ISSN 2077–9879

Eurasian Mathematical Journal

2015, Volume 6, Number 4

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia
the University of Padua

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

Aims and Scope

The Eurasian Mathematical Journal (EMJ) publishes carefully selected original research papers in all areas of mathematics written by mathematicians, principally from Europe and Asia. However papers by mathematicians from other continents are also welcome.

From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

The contents of EMJ are indexed in Scopus, Mathematical Reviews, MathSciNet, Zentralblatt Math (ZMATH), Referativnyi Zhurnal – Matematika, Math-Net.Ru.

The EMJ is included in the list of journals recommended by the Committee for Control of Education and Science (Ministry of Education and Science of the Republic of Kazakhstan).

Information for the Authors

Submission. Manuscripts should be written in LaTeX and should be submitted electronically in DVI, PostScript or PDF format to the EMJ Editorial Office via e-mail (eurasianmj@yandex.kz).

When the paper is accepted, the authors will be asked to send the tex-file of the paper to the Editorial Office.

The author who submitted an article for publication will be considered as a corresponding author. Authors may nominate a member of the Editorial Board whom they consider appropriate for the article. However, assignment to that particular editor is not guaranteed.

Copyright. When the paper is accepted, the copyright is automatically transferred to the EMJ. Manuscripts are accepted for review on the understanding that the same work has not been already published (except in the form of an abstract), that it is not under consideration for publication elsewhere, and that it has been approved by all authors.

Title page. The title page should start with the title of the paper and authors' names (no degrees). It should contain the Keywords (no more than 10), the Subject Classification (AMS Mathematics Subject Classification (2010) with primary (and secondary) subject classification codes), and the Abstract (no more than 150 words with minimal use of mathematical symbols).

Figures. Figures should be prepared in a digital form which is suitable for direct reproduction.

References. Bibliographical references should be listed alphabetically at the end of the article. The authors should consult the Mathematical Reviews for the standard abbreviations of journals' names.

Authors' data. The authors' affiliations, addresses and e-mail addresses should be placed after the References.

Proofs. The authors will receive proofs only once. The late return of proofs may result in the paper being published in a later issue.

Offprints. The authors will receive offprints in electronic form.

Publication Ethics and Publication Malpractice

For information on Ethics in publishing and Ethical guidelines for journal publication see http://www.elsevier.com/publishingethics and http://www.elsevier.com/journal-authors/ethics.

Submission of an article to the EMJ implies that the work described has not been published previously (except in the form of an abstract or as part of a published lecture or academic thesis or as an electronic preprint, see http://www.elsevier.com/postingpolicy), that it is not under consideration for publication elsewhere, that its publication is approved by all authors and tacitly or explicitly by the responsible authorities where the work was carried out, and that, if accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright-holder. In particular, translations into English of papers already published in another language are not accepted.

No other forms of scientific misconduct are allowed, such as plagiarism, falsification, fraudulent data, incorrect interpretation of other works, incorrect citations, etc. The EMJ follows the Code of Conduct of the Committee on Publication Ethics (COPE), and follows the COPE Flowcharts for Resolving Cases of Suspected Misconduct ($http://publicationethics.org/files/u2/New_Code.pdf$). To verify originality, your article may be checked by the originality detection service CrossCheck http://www.elsevier.com/editors/plagdetect.

The authors are obliged to participate in peer review process and be ready to provide corrections, clarifications, retractions and apologies when needed. All authors of a paper should have significantly contributed to the research.

The reviewers should provide objective judgments and should point out relevant published works which are not yet cited. Reviewed articles should be treated confidentially. The reviewers will be chosen in such a way that there is no conflict of interests with respect to the research, the authors and/or the research funders.

The editors have complete responsibility and authority to reject or accept a paper, and they will only accept a paper when reasonably certain. They will preserve anonymity of reviewers and promote publication of corrections, clarifications, retractions and apologies when needed. The acceptance of a paper automatically implies the copyright transfer to the EMJ.

The Editorial Board of the EMJ will monitor and safeguard publishing ethics.

Web-page

The web-page of EMJ is www.emj.enu.kz. One can enter the web-page by typing Eurasian Mathematical Journal in any search engine (Google, Yandex, etc.). The archive of the web-page contains all papers published in EMJ (free access).

