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KORDAN NAURYZKHANOVICH OSPANOV

(to the 60th birthday)

On 25 September 2015 Kordan Nauryzhanovich Ospanov, professor of the Department "Fundamental Mathematics" of the L.N. Gumilyov Eurasian National University, Doctor of Physical and Mathematical Sciences (2000), a member of the Editorial Board of our journal, celebrated his 60th birthday.

He was born on September 25, 1955, in the village Zhanatalap of the Zhanaarka district of the Karaganda region. In 1976 he graduated from the Kazakh State University, and in 1981 he completed his postgraduate studies at the Abay Kazakh Pedagogical Institute.

Scientific works of K.N. Ospanov are devoted to application of methods of functional analysis to the theory of differential equations. On the basis of a local approach to the resolvent representation he has found weak conditions for the solvability of the singular generalized Cauchy-Riemann system and established coercive estimates for its solution. He has obtained a criterion of the spectrum discreteness for the resolvent of the system and the exact in order estimates of singular values and Kolmogorov widths. He has original research results on the coercive solvability of the quasilinear singular generalized Cauchy-Riemann system and degenerate Beltrami-type system. He has established important smoothness and approximation properties of non strongly elliptic systems. K.N. Ospanov has found separability conditions in Banach spaces for singular linear and quasi-linear second-order differential operators with growing intermediate coefficients and established a criterion for the compactness of its resolvent and finiteness of the resolvent type.

His results have contributed to a significant development of the theory of two-dimensional singular elliptic systems, degenerate differential equations and non strongly elliptic boundary value problems.

K.N. Ospanov has published more than 140 scientific papers. The list of his most important publications one may see on the web-page

http://mmf.enu.kz/images/stories/photo/pasport/fm/ospanov

K.N. Ospanov is an Honoured Worker of Education of the Republic of Kazakhstan, and he was awarded the state grant "The best university teacher".

The Editorial Board of the Eurasian Mathematical Journal is happy to congratulate Kordan Nauryzkhanovich Ospanov on occasion of his 60th birthday, wishes him good health and further productive work in mathematics and mathematical education.

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ALMOST HYPOELLIPTIC OPERATORS WITH CONSTANT POWERS

V.N. Margaryan, H.G. Ghazaryan

Communicated by V.I. Burenkov

Key words: almost hypoelliptic operator (equation), operator with constant power, Sobolev spaces generated by a differential operator, (non-)degenerate operator (polynomial).

AMS Mathematics Subject Classification: 12E10.

Abstract. The concept of a formally almost hypoelliptic operator with constant power and the concept of weighted Sobolev spaces generated by such operators are introduced. We prove some properties of such operators, establish some estimates for functions in those spaces, in particular, the density of smooth functions in those spaces. We intend, in another work, using the results of this paper, to select a set of infinitely differentiable solutions for a class of almost hypoelliptic equations having constant power.

1 Weitghted function spaces generated by differential operators with constant coefficients

We use the following standard notation: $\mathbb{N}$ is the set of all natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup 0$, $\mathbb{N}_0^n = \mathbb{N}_0 \times ... \times \mathbb{N}_0$ - the set of all $n-$ dimensional multi-indices, $\mathbb{R}^n$ and $\mathbb{C}^n$ - the n - dimensional euclidian spaces of real points (vectors) $\xi = (\xi_1, ..., \xi_n)$ and complex points (vectors) $\zeta = (\zeta_1, ..., \zeta_n)$ respectively. For $\xi \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}_0^n$ we put $|\xi| = \sqrt{\xi_1^2 + ... + \xi_n^2}$, $|\alpha| = \alpha_1 + ... + \alpha_n$. Finally we put $\mathbb{R}^{n,0} = \{\xi \in \mathbb{R}^n : \xi_1 \cdot ... \cdot \xi_n \neq 0\}$ and $\mathbb{R}^{n,+} = \{\xi \in \mathbb{R}^n : \xi_j \geq 0 \ (j = 1, ..., n)\}$.

Let $g \in C^\infty = C^\infty(\mathbb{R}^n)$ be any positive function such that

a) for any $\alpha \in \mathbb{N}_0^n$ there exist positive numbers κ and κ_α such that for all $x \in R^n$

$$\kappa^{-1} e^{-\delta |x|} \leq g_\delta(x) \leq \kappa e^{-\delta |x|}; |D^\alpha g_\delta(x)| \leq \kappa_\alpha \delta^{|\alpha|} g_\delta(x), \tag{1.1}$$

where $g_\delta(x) = g(\delta x)$.

b) there exist positive numbers σ_1 and σ_2 such that for any $\delta > 0$, $x \in \mathbb{R}^n$ and $T > 0$

$$\sup_{y \in S_T} g_\delta(x + y) \leq \sigma_1 g_\delta(x); \ \sup_{y \in S_T} |g_\delta(x + y) - g_\delta(x)| \leq \sigma_2 T g_\delta(x), \tag{1.2}$$

where $S_T = \{x \in \mathbb{R}^n; |x| < T\}$.

As a function g one can take the regularization of the function $H(x) = e^{-|x|}$ when $|x| > 1$ and $H(x) = e^{-1}$ when $|x| \leq 1$.

Denote by $L_{2,\delta} = L_{2,\delta}(\mathbb{R}^n)$ the set of all measurable functions u with finite norms

$$||u\, e^{-\delta\,|x|}||_{L_2(\mathbb{R}^n)}$$

and for any $k \in \mathbb{N}$ by W_δ^k the set of all functions with finite norms

$$\sum_{|\alpha|\le k} ||(D^\alpha u)||_{L_{2,\,\delta}},$$

where $D^\alpha u$ are weak derivatives of the function u.

Note that these norm are equivalent to

$$||u||_{L_{2,\,\delta}} = \left[\int_{\mathbb{R}^n} |u(x)|^2 g_\delta(x)dx\right]^{1/2} \tag{1.3}$$

and to

$$||u||_{W_\delta^k} = \sum_{|\alpha|\le k} ||(D^\alpha u) g_\delta(x)||_{L_2}. \tag{1.4}$$

Let $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\varphi \ge 0$, $\int \varphi(x)dx = 1$, $\varepsilon > 0$, $\varphi_\varepsilon(x) = \varepsilon^{-n}\,\varphi(x/\varepsilon)$, $f_\varepsilon(x) = (f * \varphi_\varepsilon)(x)$:

Lemma 1.1. *For any $\delta > 0$, $f \in L_{2,\,\delta}$, and $\alpha \in \mathbb{N}_0^n$*

1) for any $\varepsilon > 0$, $D^\alpha f_\varepsilon \in L_{2,\,\delta}$,

2) $||f_\varepsilon - f||_{L_{2,\,\delta}} \to 0$ as $\varepsilon \to +0$.

