \{B^\sigma \cdot A^s h\} \quad (h \in E).$$

From here it follows:

$$[B \cdot A]_K^{-1}(l) = \overline{\text{co}} \, \{(B^\sigma \cdot A^s)^{-1}\} \cdot (l) = \overline{\text{co}} \, \{(A^s)^{-1} \cdot (B^\sigma)^{-1}\} \cdot (l)$$
$$\subset [\overline{\text{co}} \, \{(A^s)^{-1}\} \cdot \overline{\text{co}} \, \{(B^\sigma)^{-1}\}] \cdot (l) = [A_K^{-1} \cdot B_K^{-1}](l).$$

□

Finally, let us explain the question on the repeated K–invertibility.

Theorem 4.4. *If $A \in Isom_K(E; F)$ then $A_K^{-1} \in Isom_K(F; E)$. Moreover*

$$A_K h \subset (A_K^{-1})_K^{-1} \cdot h \quad (\forall \, h \in E). \tag{4.3}$$

Proof. Let $A_K = \{A^s\}$. Then $A_K^{-1} \supset \{(A^s)^{-1}\}$, whence it follows

$$(A_K^{-1})_K^{-1} h \supset \{((A^s)^{-1})^{-1} h\} = \{A^s h\} = A_K h.$$

□

Remark 4. *For the case $E = F = \mathbb{R}$ (i.e., the cone $\mathbb{R}^*_K$, see example 3.4) the precise equality $A_K = (A_K^{-1})_K^{-1}$ takes place:*

$$(A_K = [k_1; k_2]) \Rightarrow \left(A_K^{-1} = \left[\frac{1}{k_2}; \frac{1}{k_1}\right]\right) \Rightarrow ((A_K^{-1})_K^{-1} = [k_1; k_2] = A_K).$$

However, in general case the precise equality in (4.3) is not fulfilled.

Example 4.2. *Let us consider the following K–operator $\mathbb{R}^2 \to \mathbb{R}^2_K$:*

$$A = A_K = \left\{ A^x = \begin{pmatrix} x & 0 \\ 0 & 1-x \end{pmatrix} \right\}_{\delta \le x \le 1-\delta}.$$

Obviously, the set $\{A^x\}_{\delta \le x \le 1-\delta}$ is convex and compact, $A = A_K \in Isom_K(\mathbb{R}^2; \mathbb{R})$. Here the set of basis selectors of A can be identified with the linear segment $\{(x, y) |\, x + y = 1,\ \delta \le x \le 1 - \delta\}$. In this case the set

$$\left\{ (A^x)^{-1} = \begin{pmatrix} 1/x & 0 \\ 0 & 1/1-x \end{pmatrix} \right\}_{\delta \le x \le 1-\delta}$$

can be identified with the hyperbolic arc:

$$\gamma = \{(u, v) |\, \frac{1}{u} + \frac{1}{v} = 1,\ \frac{1}{1-\delta} \le u \le \frac{1}{\delta}\}.$$

Obviously, the set γ is not convex. Hence, the set $A_K^{-1} = \overline{co}\,\{(A^x)^{-1}\}_{\delta \le x \le 1}$ (the corresponding hyperbolic section) does not coincide with γ. From here it follows:

$$(A_K^{-1})_K^{-1} \neq\supset \gamma^{-1} = A_K.$$

5 Extremal points of the packets of linear selectors

Compactness and convexity of the packets A_K and A_K^{-1} leads to the actual problem of describing the extremal points of these sets. Such description, undoubtedly, will make it possible to apply the Krein-Milman theorem effectively.

In what follows, we denote by $\mathrm{Extr}(C)$ the set of all extremal points of C; here C is a convex compact set either from F, or from $L(E; F)$, E and F are real Banach spaces, $H = \{h_i\}_{i \in I}$ is a Hamel basis in E. First, let us obtain a description of $\mathrm{Extr}(A_K)$.