Subscription

For Institutions

- US$ 200 (or equivalent) for one volume (4 issues)
- US$ 60 (or equivalent) for one issue

For Individuals

- US$ 160 (or equivalent) for one volume (4 issues)
- US$ 50 (or equivalent) for one issue.

The price includes handling and postage.

The Subscription Form for subscribers can be obtained by e-mail:

eurasianmj@yandex.kz

The Eurasian Mathematical Journal (EMJ)
The Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana
Kazakhstan

KORDAN NAURYZKHANOVICH OSPANOV

(to the 60th birthday)

On 25 September 2015 Kordan Nauryzhanovich Ospanov, professor of the Department "Fundamental Mathematics" of the L.N. Gumilyov Eurasian National University, Doctor of Physical and Mathematical Sciences (2000), a member of the Editorial Board of our journal, celebrated his 60th birthday.

He was born on September 25, 1955, in the village Zhanatalap of the Zhanaarka district of the Karaganda region. In 1976 he graduated from the Kazakh State University, and in 1981 he completed his postgraduate studies at the Abay Kazakh Pedagogical Institute.

Scientific works of K.N. Ospanov are devoted to application of methods of functional analysis to the theory of differential equations. On the basis of a local approach to the resolvent representation he has found weak conditions for the solvability of the singular generalized Cauchy-Riemann system and established coercive estimates for its solution. He has obtained a criterion of the spectrum discreteness for the resolvent of the system and the exact in order estimates of singular values and Kolmogorov widths. He has original research results on the coercive solvability of the quasilinear singular generalized Cauchy-Riemann system and degenerate Beltrami-type system. He has established important smoothness and approximation properties of non strongly elliptic systems. K.N. Ospanov has found separability conditions in Banach spaces for singular linear and quasi-linear second-order differential operators with growing intermediate coefficients and established a criterion for the compactness of its resolvent and finiteness of the resolvent type.

His results have contributed to a significant development of the theory of two-dimensional singular elliptic systems, degenerate differential equations and non strongly elliptic boundary value problems.

K.N. Ospanov has published more than 140 scientific papers. The list of his most important publications one may see on the

http://mmf.enu.kz/images/stories/photo/pasport/fm/ospanov

K.N. Ospanov is an Honoured Worker of Education of the Republic of Kazakhstan, and he was awarded the state grant "The best university teacher".

The Editorial Board of the Eurasian Mathematical Journal is happy to congratulate Kordan Nauryzkhanovich Ospanov on occasion of his 60th birthday, wishes him good health and further productive work in mathematics and mathematical education.

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 6, Number 4 (2015), 7 – 18

ON FINITE-DIMENSIONAL BANACH SPACES IN WHICH SUNS ARE CONNECTED

A.R. Alimov

Communicated by V.S. Guliyev

Key words: sun, strict sun, bounded connectedness, (BM)-space, contractibility, nearly best approximation, ε-selection, Menger connectedness, monotone path-connectedness.

AMS Mathematics Subject Classification: 41A65.

Abstract. The present paper extends and refines some results on the connectedness of suns in finite-dimensional normed linear spaces. In particular, a sun in a finite-dimensional (BM)-space is shown to be monotone path-connected and having a continuous multiplicative (additive) ε-selection from the operator of nearly best approximation for any $\varepsilon > 0$. New properties of (BM)-space are put forward.

1 Introduction

In what follows X is a normed linear space, X_n is an X of finite dimension n. Throughout, given $x \in X$ and $r > 0$, the open ball, closed ball and sphere centred at x of radius r will be denoted by $B(x, r)$, $\mathring{B}(x, r)$ and $S(x, r)$, respectively. For brevity, $B := B(0, 1)$ and $S = S(0, 1)$ will denote the unit ball and the unit sphere.