Proof. Since for any $\alpha \in \mathbb{N}_0^n$

$$||D^\alpha f_\varepsilon||_{L_{2,\,\delta}} = ||f * D^\alpha \varphi_\varepsilon||_{L_{2,\,\delta}} = \varepsilon^{-|\alpha|}||f * (D^\alpha\varphi)_\varepsilon||_{L_{2,\,\delta}}$$

$$= \varepsilon^{-|\alpha|}\left\|\int f(x-y)(D^\alpha\varphi)_\varepsilon(y)\, g_\delta(x)dy\right\|_{L_2}$$

$$= \varepsilon^{-|\alpha|}\left\|\int (f\, g_\delta)(x-y)\, D^\alpha\varphi_\varepsilon(y)dy + \int f(x-y)[g_\delta(x) - g_\delta(x-y)]\,(D^\alpha\varphi)_\varepsilon(y)dy\right\|_{L_2},$$

by the properties of the function g_δ and by Young's inequality for some positive constants C_1 and C_2 we have

$$||D^\alpha f_\varepsilon||_{L_{2,\,\delta}} \le \varepsilon^{-|\alpha|}\left[||(f\, g_\delta) * (D^\alpha\varphi)_\varepsilon||_{L_2} + C_1 \left\|\int |f(x-y)\, g_\delta(x-y)|\,|(D^\alpha\varphi)_\varepsilon|\, dy\right\|_{L_2}\right]$$

$$\le \varepsilon^{-|\alpha|}\,[||f\, g_\delta||_{L_2} + C_1\, ||f\, g_\delta||_{L_2}]\; ||(D^\alpha\varphi)_\varepsilon\, ||_{L_1} \le C_2\, \varepsilon^{-|\alpha|}\, ||f\, g_\delta||_{L_2} < \infty \;\; \forall\varepsilon > 0,$$

which proves Statement 1). To prove Statement 2) note that

$$||f_\varepsilon - f||_{L_{2,\,\delta}} = ||[(f * \varphi_\varepsilon) - f]\, g_\delta||_{L_2}$$

$$= \left\|\int f(x-y)\,\varphi_\varepsilon(y)\, g_\delta(x)dy - \int f(x)\, g_\delta(x)\,\varphi_\varepsilon(y)dy\right\|_{L_2}$$

$$= \left\| \int [(f\,g_\delta)(x-y) - (f\,g_\delta)(x)]\,\varphi_\varepsilon(y)dy \right.$$

$$\left. + \int f(x-y)\,[g_\delta(x) - g_\delta(x-y)]\varphi_\varepsilon(y)dy \right\|_{L_2}. \tag{1.5}$$

Since $f\,g_\delta \in L_2$, by mean continuity of functions from L_2 and by generalized Minkowski's inequality we have as $\varepsilon \to +0$

$$\left\| \int [(f\,g_\delta)(x-y) - (f\,g_\delta)(x)]\,\varphi_\varepsilon(y)dy \right\|_{L_2}$$
$$\le \sup_{y \in supp\varphi_\varepsilon} ||(f\,g_\delta)(\cdot - y) - (f\,g_\delta)(\cdot)||_{L_2} \to 0.$$

Since $(f\,g_\delta) \in L_2$, by the properties of the function g_δ and by Young's inequality, for the second term of (1.5) we have with a positive constant $C_3 = C_3(g)$

$$\left\| \int f(x-y)\,[g_\delta(x) - g_\delta(x-y)]\varphi_\varepsilon(y)dy \right\|_{L_2}$$
$$\le C_3\,\varepsilon \left\| \int |(f\,g_\delta)(x-y)|\,\varphi_\varepsilon(y)dy \right\|_{L_2} \le C_3\,\varepsilon\,||f\,g_\delta||_{L_2} \to 0$$

as $\varepsilon \to +0$, which proves the lemma. □

Let

$$Q(D) = \sum_{|\nu| \le m} \gamma_\nu D^\nu$$

be a linear differential operator with constant coefficients and

$$Q(\xi) = \sum_{|\nu| \le m} \gamma_\nu \xi^\nu$$

be its characteristic polynomial (symbol), where for $x \in \mathbb{R}^n,\ \xi \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}_0^n$: $D_j := \frac{1}{i}\frac{\partial}{\partial x_j}\ \ (j = 1, ..., n),\ D^\alpha := D_1^{\alpha_1}, ..., D_n^{\alpha_n},\ \xi^\alpha = \xi_1^{\alpha_1} \cdot ... \cdot \xi_n^{\alpha_n}$.

For a multi-index α we denote by $Q^{(\alpha)}(D)$ the operator with characteristic polynomial $Q^{(\alpha)}(\xi) := D^\alpha Q(\xi)$.

Let us introduce the following Sobolev-type weighted spaces generated by the operator $Q(D)$

$$H_1(Q,\,\delta) := \{u;\, ||u||_{H_1(Q,\,\delta)} := ||u||_{L_{2,\,\delta}} + ||Q(D)u||_{L_{2,\,\delta}} < \infty\},$$

$$H_2(Q,\,\delta) := \{u;\, ||u||_{H_2(Q,\,\delta)} := ||u||_{L_{2,\,\delta}} + ||Q(D)(u\,g_\delta)||_{L_2} < \infty\},$$

$$H_3(Q,\,\delta) := \{u;\, ||u||_{H_3(Q,\,\delta)} := \sum_{|\alpha| \le m} ||Q^{(\alpha)}(D)(u\,g_\delta)||_{L_2} < \infty\},$$

$$H_4(Q,\,\delta) := \{u;\, ||u||_{H_4(Q,\,\delta)} := \sum_{|\alpha| \le m} ||Q^{(\alpha)}(D)u||_{L_{2,\,\delta}} < \infty\}.$$

Lemma 1.2. *Let* $\operatorname{ord} Q = m$. *Then for any* $\delta_0 > 0$ *there exists a constant* $c = c(\delta_0) > 0$ *such that for all* $\delta \in (0, \delta_0)$ *and* $u \in W_\delta^m$

$$c^{-1} ||u||_{H_4(Q,\,\delta)} \le ||u||_{H_3(Q,\,\delta)} \le c\, ||u||_{H_4(Q,\,\delta)}. \tag{1.6}$$

Proof. By applying the Leibnitz formula and the properties of the function g, we obtain with some positive constants $C_1, C_2 = C_2(\delta_0, m)$ for all $u \in W_\delta^m$

$$||u||_{H_3(Q,\delta)} = \sum_\alpha ||Q^{(\alpha)}(D)(u\, g_\delta)||_{L_2} = \sum_\alpha || \sum_\beta \frac{1}{\beta!}[Q^{(\alpha+\beta)}(D)u]\, D^\beta g_\delta||_{L_2}$$

$$\le \sum_\alpha \sum_\beta \frac{\delta^{|\beta|}}{\beta!} ||[Q^{(\alpha+\beta)}(D)u]\, (D^\beta g)_\delta||_{L_2}$$

$$\le C_1 \sum_\alpha \sum_\beta \frac{\delta^{|\beta|}}{\beta!}||[Q^{(\alpha+\beta)}(D)u]\, g_\delta||_{L_2} \le C_2\, ||u||_{H_4(Q,\delta)},$$

which proves the right-hand-side inequality in (1.6). To prove the left-hand-side inequality it suffices to prove that for each $k = 0, 1, ..., m$ there exists a number $C_3^k = C_3^k(\delta_0, m) > 0$ such that for all $\delta \in (0, \delta_0)$ and $u \in W_\delta^m$

$$\sum_{m-k \le |\alpha| \le m} ||Q^{(\alpha)}(D)u||_{L_{2,\,\delta}} \le C_3^k \sum_{m-k \le |\alpha| \le m} ||Q^{(\alpha)}(D)(u\, g_\delta)||_{L_2}. \tag{1.7_k}$$