Theorem 5.1. *Let $A \in L_K(E; F)$, $A_K = \{A^s\}$ be its s–representation. Then*

$$(A^s \in Extr(A_K)) \Leftrightarrow (\forall\, h_i \in H :\ A^s h_i \in Extr(Ah_i)). \tag{5.1}$$

Proof. First, let us suppose the condition in the right-hand side of (5.1) not to be satisfied:

$$\exists\, h_i \in H :\ A^s h_i \notin \mathrm{Extr}(Ah_i).$$

Hence, $A^s h_i$ is a *C–inner point* (see [5]) of some segment in A_K:

$$A^s_{h_i} = (1-\lambda) a_i^1 + \lambda \cdot a_i^2 \quad (0 < \lambda < 1;\ a_i^1, a_i^2 \in A_K^{h_i}; a_i^1 \neq a_i^2).$$

Set:

$$A_1^s h_j = \begin{cases} a_i^1, & j = i; \\ A^s h_j, & j \neq i, \end{cases} \qquad A_2^s h_j = \begin{cases} a_i^1, & j = i; \\ A^s h_j, & j \neq i, \end{cases} \qquad (A_1^s, A_2^s \in A_K)$$

Then $A^s h_j = (1-\lambda)A_1^s h_j + \lambda \cdot A_2^s h_j$ $(\forall\, h_j \in H)$, whence $A^s = (1-\lambda)A_1^s + \lambda \cdot A_2^s$ follows. So, $A^s \notin \mathrm{Extr}(A_K)$.

Conversely, suppose $A^s \notin \mathrm{Extr}(A_K)$. Then A^s is a C–inner point of some segment in A_K:

$$A^s = (1-\lambda)A_1^s + \lambda \cdot A_2^s \quad (0 < \lambda < 1;\ A_1^s, A_2^s \in A_K;\ A_1^s \neq A_2^s).$$

Since $A_1^s \neq A_2^s$ then $A_1^s h_i \neq A_2^s h_i$ for some $h_i \in H$. Then from the equality $A^s h_i = (1-\lambda)A_1^s h_i + \lambda \cdot A_2^s h_i$ it follows that $A^s h_i$ is a C–inner point of the segment $[A_1^s h_i; A_2^s h_i] \subset Ah_i$. Thus, $A^s h_i \notin \mathrm{Extr}(Ah_i)$ and the condition from the right in (5.1) is not satisfied. □

Now, applying the Krein-Milman theorem immediately yields

Corollary 5.1. *Denote by A_K^e the set of all basis selectors from A_K, that satisfy the condition in the right-hand side of (5.1). Then*

$$A_K = \overline{co}(A_K^e).$$

Now, let's pass to the case of the K–invertible K–operator $A \in \mathrm{Isom}_K(E;F)$. Thus, $A_K \in \mathrm{Isom}_K(E;F)$, $A_K^{-1} \in \mathrm{Isom}_K(F;E)$. Above all, we are interested in the question on the connection between the sets $\mathrm{Extr}(A_K)$ and $\mathrm{Extr}(A_K^{-1})$.

Theorem 5.2. *If $A \in Isom(E;F)$, then the following inclusion*

$$(Extr\, A_K)^{-1} \subset Extr(A_K^{-1}). \tag{5.2}$$

takes place.

Proof. Let A_e be an extremal point of A_K, H be a closed hyperplane of support in $L(E;F)$, that passes through A_e, H_+ be a corresponding half-space in $L(E;F)$, containing A_K. Without loss of generality, it can be assumed $0 \notin H$. First, prove that the set $(H_+)^{-1}$ is a convex in some neighborhood of A_e^{-1}. Take the notation $\tilde{H}_+ = H_+ \backslash H$, $\tilde{H}_+^{-1} = H_+^{-1} \backslash H^{-1}$.

Let A_1, $A_2 \in H \cap \mathrm{Isom}(E;F)$, $A_1 \neq A_2$. Now, let's join the points A_1^{-1}, $A_2^{-1} \in H^{-1} \cap \mathrm{Isom}(F;E)$ by a segment and prove that $(A_1^{-1}; A_2^{-1}) \subset H_+^{-1}$.

Let, on the contrary, some point $B^{-1} \in (A_1^{-1}; A_2^{-1}) \cap H^{-1}$ exist. Hence,

$$B \in (A_1^{-1}; A_2^{-1}) \cap (A_1; A_2).$$

Thus, for some $0 \leq \lambda, \mu \leq 1$ the following equality holds:

$$(\lambda A_1^{-1} + (1-\lambda)A_2^{-1})^{-1} = (\mu A_1 + (1-\mu)A_2),$$

whence it follows

$$I_F = (\mu A_1 + (1-\mu)A_2)(\lambda A_1^{-1} + (1-\lambda)A_2^{-1})$$

$$= (2\lambda\mu + 1 - \lambda\mu) \cdot I_F + \mu(1-\lambda)A_1A_2^{-1} + \lambda(1-\mu)A_2A_1^{-1}.$$

From here, denoting $T = A_1A_2^{-1}$, we come to the operator square equation:

$$\mu(1-\lambda)T^2 + (2\lambda\mu - \lambda - \mu)T + \lambda(1-\mu)I_F = 0,$$

that has solutions $T_1 = I_F$ and $T_2 = (\lambda(1-\mu)/\mu(1-\lambda)) \cdot I_F$ (which coincide if $\lambda = \mu$). Denote by $t = \lambda(1-\mu)/\mu(1-\lambda)$ and consider both cases.