The best approximation, that is, the distance from a given element x of a normed linear space X to a given nonempty set $M \subset X$ is, by definition,

$$\rho(x, M) := \inf_{y \in M} \|x - y\|.$$

The set of all *nearest points* (elements of best approximation) in M for a given element x is denoted by $P_M x$. In other words,

$$P_M x := \big\{ y \in M \mid \rho(x, M) = \|x - y\| \big\}.$$

Given a subset $\emptyset \neq M \subset X$, a point $x \in X \setminus M$ is called a *solar point* if there exists a point $y \in P_M x \neq \emptyset$ (a *luminosity point*) such that

$$y \in P_M\big((1 - \lambda)y + \lambda x\big) \quad \text{for all} \quad \lambda \geq 0 \tag{1.1}$$

(geometrically, this means that there is a 'sun' ray emanating from y and passing through x such that y is a nearest point in M for any point from the ray).

A point $x \in X \setminus M$ is called a *strict solar point* if $P_M x \neq \emptyset$ and if condition (1.1) is satisfied for any point $y \in P_M x$. A closed set $M \subset X$ is called a *sun* (respectively, a *strict sun*) if any point $x \in X \setminus M$ is a solar point (respectively, a strict solar point) of M. The concept of a 'sun' was introduced by N. V. Efimov and S. B. Stechkin. For a recent survey on suns and approximative and geometrical properties thereof, consult [9].

Throughout we shall be concerned with structural characteristics of 'suns' with focus mainly on connectedness, acyclicity, cell-likeness, contractibility and the existence of a continuous ε-selections for any $\varepsilon > 0$ from the operator of nearly best approximation.

2 Definitions and notation

A *bar* is, by definition, an intersection of extreme hyperstrips of the form

$$\{x \in X \mid a \leq f(x) \leq b\},\ -\infty \leq a \leq b \leq +\infty,\ f \in \operatorname{ext} S^*,$$

which are generated in the space X by extreme functionals from the dual unit sphere S^*; see, for example, [8] for more on bars and their separation properties. By Krein–Milman's theorem it follows that any closed unit ball is a bar. If $S^* = \operatorname{ext} S^*$ (i.e., if the norm of the dual space is rotund), then any bar is a closed convex set, and vice versa. In $C(Q)$, Q is a metrizable compact set, any bar is of the form

$$\Pi = [\![f_1, f_2]\!] := \{f \in C(Q) \mid f(t) \in [f_1(t), f_2(t)], t \in Q\},$$

where $f_1, f_2 : Q \to \overline{\mathbb{R}}$, $f_1 \leq f_2$, f_1 is upper semicontinuous on Q, and f_2 is lower semicontinuous (see [8], [9, § 7]).

Following Vlasov [25] if Q denotes some property (for example, 'connected') we say that a closed set M is:

- P-Q if for all $x \in X$ the set $P_M x$ is nonempty and has property Q;
- B-Q if $M \cap B(x, r)$ has property Q for all $x \in X$, $r > 0$;
- $\mathring{B}$-Q if $M \cap \mathring{B}(x, r)$ has property Q for all $x \in X$, $r > 0$;
- *extremally* Q if $M \cap \Pi$ has property Q for any bar Π in X.

For example, a closed subset of a finite-dimensional space is P-nonempty or is an existence (proximinal) set.

Remark. B-connected subsets are sometimes called V-connected (here the letter 'V' comes from L. P. Vlasov's works, who denoted balls by $V(x, r)$). Our term B-connectedness is in line with more familiar notation $B(x, r)$ for balls and also with the concept of 'bounded connectedness', which was introduced by D. Wulbert in the 1960s.

We shall require below two types of connectedness of sets: the m-connectedness (or the Menger connectedness [14]) and the monotone path-connectedness [1], the latter is a sharpening of the path-connectedness—as distinct from the path-connectedness, two points of a monotone path-connected set can be joined by a *monotone* continuous curve (see below).