We prove by induction in k. Since $Q^{(\alpha)}(\xi) \equiv \mathit{const}$ for any $\alpha \in \mathbb{N}_0^n : |\alpha| = m$, inequality (1.7_0) is obvious. Assume that inequalities (1.7_k) hold for all $k \le k_0 < m$; we will prove it for $k = k_0 + 1$. By applying the Leibnitz formula, the properties of the function g and the induction hypothesis we have with a constant $C_4^{k_0} = C_4^{k_0}(\delta_0, m) > 0$

$$\sum_{m-(k_0+1) \le |\alpha| \le m} ||Q^{(\alpha)}(D)u||_{L_{2,\,\delta}} = \sum_{m-k_0 \le |\alpha| \le m} ||[Q^{(\alpha)}(D)u]||_{L_{2,\,\delta}}$$

$$+ \sum_{|\alpha| = m-(k_0+1)} ||[Q^{(\alpha)}(D)u]||_{L_{2,\,\delta}} = \sum_{m-k_0 \le |\alpha| \le m} ||[Q^{(\alpha)}(D)u]||_{L_{2,\,\delta}}$$

$$+ \sum_{|\alpha| = m-(k_0+1)} ||[Q^{(\alpha)}(D)(u\, g_\delta) - \sum_{\beta \ne 0} \frac{1}{\beta!}[Q^{(\alpha+\beta)}(D)u]\, [D^\beta g_\delta]||_{L_2}$$

$$\le C_3^{k_0} \sum_{|\alpha| \ge m-k_0} ||[Q^{(\alpha)}(D)(u\, g_\delta)||_{L_2} + \sum_{|\alpha| = m-(k_0+1)} ||Q^{(\alpha)}(D)(u g_\delta)||_{L_2}$$

$$+ \sum_{|\alpha| = m-(k_0+1)} \sum_{\beta \ne 0} \frac{\delta^{|\beta|}}{\beta!}||[Q^{(\alpha+\beta)}(D)u]\, (D^\beta g)_\delta||_{L_2}$$

$$\le C_4^{k_0} \sum_{m-(k_0+1) \le |\alpha| \le m} ||[Q^{(\alpha)}(D)(u\, g_\delta)||_{L_2},$$

which proves (1.7_k) and completes the proof. □

Definition 1. We say that the differential operator $R_1(D)$ (the polynomial $R_1(\xi)$) with constant coefficients is **more powerful** than the differential operator $R_2(D)$ (the polynomial $R_2(\xi)$) and write $R_2 < R_1$ if for some $C > 0$

$$| R_2(\xi) | \leq C[| R_1(\xi) | + 1] \ \forall \xi \in \mathbb{R}^n.$$

If $|R_2(\xi)|/[| R_1(\xi) | + 1] \to 0$ *as* $|\xi| \to \infty$ *we write* $R_2 << R_1$

Definition 2. The differential operator $R(D)$ (the polynomial $R(\xi)$) with constant coefficients is called **almost hypoelliptic** (see [10] or [5]) if $D^\alpha R < R$ for all $\alpha \in \mathbb{N}_0^n$.

Lemma 1.3. *Let $Q(D)$ be an almost hypoelliptic operator with constant coefficients of order m. There exist numbers $\delta_0 = \delta_0(Q) \in (0,1)$ and $c = c(Q) > 0$ such that for any $\delta \in (0, \delta_0)$ and all $u \in W_\delta^m$*

$$c^{-1} \, ||u||_{H_2(Q,\delta)} \leq ||u||_{H_1(Q,\delta)} \leq c \, ||u||_{H_2(Q,\delta)} \tag{1.8}$$

Proof. It is obvious that $||u||_{H_1(Q,\delta)} \leq ||u||_{H_4(Q,\delta)}$ for any $u \in W_\delta^m$, hence by Lemma 1.2 $||u||_{H_1(Q,\delta)} \leq C_1 \, ||u||_{H_3(Q,\delta)}$ with a constant $C_1 > 0$.

From this, applying the Fourier transform, Parseval's equality and almost hypoellipticity of $Q(D)$, we have with a constant $C_2 > 0$ for all $u \in W_\delta^m$

$$||u||_{H_1(Q,\delta)} \leq C_1 \, ||u||_{H_3(Q,\delta)} = C_1 \sum_\alpha ||Q^{(\alpha)}(\xi) F(u \, g_\delta)||_{L_2} \leq C_2 \, ||[|Q(\xi)|$$
$$+1] F(u \, g_\delta)||_{L_2} \leq C_2 \, [||| Q(D)(u \, g_\delta)||_{L_2} + ||F(u \, g_\delta)||_{L_2}] = C_2 \, ||u||_{H_2(Q,\delta)},$$

where $F(f)$ is the Fourier transform of a function $f \in L_2$. This proves the right - hand side of (1.8).

To prove the left-hand side of (1.8), we apply the Leibnitz formula and properties (1.1) of the function g. We have with a constant $C_3 > 0$ for all $u \in W_\delta^m$

$$||u||_{H_2(Q,\delta)} = ||u \, g_\delta||_{L_2} + ||Q(D)(u \, g_\delta)||_{L_2} \leq ||u \, g_\delta||_{L_2} + ||[Q(D)u] \, g_\delta)||_{L_2}$$
$$+ \sum_{\alpha \neq 0} \frac{1}{\alpha!} ||[Q^{(\alpha)}(D)u] \, D^\alpha g_\delta)||_{L_2} \leq ||u||_{H_1(Q,\delta)}$$
$$+C_3 \sum_{\alpha \neq 0} \frac{\delta^{|\alpha|}}{\alpha!} ||[Q^{(\alpha)}(D)u] \, g_\delta)||_{L_2}. \tag{1.9}$$

By Lemma 1.2 from here we have with a constant $C_4 > 0$ for any $\delta \in (0, 1)$ and for all $u \in W_\delta^m$

$$||u||_{H_2(Q,\delta)} \leq ||u||_{H_1(Q,\delta)} + \delta C_3 \sum_{\alpha \neq 0} ||[Q^{(\alpha)}(D)u] \, g_\delta)||_{L_2}$$
$$\leq ||u||_{H_1(Q,\delta)} + \delta \, C_3 \, ||u||_{H_4(Q,\delta)} \leq ||u||_{H_1(Q,\delta)} + \delta \, C_4 \, ||u||_{H_3(Q,\delta)}$$
$$= ||u||_{H_1(Q,\delta)} + \delta \, C_4 \, [\sum_\alpha ||[Q^{(\alpha)}(D)(u \, g)||_{L_2} + ||u \, g||_{L_2}].$$