(a). $T = A_1A_2^{-1} = I_F$. Then $A_1 = A_2$ and we arrive at a contradiction.

(b). $T = A_1A_2^{-1} = t \cdot I_F$, $t \neq 1$. Then $A_1 = t \cdot A_2$ whence it follows $0 \in H$, that contradicts our hypothesis.

Thus, the set H_+^{-1} is a convex (locally, near A_e^{-1}) and contains A_K^{-1}.

Now, let us pass some hyperplane of support P to the set H_+^{-1} through the point A_e^{-1}. Since $A_K^{-1} \subset \tilde{H}_+^{-1} \cup \{A_e^{-1}\}$, P is a hyperplane of support to A_K^{-1} and $P \cap A_K^{-1} = \{A_e^{-1}\}$. So, $A_e^{-1} \in \mathrm{Extr}(A_K^{-1})$. □

Remark 5. *Estimate (5.2) assumes a strong inclusion. So, for example, in Example 4.2 considered above, the set $(Extr\, A_K)^{-1}$ consists of two endpoints of the hyperbolic arc $\gamma : \frac{1}{u} + \frac{1}{v} = 1$, $\frac{1}{1-\delta} \leq u \leq \frac{1}{\delta}$. At the same time, the set Extr(A_K^{-1}) is the whole hyperbolic arc γ, including all intermediate points.*

Corollary 5.2. *Under the assumptions of Theorem 5.2 the following inclusion*

$$\overline{co}\,(Extr\, A_K)^{-1} \subset A_K^{-1}$$

takes place.

Proof. It suffices to pass to the operation $\overline{\mathrm{co}}$ in estimate (5.2) and then to apply the Krein-Milman theorem to the right-hand side of the obtained estimate. □

Corollary 5.3. *Under the assumptions of Theorem 5.2 the following inclusion*

$$A_K \subset \overline{co}\,(Extr\, A_K^{-1})^{-1}$$

takes place.

Proof. Here, before application of the Krein-Milman theorem, to the left-hand side of (5.2) it is necessary to pass to the inverse values in (5.2). □

Now let us consider a question on sufficient condition of K–invertibility, namely, on K–analogue of the classical von Neumann theorem (see, e.g., [3]).

Theorem 5.3. *Let $A \in L_K(E)$. If $A = I - B$, where $\|B_K\| < 1$, then A is K–invertible. Moreover, the following estimate*

$$A_K^{-1}h \subset (I + \sum_{n=1}^{\infty} B_K^n)h \quad (\forall\, h \in E) \tag{5.3}$$

takes place. Here in (5.3) the power of the K–operator is meant with respect to the K–product (see [14]), and the convergence of the power series in (5.3) is meant with respect to the cone-norm in $L_K(E)$.

Proof. 1) Since $(A = I - B) \Rightarrow (A_K = I - B_K)$, denoting by $A_K = \{A^s\}$, $B_K = \{B^s\}$, we obtain:

$$A^s = I - B^s, \quad \|B_s\| \leq 1 - \varepsilon.$$

From here, using the classical von Neumann theorem implies invertibility for all the selectors $A^s \in A_K$ and validity of the equalities

$$(A^s)^{-1} = (I - B^s)^{-1} = \sum_{n=0}^{\infty} (B^s)^n \quad (A^s \in A_K,\, B^s \in B_K).$$

Hence,

$$A_K^{-1}h = \overline{\mathrm{co}}\{(A^s)^{-1}h\} = \overline{\mathrm{co}}\left\{\sum_{k=0}^{\infty}(B^s)^n h\right\} \subset \sum_{n=0}^{\infty} \overline{\mathrm{co}}\{(B^s)^n h\}. \tag{5.4}$$