Following Brown [14], given a bounded set $\emptyset \neq M \subset X$ we let $\mathrm{m}(M)$ denote the *Banach–Mazur hull* (or the *ball hull*) of M, which is the intersection of all closed balls containing M. A set $M \subset X$ is called m-*connected* (*Menger-connected*) (see Brown [14]) if $\mathrm{m}(\{x, y\}) \cap M \neq \{x, y\}$ for any distinct points $x, y \in M$. For brevity, we write $\mathrm{m}(\{x, y\}) = \mathrm{m}(x, y)$. Menger-connected sets are frequently found to be metrically convex with respect to the so-called Brown-associated norm (see [14], [1]), which enables one to use the machinery of metric convexity in this context. This important observation had evolved from the pioneering paper by Berens and Hetzelt [11], who characterized the suns in $\ell^\infty(n)$ in terms of ℓ^1-convexity. Thus, Berens and Hetzelt were first to find a nontrivial example of a Banach space of dimension ≥ 2 in which any sun is P-acyclic, thereby reversing for $\ell^\infty(n)$ the famous Vlasov's theorem ([25, Theorem 4.4]) that states that any P-acyclic boundedly compact subset of a Banach space is a sun. As of now, only few nontrivial examples are known of spaces in which the conversion of Vlasov's theorem holds.

An account of some properties of m-connected sets may be found in Brown [14], [15], Franchetti and Roversi [18], and Alimov [1], [2], [3].

Let $k(\tau)$, $0 \leq \tau \leq 1$, be a continuous curve in a normed linear space X. A curve $k(\cdot)$ is said to be *monotone* [7] if $f(k(\tau))$ is a monotone function in τ for any $f \in \operatorname{ext} S^*$.

A closed subset $M \subset X$ is called *monotone path-connected* [7] if any two points in M can be joined by a continuous monotone curve (arc) $k(\cdot) \subset M$. We note that a monotone path-connected set is always B-monotone path-connected. In particular, a monotone path-connected set is always m-connected [9]; a closed m-connected set may fail to be connected [18] (but only if $\dim X = \infty$). For more on monotone path-connected and m-connected sets, see [9, § 8] and [1].

We need to recall a few more definitions. Let A be a nontrivial arbitrary abelian group. A space (throughout, all spaces are assumed to be metrizable) is called *acyclic* if its Čech *co*homology group with coefficients from A is trivial (it has no cycles besides the boundary). Thus, the definition of acyclicity depends on the group of coefficients in question. However, if a (co)homology has compact support (satisfying the compact supports axiom) and if the coefficients of the (co)homology group lie in a filed, then the notions of homological and cohomological acyclicity coincide. Below, unless otherwise stated, the acyclicity will be understood in the sense of Čech cohomology with coefficients in an arbitrary abelian group (*cf.* also Brown [14], the comment after Theorem 2.5).

A compact space Y is called *cell-like* (or having the shape of a point) if there exists an ANR-space (absolute neighbouring retract) Z and an embedding $i : Y \to Z$ such that the image $i(Y)$ is contractible in any of its neighbourhoods $U \subset Z$; a cell-like set needs not be contractible. From the well-known Hyman's characterization of R_δ-sets it readily follows that an R_δ-set is always cell-like (see [1] for the references). But since any mapping of a compact of trivial shape into an ANR is homotopically trivial, a compact of trivial shape (cell-like) is contractible in each of its neighbourhoods in any ambient ANR. As a cor, we have that the classes of R_δ-sets and cell-like (having the shape of a point) compact spaces coincide. Finally, we note that the cell-likeness implies the acyclicity (with respect to any continuous (co)homology theory).

3 Synopsis

By the geometric form of the Hahn–Banach theorem it easily follows that any convex proximinal set is a strict sun. In the 1960s V. Klee and independently N. V. Efimov and S. B. Stechkin proved a partial converse to this result, namely: in a smooth normed linear space any sun is convex (this result was proved originally for strict suns, the extension to the case of suns is trivial; for more details see [9]). Consequently, the problem of connectedness of suns is vacuous in any smooth space.

The study of the problem of connectedness (and in particular, the B-connectedness) of suns in arbitrary normed spaces was initiated, as is natural, in the case $\dim X = 2$. As distinct from the case of spaces of dimension ≥ 3, the two-dimensional setting is fairly transparent: if M is a sun in a two-dimensional X_2 and $x \notin M$, then $P_M x$ is either a point, a closed interval or a union of two intervals with ends at one point (Berens and Hetzelt [19]; for the general asymmetric case see Alimov, see e.g. [9]). In other words, in the two-dimensional setting the suns are P- and B-contractible). Moreover, for any M in X_2 there is a ray selection of the nearest point mapping P_M which is strongly contractive with respect to the so-called Radon transformed norm [19] (or with respect to the Brown associated norm $|\cdot|$; see [9]). Note that in view of Lemma 1 of [5] a similar result also holds in an arbitrary normed linear space X: any sun lying in a two-dimensional subspace of X is B-contractible.