From here, applying the Fourier transform, Parseval's equality and almost hypoellipticity of $Q(D)$, we have with a positive constant C_5 for all $u \in W_\delta^m$

$$||u||_{H_2(Q,\delta)} \leq ||u||_{H_1(Q,\delta)} + \delta\, C_4 \left[\sum_\alpha ||Q^{(\alpha)}(\xi) F(u\, g_\delta)(\xi)||_{L_2} + ||u\, g_\delta||_{L_2} \right]$$

$$\leq ||u||_{H_1(Q,\delta)} + \delta\, C_5\, [||Q(\xi) F(u\, g_\delta)||_{L_2} + ||F(u\, g_\delta)||_{L_2} + \delta\, C_4\, ||u\, g_\delta||_{L_2}$$

$$= ||u||_{H_1(Q,\delta)} + \delta\, C_5\, [||Q(D)(u\, g_\delta)||_{L_2} + \delta\,(C_4 + C_5)\, ||u\, g_\delta||_{L_2}.$$

Hence for any $\delta \in (0, 1/2\, C_5)$ we have with a constant $C_6 > 0$ for all $u \in W_\delta^m$

$$||u||_{H_2(Q,\delta)} \leq 2\, ||u||_{H_1(Q,\delta)} + \frac{C_4}{C_5}\, ||u\, g_\delta||_{L_2} \leq C_6\, ||u||_{H_1(Q,\delta)},$$

from which the left-hand side of (1.8) immediately follows. □

For any $\delta > 0$ we denote $W_\delta^\infty = \bigcap_{k=0}^{\infty} W_\delta^k$.

Lemma 1.4. *1) For any $\delta > 0$ and any linear differential operator $Q(D)$ with constant coefficients the set W_δ^∞ is dense in $H_j(Q,\delta)$ $(j = 1, 3, 4)$.*

2) For almost hypoelliptic operator $Q(D)$ there exist a number $\delta_0 > 0$ such that W_δ^∞ is dense in $H_2(Q,\delta)$ for any $\delta \in (0, \delta_0)$.

Proof. Since the density W_δ^∞ in $H_j(Q,\delta)$ $j = 1, 3, 4$ is proved by the same method, we prove this density only in H_1.

Let $u \in H_1(Q,\delta)$, i.e. $u \in L_{2,\,\delta}$ and $Q(D)u \in L_{2,\,\delta}$. Then by Statement 1) of Lemma 1.1 $u_\varepsilon \in W_\delta^\infty$ and (see. Lemma 5.2 in [1]) $Q(D)u_\varepsilon = (Q(D)u)_\varepsilon \in W_\delta^\infty$ for any $\varepsilon > 0$. Therefore by statement 2) of Lemma 1.1 we obtain

$$||u_\varepsilon - u||_{H_1(Q,\delta)} = ||u_\varepsilon - u||_{L_{2,\,\delta}} + ||Q(D)u_\varepsilon - Q(D)u||_{L_{2,\,\delta}} \to 0$$

as $\varepsilon \to 0$, which proves the first part of the lemma.

The second part of the lemma is proved similarly by applying Lemma 1.3. □

As an immediate corollary of Lemmas 1.2 - 1.4 and L. Hörmander's theorem (see [8] or [12, Theorem 12.2 and Corollary 2]) we get

Corollary 1.1. *Let $Q(D)$ be an almost hypoelliptic operator. There exists a number $\delta_0 = \delta_0(Q) > 0$ such that all spaces $H_j(Q,\delta)$ $(j = 1, ..., 4)$ coincide for any $\delta \in (0, \delta_0)$.* □

2 Some properties of almost hypoelliptic polynomials with constant coefficients

We begin with some results on almost hypoelliptic polynomials with real constant coefficients. Denote by $Pol(n,m)$ the set of all polynomials P in n variables of deg $P \leq m$ and by $I_n = I_{n,m}$ the set of all polynomials $P \in Pol(n,m)$ satisfying the condition:

$$|P(\xi)| \to \infty \quad \text{as} \quad |\xi| \to \infty.$$

For any $P \in I_n$ with $n > 1$, up to multiplying by -1, there exist positive constants $\varepsilon_0 = \varepsilon_0(P)$ and $M_0 = M_0(P)$ such that

$$P(\xi) \geq \varepsilon_0 \ \ \forall \xi \in \mathbb{R}^n : |\,\xi| \geq M_0. \tag{2.1}$$

Therefore, in the sequel, without loss of generality, we can assume that any polynomial $P \in I_n$ satisfies condition (2.1).

Let $P \in Pol(n, m)$, $\xi \in \mathbb{R}^n$ and $d_P(\xi)$ denote the distance from ξ to the surface $D(P) = \{\zeta \in \mathbb{C}^n : P(\zeta) = 0\}$. Then there exists a constant $c = c(m, n) > 0$ such that for all polynomials $P \in Pol(n, m)$ and for all $\xi \in \mathbb{R}^n$ such that $P(\xi) \neq 0$ we have

$$c^{-1} \leq d_P(\xi) \sum_{\alpha \neq 0} |P^{(\alpha)}(\xi)/P(\xi)|^{1/|\alpha|} \leq c \tag{2.2}$$

(see [8], Lemma 11.1.4). Hence for any almost hypoelliptic polynomial $P \in I_n$ there exists a positive constant $\varepsilon_1 = \varepsilon_1(P)$ such that

$$d_P(\xi) \geq \varepsilon_1 \ \ \forall \xi \in \mathbb{R}^n : |\,\xi| \geq M_0. \tag{2.3}$$

Lemma 2.1 *Let $P \in I_n$ be an almost hypoelliptic polynomial and let $P < Q < P$. Then*

1) $Q \in I_n$ and Q is almost hypoelliptic,

2) there exists a number $\theta > 0$ such that

$$\theta^{-1} \leq d_P(\xi)/d_Q(\xi) \leq \theta \ \ \forall \xi \in \mathbb{R}^n : |\,\xi| \geq M_0, \tag{2.4}$$

3) *there exist positive numbers c_1, c_2, and δ_1 such that for any $\delta \in (0, \delta_1)$*

$$||Q(D)u||_{L_{2,\,\delta}} = ||[Q(D)u]\, g_\delta||_{L_2} \leq c_1\, ||u||_{H_2(P,\delta)} \ \ \forall u \in H_2(P, \delta), \tag{2.5}$$

$$||Q(D)(u\, g_\delta)||_{L_2} \leq c_2\, ||u||_{H_2(P,\delta)} \ \ \forall u \in H_2(P, \delta). \tag{2.6}$$

Proof. The first part of Statement 1) is obvious. To prove the second part of Statement 1) note that by Lemma 10.4.2 in [8] there exsists a constant $C_1 > 0$ such that for every polynomial $R \in Pol(n, m)$,

$$C_1^{-1}\, \tilde{R}(\xi, t) \leq \sup_{|\eta|<t} |R(\xi + \eta)| \leq C_1\, \tilde{R}(\xi, t) \ \ \forall \xi \in \mathbb{R}^n, t > 0, \tag{2.7}$$

where we have used the Hörmander function

$$\tilde{R}(\xi, t) = [\sum_{\alpha} |R^{\alpha}(\xi)|^2 t^{2\,|\alpha|}]^{1/2}, \ \ \tilde{R}(\xi) = \tilde{R}(\xi, 1).$$

Since the polynomial $P \in I_n$ is almost hypoelliptic and the function $d_P(\xi)$ satisfies inequality (2.3), hence one can rewrite inequality (2.7) for polynomial P as

$$C_1^{-1}\, \tilde{P}(\xi, t) \leq \sup_{|\eta|<t} |P(\xi + \eta)| \leq C_1\, \tilde{P}(\xi, t) \ \ \forall \xi \in \mathbb{R}^n \tag{2.7'}$$

for any $t = d_P(\xi) \geq \varepsilon$ with some $\varepsilon > 0$.