2) Next, let us obtain an estimate of the right-hand side of (5.4) via the powers B_K^n. In the case $n = 2$ we have:

$$B_K^2 h = \overline{\mathrm{co}}(\bigcup_{B_K h} B_K k) = \overline{\mathrm{co}}\{B^{s_1}(B^{s_2}h)\} \supset \overline{\mathrm{co}}\{(B^s)^2 h\}.$$

From here, it easily follows by induction that

$$B_K^n h \supset \overline{\mathrm{co}}\{(B^s)^n h\} \supset (\overline{\mathrm{co}}\{(B^s)^n\}) \cdot h. \tag{5.5}$$

Finally, (5.4) and (5.5) imply

$$A_K^{-1}h \subset \sum_{n=0}^{\infty} B_K^n h \subset (\sum_{n=0}^{\infty} B_K^n)h,$$

whence the estimate (5.2) follows. □

By applying the Krein-Milman theorem, now it is easy to obtain

Theorem 5.4. *Let, under the assumptions of Theorem 5.3, the inequality*

$$\|A_e - I\| \leq 1 - \varepsilon$$

hold for all extremal points $A_e \in \mathrm{Extr}\, A_K$. Then A is K–invertible.

Proof. Let $A^s \in \mathrm{co}(\mathrm{Extr}(A_K))$. Then $A^s = \sum\limits_{k=1}^{n} \lambda_k \cdot A_e^{s_k}$, where $\lambda_k \geq 0$, $\sum\limits_{k=1}^{n} \lambda_k = 1$, $A_e^{s_k} \in \mathrm{Extr}(A_K)$. Hence,

$$\|A^s - I\| = \|\sum_{k=1}^{\infty} \lambda_k (A_e^{s_k} - I)\| \leq \sum_{k=1}^{n} \lambda_k \cdot \|A_e^{s_k} - I\| \leq \left(\sum_{k=1}^{n} \lambda_k\right) \cdot (1 - \varepsilon) = 1 - \varepsilon.$$

Thus, the inequality $\|A^s - I\| \leq 1 - \varepsilon$ holds for all $A^s \in \mathrm{co}(\mathrm{Extr}\, A_K)$. From here, by continuity of the operator norm, it follows the validity of this inequality for all $A^s \in \overline{\mathrm{co}}(\mathrm{Extr}\, A_K) = A_K$, by virtue of the Krein-Milman theorem. It remains to apply Theorem 5.3. □

In conclusion, let us give an example of an „incomplete" packet of invertible basis selectors, the s–representation of which contains an irreversible selector.

Example 5.1. *Let $E = F = \mathbb{R}^2$. For $(x, y) \in E$ the K–operator $B : E \to F_K$ is defined as follows:*

$$B(x, y) = \left\{ \frac{1}{2}((1 + \lambda - \lambda\mu)x,\ (1 + \lambda\mu)y) |\, 0 \le \lambda,\ \mu \le 1 \right\}; \quad A = I - B.$$

Geometrically $A(x, y)$ represents a right triangle with the right angle vertex $(x/2; y/2)$ and legs $x/2$ and $y/2$. The operator $A = I - B$ can be represented as a set of the invertible linear selectors $A = \{I - B_{\lambda\mu} |\, 0 \le \lambda,\ \mu \le 1\}$, with $B_{\lambda\mu}(x, y) = \frac{1}{2}((1 + \lambda - \lambda\mu)x, (1 + \lambda\mu)y)$, where all $(I - B_{\lambda\mu})$ are invertible.

At the same time, s-representation A_K takes form $A_K = \{I - C_{\lambda\mu} |\, 0 \le, \mu \le 1\}$, where

$$C_{\lambda\mu}(x, y) = \frac{1}{2}((1 + \lambda)x, (1 + \mu)y).$$

Geometrically $C_{\lambda\mu}(x, y)$ represents a rectangle with the south-western vertex $(x/2, y/2)$ and sides $x/2$ and $y/2$. Here the selector $I - C_{11}$ is irreversible, hence A_K is not K–invertible.

6 Conclusion

Thus, in the paper a sufficiently complete description was exposed of the packet of basis selectors for an arbitrary sublinear compact-valued operator. It enabled us to give a rather complete description of the inverse selector representation.

It appears that a base has been built that will allow, by using the concepts of compact subdifferentiability and subsmoothness, to obtain the appropriate form of inverse function and implicit function theorems. The authors expect to investigate this problem in a reasonable time.

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