Unlike the case of X_2, a sun in X_3 may fail to be a B-retract. Indeed, Berens and Hetzelt [11, p. 284] have constructed an example of a sun in $\ell^\infty(3)$ which fails to have a continuous selection from the metric projection. Clearly, such a set is not a B-retract. One may also easily construct an example of a strict sun (a Kolmogorov set) in $\ell^\infty(3)$ on which the metric projection is not lower semi-continuous.

Open problem. It is a long-standing open question whether a sun in X_n, $n \geq 3$, is B-connected. For the strict suns the answer here is affirmative—a boundedly compact strict sun in a normed linear space is B-path-connected (see Theorem 7.10 of [9]). However, it is unclear whether a strict sun in X_n, $n \geq 4$, is B-contractible (the author has recently shown that this is so in X_3) or even P-acyclic.

In the general finite-dimensional setting, the first results on the connectedness of suns were obtained by D. Braess, B. Brosowski and F. Deutsch, Ch. Dunham, and V. A. Koshcheev in the 1970s (see [9]). In particular, Koshcheev [20, Theorem 6] proved that in a finite-dimensional normed linear space every sun is connected. Brown [15] extended this result by showing that suns in X_n are path-connected and locally path-connected. In an infinite-dimensional normed space a compact sun is known to be connected (Koshcheev [20]).

In their attempt to solve the problem of B-connectedness of suns in the multivariate setting, Berens and Hetzelt [11] put forward the first nontrivial example of a ('non-nonsquare') space of arbitrary finite dimension ≥ 3 in which any sun is B-cell-like (and even B-contractible, as follows from Theorem 5.1). Namely, they proved [11] the ℓ^1-connectedness (in the sense of Menger) of an arbitrary sun in $\ell^\infty(n)$, which implies its bounded cell-likeness [11], P- and B-acyclicity [14], and moreover, extreme monotone path-connectedness, and extreme bounded cell-likeness [1]. Subsequently, the problem of B-connectedness and B-acyclicity of suns was studied by A. L. Brown and A. R. Alimov (see, e.g., [14], [13], [7], [1] and the recent survey [9]).

Pursuing the aim of reverting the aforementioned Vlasov's theorem on solarity of P-acyclic sets, Brown [14] introduced the class of finite-dimensional (BM)-spaces (to be defined later) and showed that a sun in a finite-dimensional (BM)-space is m-connected (Menger-connected) and $|\cdot|$-convex with respect to the so-called Brown associated norm $|\cdot|$. Later Brown [15] showed that a closed m-connected subset of a finite-dimensional Banach space is infinitely connected (and hence B-cell-like and B-acyclic (relative to any continuous theory of (co)homologies)).

The author [1] has partially extended this result by showing that a boundedly compact m-connected subset of a separable Banach space is monotone path-connected, and moreover, B-cell-like and boundedly extremally cell-like (and hence a sun). A slightly weaker result in this direction was obtained earlier by Franchetti and Roversi who showed that a boundedly compact m-connected set in a certain class of infinite-dimensional Banach spaces is P-acyclic [18, Theorem 6.3] and hence a sun [18, Theorem 7.4]. In c_0 the following 'almost complete' result holds [4]: 1) a sun in c_0 is monotone path-connected; 2) an approximatively compact m-connected subset of c_0 is a sun.

Not much is known about disconnected suns in infinite-dimensional spaces. The only example of a disconnected sun in some renormed infinite-dimensional subspace of the space $C[0,1]$ was constructed by Koshcheev (see [9, § 7.3]). Observe that the sun in Koshcheev's example is not a strict sun and is not approximatively compact.

We also note that there exist examples of finite-dimensional spaces in which there are non-m-connected (and hence, non-monotone path-connected) suns (see Brown [14, Theorem 4.3], Franchetti and Roversi [18, p. 19], and even non-monotone path-connected Chebyshev suns (Alimov [1])).

Many classical nonlinear families of functions in $C(Q)$ have solar properties (see, for example, [9]): these include the (generalized) rational functions, sums of exponentials with (nonnegative or arbitrary) coefficients, γ-polynomials, etc.