Hence by the assumptions of the lemma we have with some constants $\mathbb{C}_j > 0$ $(j = 2, ..., 7)$ for all $\xi \in \mathbb{R}^n : |\xi| \geq M_0 + 1$

$$\tilde{Q}(\xi) \leq C_2 \sup_{|\eta|<1} |Q(\xi+\eta)| \leq C_3 \sup_{|\eta|<1} [|P(\xi+\eta)| + 1] \leq C_4\, \tilde{P}(\xi) \leq$$

$$\leq C_5\,(|P(\xi)| + 1) \leq C_6\,|P(\xi)| \leq C_7\,[|Q(\xi)| + 1],$$

which proves almost hypoellipticity of Q.

Since by (2.3) $d_P(\xi) \geq \varepsilon_1$ for $|\,\xi| \geq M_0$, to prove inequality (2.4) we can once more use inequality (2.7') for $t = d_P(\xi)$. We obtain with some constants $C_j > 0$ $(j = 8, ..., 13)$ for $|\,\xi| \geq M_0$

$$\tilde{Q}(\xi, d_P(\xi)) \leq C_8 \sup_{|\eta|<d_P(\xi)} |Q(\xi+\eta)| \leq C_9 \sup_{|\eta|<d_P(\xi)} [|P(\xi+\eta)| + 1] \leq$$

$$\leq C_{10}\,[\tilde{P}(\xi, d_P(\xi)) + 1] \leq C_{11}\,[|P(\xi)| + 1] \leq C_{12}\,|P(\xi)| \leq$$

$$\leq C_{13}\,|Q(\xi)| \;\; \forall \xi \in \mathbb{R}^n : |\xi| \geq M_0.$$

Since the functions $\tilde{Q}(\xi, t)$ and $\sum_\alpha |Q^{(\alpha)}(\xi)| t^{|\alpha|}$ are equivalent, we can rewrite the last inequality as

$$\sum_\alpha |Q^{(\alpha)}(\xi)|\, d_P^{|\alpha|}(\xi) \leq C_{13}\,|Q(\xi)| \;\; \forall \xi \in \mathbb{R}^n : |\,\xi| \geq M_0.$$

Then for any $0 \neq \alpha \in \mathbb{N}_0^n$ and for these ξ we have

$$d_P^{|\alpha|}(\xi)\,|Q^{(\alpha)}(\xi)/Q(\xi)| \leq C_{13}.$$

This implies that

$$d_P(\xi) \sum_{\alpha \neq 0} |Q^{(\alpha)}(\xi)/Q(\xi)|^{1/|\alpha|} \leq C_{14} \;\; \forall \xi \in \mathbb{R}^n : |\,\xi| \geq M_0$$

for a constant $C_{14} > 0$.

Applying inequality (2.2) for the polynomial Q from here we get for any $\xi \in \mathbb{R}^n$: $|\,\xi| \geq M_0$ with the constant $C_{15} = 1/(c\,C_{14})$

$$d_Q(\xi) \geq \frac{1}{c \left[\sum_{\alpha \neq 0} |Q^{(\alpha)}(\xi)/Q(\xi)|^{1/|\alpha|} \right]} \geq C_{15}\, d_P(\xi).$$

In this connection note that for any polynomial Q with degree $Q \geq 1$

$$\sum_{\alpha \neq 0} |Q^{(\alpha)}(\xi)/Q(\xi)|^{1/|\alpha|} \neq 0 \;\; \text{for}\, \xi \in \mathbb{R}^n.$$

In the same manner we can see that $d_P(\xi) \geq C_{16}\, d_Q(\xi)$ with a constant $C_{16} > 0$ and for the same $\xi \in \mathbb{R}^n$. Last inequalities lead to inequality (2.4).

Inequality (2.6) immediately follows by the condition $Q < P$ by applying Parseval's equality. To prove inequality (2.5) first note that by (2.7) for $t = 1$ we have with some positive constants C_j $(j = 17, ..., 20)$

$$\tilde{Q}(\xi) \le C_{17} \sup_{|\eta|\le 1} |Q(\xi+\eta)| \le C_{18} \sup_{|\eta|\le 1} [|P(\xi+\eta)|+1] \le$$
$$\le C_{19}\, \tilde{P}(\xi) \le C_{20}\, [|P(\xi)|+1] \ \ \forall \xi \in \mathbb{R}^n. \tag{2.8}$$

Let number $\delta_0 \in (0, 1)$ be as in Corollary 1.1. Applying Corollary 1.1, Parseval's equality and inequality (2.8) we get with some positive constants C_{21}, C_{22} for all $\delta \in (0, \delta_0)$

$$||Q(D)u||_{L_{2,\,\delta}} \le ||u||_{H_4(Q,\delta)} \le C_{21}\, ||u||_{H_3(Q,\delta)} = C_{21} \sum_{\alpha} ||Q^{(\alpha)}(D)(u\, g_\delta)||_{L_2}$$
$$= C_{21} \sum_{\alpha} ||Q^{(\alpha)}(\xi) F(u\, g_\delta)||_{L_2} \le C_{21}\, [||P(\xi)F(u\, g_\delta)||_{L_2} + ||F(u\, g_\delta)||_{L_2}]$$
$$= C_{22}\, ||u||_{H_2(P,\delta)} \ \ \forall u \in H_2(P, \delta).$$

□

Lemma 2.2. *Let a polynomial $Q(\xi)$ satisfy the condition*

$$|\xi| \le c[|Q(\xi)|+1] \ \ \forall \xi \in \mathbb{R}^n \tag{2.9}$$

with a number $c = c(Q) > 0$. Then for any $\Delta > 0$ and $k \in \mathbb{N}$ there exists a number $C = C(Q, \Delta, k) > 0$ such that for any $\delta \in (0, \Delta)$ and for all $u \in W_\delta^\infty$

$$\sum_{|\alpha|=k} ||D^\alpha u||_{L_{2,\,\delta}} \le C \left[\sum_{|\beta|=k-1} ||D^\beta u||_{H_2(Q,\delta)} + \sum_{|\gamma|=k-1} ||D^\gamma u||_{L_{2,\,\delta}} \right]. \tag{2.10}$$

Proof. Let $\alpha \in \mathbb{N}_0^n$, $|\alpha| = k$. Represent α as $\alpha = \mu_\alpha + \nu_\alpha$, where $|\mu_\alpha| = k-1$, $|\nu_\alpha| = 1$. Then using the properties of the function g, we get with a constant $C_1 > 0$

$$\sum_{|\alpha|=k} ||D^\alpha u||_{L_{2,\,\delta}} = \sum_{|\alpha|=k} ||(D^{\mu_\alpha+\nu_\alpha} u)\, g_\delta||_{L_2} \le \sum_{|\alpha|=k} ||D^{\nu_\alpha}[(D^{\mu_\alpha} u)\, g_\delta]$$
$$-(D^{\mu_\alpha} u)\, D^{\nu_\alpha}\, g_\delta||_{L_2} \le \sum_{|\alpha|=k} ||D^{\nu_\alpha}[(D^{\mu_\alpha} u)\, g_\delta]||_{L_2} + C_1\, \delta \sum_{|\alpha|=k-1} ||D^{\mu_\alpha} u||_{L_{2,\,\delta}}.$$

Applying the Fourier transform and the Parseval equality, by the properties of the polynomial $Q(\xi)$ we get with some positive constants C_2 and C_3

$$\sum_{|\alpha|=k} ||D^\alpha u||_{L_{2,\,\delta}} \le \sum_{|\alpha|=k} |||\xi^{\nu_\alpha}|\, F[(D^{\mu_\alpha} u)\, g_\delta]|\, ||_{L_2} + C_1\, \delta \sum_{|\alpha|=k} ||D^{\gamma_\alpha} u||_{L_{2,\,\delta}}$$
$$\le C_2 \sum_{|\alpha|=k} ||[|Q(\xi)|+1]\, |F[(D^{\mu_\alpha} u)\, g_\delta]|\, ||_{L_2} + C_1\, \delta \sum_{|\alpha|=k-1} ||D^{\mu_\alpha} u||_{L_{2,\,\delta}}$$
$$\le C_3\, [\sum_{|\beta|=k-1} ||D^\beta u||_{H_2(Q,\delta)} + \sum_{|\mu|=k-1} ||D^\mu u||_{L_{2,\,\delta}}] \ \ \forall u \in W_\delta^\infty.$$

3 Differential operators with variable coefficients

Let

$$P(x,D)=\sum_{\alpha\in(P,x)}\gamma_\alpha(x)D^\alpha$$

be a linear differential operator with coefficients, defined on a domain $\Omega\subset\mathbb{R}^n$ and

$$P(x,\xi)=\sum_{\alpha\in(P,x)}\gamma_\alpha(x)\xi^\alpha$$

be its characteristic polynomial (complete symbol), where (P,x) is a finite set in $\mathbb{N}_0^n$.

It is assumed that for any $\alpha\in(P,x)$ there exists $x\in\Omega$ such that $\gamma_\alpha(x)\neq 0$.

Definition 3. A linear differential operator $P(x,D)$ (and the corresponding polynomial $P(x,\xi)$ in ξ) with the coefficients defined in $\Omega\subset\mathbb{R}^n$ is said to have constant power in Ω (see [16] or [11]) if for arbitrary $x,y\in\Omega$ there exists a constant $C(x,y)>0$ such that

$$|P(x,\xi)|/[|P(y,\xi)|+1]\leq C(x,y)$$

for all $\xi\in\mathbb{R}^n$, or, which is the same, the polynomials $P(x,\cdot)$ and $P(y,\cdot)$ have the same power:

$$P(x,\cdot)<P(y,\cdot)<P(x,\cdot).$$

If there exists a constant $C>0$ such that

$$C^{-1}\leq[|P(x,\xi)|+1]/[|P(y,\xi)|+1]\leq C$$

for all $x,y\in\Omega$ and $\xi\in\mathbb{R}^n$, we say that the operator $P(x,D)$ (polynomial $P(x,\xi)$) has uniformly constant power in Ω.

In [3] and [11] there were found some conditions under which a polynomial $P(x,\xi)$ has (uniformly) constant power in Ω.

An operator $P(x,D)$ (a polynomial $P(x,\xi)$)) with constant power we call **formally almost hypoelliptic** in Ω, if the operator $P(x^0,D)$ with constant coefficients is almost hypoelliptic (see Definition 1.2) for any $x^0\in\Omega$.

We begin with a simple but general result on linear differential operators with constant powers (for operators with constant strength in the sense of Hörmander see [9], Lemma 11.3.2. and [2])

Lemma 3.1. *Let $P(x,D)$ have constant power in Ω. With a fixed $x^0\in\Omega$ set $P_0(D)=P(x^0,D)$ and let $P_j(D)$ $(j=0,1,...,r)$ be a basis in the finite dimensional vector space of operators with constant coefficients which are less powerful than $P_0(D)$. Then for all $x\in\Omega$ we have*

$$P(x,D)=P_0(D)+\sum_{j=0}^{r}a_j(x)\,P_j(D),\tag{3.1$'$}$$

where the coefficients $a_j(x)=a_j(x^0,x)$ are uniquely determined, vanish at x^0, and have the same differentiability and continuity properties as the coefficients of $P(x,D)$.

Let as above I_n denote the set of polynomials $R(\xi) = R(\xi_1, ..., \xi_n)$ such that $|R(\xi)| \to \infty$ as $|\xi| \to \infty$, and let $I_n(\Omega)$ be the set of polynomials $R(x,\xi) = R(x, \xi_1, ..., \xi_n)$ with coefficients defined in Ω, such that $R(x, \cdot) \in I_n$ for all $x \in \Omega$ (see, for instanse [14], [15], or [6])

In [6] there were found some conditions under which an almost hypoelliptic polynomial $R \in I_n$.

Consider the class $A = A(\mathbb{R}^n)$ of all formally almost hypoelliptic operators $P(x,D) = \sum_\alpha \gamma_\alpha(x) D^\alpha$ (polynomials $P(x,\xi)$) with coefficients in $C^\infty = C^\infty(\mathbb{R}^n)$ and with constant power in $\mathbb{R}^n$, satisfying the following conditions: there exists a point $x^0 \in \mathbb{R}^n$ such that in representation (3.1'):

1) $P_0 \in I_n$, and $P_j << P_0$ $(j = 1, ..., r)$, (see Definition 1.1),

2) $a_0(x) \equiv 0$, i.e. the operator P has the form

$$P(x,D) = P_0(D) + \sum_{j=1}^{r} a_j(x)\, P_j(D), \tag{3.1}$$

3) for any $\alpha \in \mathbb{N}_0^n$ there exists a number $c_\alpha > 0$ such that

$$|D^\alpha a_j(x)| \le c_\alpha \;\; \forall x \in \mathbb{R}^n \;\, (j = 1, ..., r). \tag{3.2}$$

4) there exists a number $c > 0$ such that

$$|\xi| \le c\,[1 + |P_0(\xi)| \;\forall \xi \in \mathbb{R}^n. \tag{3.3}$$

To give an example of such operator (polynomial) we present some futher concepts:

Definition 4. (see [13] or [7]) The Newton polyhedron $\Re(\aleph)$ of a given collection of multi-indices $\aleph = \{\alpha^j\}_1^N$ is the smallest convex polyhedron in $\mathbb{R}^{n,+}$ containing all multi-indices α^j $(j = 1, ..., N)$. The Newton polyhedron $\Re(x,P)$ of an operator $P(x,D)$ (and a polynomial $P(x,\xi)$) is, by definition, the Newton polyhedron of the collection (P,x) (see [13] or [11]).

A polyhedron $\Re$ with vertices in $\mathbb{N}_0^n$ is called complete, if $\Re$ has a vertex at the origin and also it has vertices on each coordinate axis.

Let $\Re$ be a complete polyhedron. A set $\Gamma \subset \Re$ is called a face of $\Re$, if there exist a unit vector $\lambda = (\lambda_1, ..., \lambda_n)$ and a number $d = d(\lambda, \Gamma) \ge 0$ such that $(\lambda, \alpha) = (\lambda_1\alpha_1 + ... + \lambda_n\alpha_n) = d$ for all points $\alpha \in \Gamma$, while $(\lambda, \beta) < d$ for $\beta \in \Re \setminus \Gamma$. The unit vector λ is called an outword normal ($\Re$ - normal) of the face Γ. The set of all $\Re$ - normals of Γ we denote by $\Lambda(\Gamma)$.

Definition 5. A face Γ of a complete polyhedron $\Re$ is called **principal**, if there exists a $\lambda \in \Lambda(\Gamma)$ with at least one positive coordinate. If in $\Lambda(\Gamma)$ there exists a λ with nonnegative (positive) coordinates, then we call the face Γ **regular** (**completely regular**). A point $\alpha \in \Re$ is called principal (regular, completely regular) if α belongs to a principal (regular, completely regular) face of $\Re$. A complete polyhedron $\Re$ is called **regular** (**completely regular**) if all its $(n-1)-$dimensiomal non-coordinate faces are regular (completely regular). The set of all principal points $\alpha \in \Re \cap N_0^n$ is denoted by $\Re'$.

It is proved in [4] that

1) the Newton polyhedron $\Re(x,P)$ of an operator $P(x,D)$ (and a polynomial $P(x,\xi)$) with constant power in Ω does not depend on the point $x\in\Omega:\ \Re(x,P)=\Re(P)\ \forall x\in\Omega$,

2) if an operator $P(x,D)$ with constant power in Ω is formally almost hypoelliptic in Ω then $\Re(P)$ is regular.

Example 3.1 Let $\Re=\Re(P_0)$ be the regular Newton polyhedron of a polynomial P_0. Let there exist a set of multi-indices $B\subset\Re\cap\mathbb{N}_0^n$ and a number $\sigma=\sigma(\Re,B)>0$ such that

$$\sum_{\nu\in B}|\xi^\nu|\le\sigma[|P_0(\xi)|+1]\ \ \forall\xi\in R^n. \tag{3.4}$$

Estimate (3.4) for $B=\Re$ have been proved by V.P. Mikhailov in [13] for so-called non-degenerate polynomials. In [3], for degenerate polynomials, conditions were found under which estimate (3.4) is true for a set $B\subset\subset\Re$ with the complete Newton polyhedron $\Re(B)$ and conditions under which P_0 is almost hypoelliptic.

When the Newton polyhedron $\Re(B)$ is complete (see, for instance [11], or [3]) and $\Re(P_j)\subset\Re'(B)$ $(j=1,...,r)$ then is easy to verify that $P_0\in I_n$ and $P_j<<P_0$ $(j=1,...,r)$. Thus, any formally almost hypoelliptic in Ω operator $P(x,D)$, satisfying estimate (3.4) and the conditions $\Re(P_j)\subset\Re'(B)$ $(j=1,...,r)$, with complete polyhedrons $\Re(P)$ and $\Re(B)$, and represented in form (3.1) belongs to A.

In the sequel we assume (see Corollary 1.1) that the number $\delta_0:=\delta_0(P_0)$ is fixed, so that all spaces $H_j(P_0,\delta)$ $(j=1,...,4)$ coincide for any $\delta\in(0,\delta_0)$ and denote $H(P_0,\delta)\equiv H_j(P_0,\delta)$ $(j=1,...,4)$.

Theorem 3.1. *Let $P\in A$ be an operator, represented in form (2.1). There exists a number $c>0$ such that for any $\delta\in(0,\delta_0)$ and for all $u\in H(P_0,\delta)$*

$$c^{-1}\,||u||_{H(P_0,\,\delta)}\le||P(x,D)u||_{L_{2,\,\delta}}+||u||_{L_{2,\,\delta}}\le c\,||u||_{H(P_0,\,\delta)}, \tag{3.5}$$

$$c^{-1}\,||u||_{H(P_0,\,\delta)}\le||P(x,D)(u\,g_\delta)||_{L_2}+||u\,g_\delta||_{L_2}\le c\,||u||_{H(P_0,\,\delta)}. \tag{3.6}$$

Proof. By Corollary 1.1 it suffices to prove estimates (3.5) - (3.6) for $H_2(P_0,\,\delta)$. The right-hand sides of (3.5) - (3.6) immediately follow by uniform boundedness of the coefficients $\{a_j(x)\}$ and Corollary 1.1. Let us prove the left-hand sides of these estimates.

Since $P_j<<P_0$ $(j=1,...,r)$, for any $\varepsilon>0$ there exists a constant $C_1=C_1(\varepsilon,\max_j\sup_x|a_j(x)|)>0$ such that for any $\delta\in(0,\delta_0)$ and for all $u\in H_2(P_0,\,\delta)$

$$\sum_{j=1}^r||a_j(x)\,P_j(D)(u\,g_\delta)||_{L_2}\le\varepsilon\,||P_0(D)(u\,g_\delta)||_{L_2}+C_1\,||(u\,g_\delta)||_{L_2}.$$

Therefore for all $u\in H_2(P_0,\,\delta)$

$$||P(x,D)(u\,g_\delta)||_{L_2}\ge||P_0(D)(u\,g_\delta)||_{L_2}-\varepsilon\,||P_0(D)(u\,g_\delta)||_{L_2}-C_1\,||(u\,g_\delta)||_{L_2}.$$

This implies the left-hand side of (3.6) if we take $\varepsilon<1$. Let us prove the left-hand sides of (3.5).

By assumption of the theorem $P_j<<P_0$ $(j=1,...,r)$, consequently $D^\alpha P_j<<P_0$ $(j=1,...,r)$ for any $\alpha\in\mathbb{N}_0^n$, and we have that with some constant $C_2>0$

$$\sum_{j=1}^{r} ||a_j(x)\, P_j(D)u||_{L_{2,\,\delta}} \le C_2 \sum_{j=1}^{r} ||P_j(D)u||_{L_{2,\,\delta}}$$

$$\le C_2 \sum_{j=1}^{r} ||u||_{H_3(P_j,\,\delta)} \ \ \forall u \in W,$$

hence for any $\varepsilon > 0$ there is a number $C_3 = C_3(\varepsilon) > 0$ such that for all $u \in W_\delta^\infty$ we have (see Corollary 1.1)

$$||P(x, D)u||_{L_{2,\,\delta}} \ge ||P_0(D)u||_{L_{2,\,\delta}} - C_2 \sum_{j=1}^{r} ||u||_{H_3(P_j,\,\delta)}$$

$$\ge ||P_0(D)u||_{L_{2,\,\delta}} - \varepsilon\, r\, C_2\, ||P_0(D)(u\, g_\delta)||_{L_{2,\,\delta}} - r\, C_2\, C_3 ||u\, g_\delta||_{L_2}.$$

Applying once more Corollary 1.1, with a constant $C_4 > 0$ we have that for all $u \in W_\delta^\infty$

$$||P(x, D)u||_{L_{2,\,\delta}} \ge ||P_0(D)u||_{L_{2,\,\delta}} - \varepsilon\, C_4\, ||P_0(D)u||_{L_{2,\,\delta}} - C_4\, C_3\, ||u\, g_\delta||_{L_2}.$$

Since by Lemma 1.4 the set W_δ^∞ is dense in $W(P_0,\, \delta)$, taking $\varepsilon \in (0, 1/(2C_4))$, from here we get the left-hand side of (3.5). □

Theorem 3.2. *Let P_0 be an operator, satisfying the assumptions of Teorem 3.1, and $\alpha \in \mathbb{N}_0^n$. Then there exist numbers $\delta_1 \in (0, \delta_0)$ and $C = C(\alpha, \delta_0) > 0$ such that for any $\delta \in (0, \delta_1)$*

$$||D^\alpha u||_{H(P_0,\,\delta)} \le C\, [||D^\alpha[P(x, D)u]||_{L_{2,\,\delta}} + \sum_{\gamma \le \alpha} ||D^\alpha u||_{L_{2,\,\delta}}$$

$$+ \sum_{0 \ne \gamma \le \alpha} ||D^{\alpha-\gamma} u||_{H(P_0,\,\delta)}] \ \ \forall u \in W_\delta^\infty. \tag{3.7}$$

Proof. By Corollary 1.1 we can prove this inequality for $H(P_0,\, \delta) = H_2(P_0,\, \delta)$. Estimate (3.7) for $\alpha = 0$ immediately follows from estimate (3.5). Let $0 \ne \alpha \in \mathbb{N}_0^n$. Since $D^\alpha u \in W_\delta^\infty$ by (3.5) we have with a constant $C_1 > 0$

$$||P(x, D)[(D^\alpha u) g_\delta]||_{L_2} + ||D^\alpha u||_{L_{2,\,\delta}} \ge C_1\, ||D^\alpha u||_{H_2(P_0,\,\delta)} \ \ \forall u \in W_\delta^\infty. \tag{3.8}$$

On the other hand, applying the properties of the function g and representation (3.1), by the Leibnitz formula we get with a constant $C_2 > 0$ for all $u \in W_\delta^\infty$

$$||P(x, D)[(D^\alpha u) g_\delta]||_{L_2} + ||D^\alpha u||_{L_{2,\,\delta}} \le ||[P(x, D)(D^\alpha u)] g_\delta]||_{L_2}$$

$$+ \sum_{\beta \ne 0} \frac{1}{\beta!} ||[P^{(\beta)}(x, D)\, D^\alpha u]\, D^\beta g_\delta||_{L_2} + ||D^\alpha u\, g_\delta||_{L_2}$$

$$\leq ||D^{\alpha}[P(x,D)u]\, g_{\delta}||_{L_2} + \sum_{j=1}^{r} \sum_{0\neq\gamma\leq\alpha} ||[a_j^{\gamma}(x) D^{\alpha-\gamma} P_j(D)u]\, g_{\delta}||_{L_2}$$

$$+C_2 \sum_{\beta\neq 0} \frac{\delta^{|\beta|}}{\beta!} ||[P^{(\beta)}(x,D) D^{\alpha}u]\, g_{\delta}||_{L_2} + ||(D^{\alpha}u)\, g_{\delta}||_{L_2}. \tag{3.8$'$}$$

Applying properties (3.2) of the coefficients $\{a_j\}$, and Lemma 1.4, for the second term of the right-hand side of (3.8') we get with a constant $C_3 = C_3(\alpha) > 0$ for all $u \in W_{\delta}^{\infty}$

$$\sum_{j=1}^{r} \sum_{0\neq\gamma\leq\alpha} ||a_j^{(\gamma)}(x)[P_j(D) D^{\alpha-\gamma}u]\, g_{\delta}||_{L_2} \leq C_3 \sum_{0\neq\gamma\leq\alpha} ||D^{\alpha-\gamma}u||_{H_2(P_0,\,\delta)}. \tag{3.9}$$

Since $P_j^{(\beta)} < P_0$ $(j = 1, ..., r)$ for any $\beta \in \mathbb{N}_0^n$, by Lemma 1.5 and by properties (3.2) of the coefficients $\{a_j\}$, for the third term of right-hand side of (3.8') we have with a constant $C_4 > 0$ for all $u \in W_{\delta}^{\infty}$

$$C_2 \sum_{\beta\neq 0} \frac{\delta^{|\beta|}}{\beta!} ||[P^{(\beta)}(x,D) D^{\alpha}u]\, g_{\delta}||_{L_2} \leq C_2\, \delta \sum_{\beta\neq 0} [\frac{1}{\beta!} \{[P_0^{\beta}(D)\, D^{\alpha}u]\, g_{\delta}||_{L_2}$$

$$+\sum_{j=1}^{r} ||P_j^{\beta}(D)\, D^{\alpha}u]\, g_{\delta}||_{L_2}\} \leq C_4\, \delta\, [||D^{\alpha}u||_{H_2(P_0,\delta)} + ||D^{\alpha}u||_{L_2,\,\delta}]. \tag{3.10}$$

From (3.8) - (3.10) we get

$$||P(x,D)[(D^{\alpha}u) g_{\delta}]||_{L_2} + ||D^{\alpha}u||_{L_2,\,\delta} \leq ||D^{\alpha}[P(x,D)u]\, g_{\delta}||_{L_2}$$

$$+C_3 \sum_{0\neq\gamma\leq\alpha} ||D^{\alpha-\gamma}u||_{H_2(P_0,\delta)} + C_4\, \delta\, [||D^{\alpha}u||_{H_2(P_0,\delta)} + ||D^{\alpha}u||_{L_2,\,\delta}] + ||D^{\alpha}u||_{L_2,\,\delta}.$$

Choose a number $\delta_1 > 0$ so that $\delta_1 C_4 < C_1/2$, from here and (3.8) we get for any $\delta \in (0, \delta_1)$ and for all $u \in W_{\delta}^{\infty}$

$$\frac{C_1}{2} ||D^{\alpha}u||_{H_2(P_0,\delta)} \leq ||D^{\alpha}[P(x,D)u]\, g_{\delta}||_{L_2}$$

$$+C_3 \sum_{0\neq\gamma\leq\alpha} ||D^{\alpha-\gamma}u||_{H_2(P_0,\delta)} + (\frac{C_1}{2} + 1) \sum_{\gamma\leq\alpha} ||D^{\alpha}u||_{L_2,\,\delta}.$$

This implies estimate (3.7) and Theorem 2.2 is proved. □

By this theorem and Lemma 2.2 we get

Corollary 3.1. *Let P be an operator, satisfying the assunptions of Teorem 3.1, and $k \in \mathbb{N}$. Then there exist positive numbers δ_1 and C such that for any $\delta \in (0, \delta_1)$*

$$\sum_{|\alpha|\leq k} ||D^{\alpha}u||_{H(P_0,\,\delta)} \leq C\, [||D^{\alpha}[P(x,D)u]||_{L_2,\,\delta}$$

$$+ \sum_{|\gamma|\leq k-1} ||D^{\gamma}u||_{H(P_0,\,\delta)} \quad \forall u \in W_{\delta}^{\infty}.$$

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