4 (BM)-spaces

In this section we recall the definition of (BM)-spaces and present some new properties thereof.

The concept of (BM)-spaces, which were introduced by Brown [14], proved instrumental in studying connectedness properties of suns. For such spaces it was found possible to carry over a number of nontrivial results on geometrical and topological properties of suns from the space $X = \ell^\infty(n)$ to more abstract spaces. For example, any sun in a finite-dimensional (BM)-space is known to be m-connected [14]. Moreover, polyhedral finite-dimensional (BM)-spaces are characterized by the fact that any sun in such a space is m-connected [13].

A point $x \in S$ will be said a (BM)-point [18] if it has the property that:

$$B \cap \big(\mathrm{m}(x,y) \setminus \{x\}\big) \neq \emptyset, \quad \text{whenever} \quad [x, x-y] \cap \mathring{B} = \emptyset. \tag{$*$}$$

A space X is said to be a (BM)-*space* if every $x \in S$ is a (BM)-point.

The class of (BM)-spaces contains the smooth spaces, the two-dimensional polyhedral spaces, the space $\ell^\infty(n)$, and more generally, any space of the form (4.1) (see [14], [13]). The space $\ell^1(n)$, $n \geq 3$, is not a (BM)-space.

An equivalent formulation of the (BM)-property $(*)$ in terms of tangent functionals is as follows [18, Lemma 8.1]:

$$B \cap \big(\mathrm{m}(x,y) \setminus \{x\}\big) \neq \emptyset, \quad \text{whenever} \quad \|y\| > 1 \quad \text{and} \quad \tau^-(x,y) = 0. \tag{$**$}$$

Here,

$$\tau^+(x,y) := \lim_{t\to 0+} \frac{\|x+ty\|-\|x\|}{t} = \sup_{f\in J_x} f(y),$$
$$\tau^-(x,y) := \lim_{t\to 0-} \frac{\|x+ty\|-\|x\|}{t} = \inf_{f\in J_x} f(y)$$

are the tangent functionals on $X \times X$, $J_x := \{f \in X^* \mid \|f\| = 1,\ f(x) = \|x\|\}$.

It is also worth noting that if $\tau^+(x,y) < 1$, then $B \cap (x,y] \neq \emptyset$. As a result, if $x \in S$ is a smooth point, then $\tau^+(x,\cdot) = \tau^-(x,\cdot)$, and hence x is a (BM)-point, because $(x,y) \subset \mathrm{m}(x,y)$. Moreover, the following result holds ([18, Theorem 8.1]): if x is not a smooth point of S, then x is a (BM)-point if and only if

$$B \cap (M(x,y) \setminus \{x\}) \neq \emptyset, \text{whenever } y \in X \text{such that } \ \tau^-(x,y) = 0 \ \text{ and } \ \tau^+(x,y) \geq 1.$$

Franchetti and Roversi [18] showed that the sublattices with unity and the closed ideals of $C(Q)$ are (BM)-spaces and also that the c_0-sum of (BM)-spaces is a (BM)-space. A characterization of two-dimensional (BM)-spaces was obtained by Brown [14]. In [13] Brown has shown that the polyhedral (BM)-spaces of finite dimension are exactly the ℓ^∞-direct sums

$$X = X_1 \oplus_\infty \cdots \oplus_\infty X_r \tag{4.1}$$

of a finite number of symmetric polyhedral spaces $X_1, \dots, X_r$ of dimension 1 or 2 (the equality is understood to mean isometric isomorphism). Three-dimensional (BM)-spaces were characterized by Brown [16]: any such a space is either smooth or is of the form $Y \oplus_\infty \mathbb{R}$ for some (BM)-space Y of dimension two.

In the following two results we establish two more properties of polyhedral (BM)-spaces. We first give necessary definitions.

Let $n, k \in \mathbb{N}$, $n > k \geq 2$. By definition, a normed linear space X has the *$n.k$-intersection property* ($X \in (n.k.I.P)$) if, for any n closed balls $B(a_i, r_i)$, $i = 1, \dots, n$, such that $\cap_{r=1}^k B(a_{i_r}, a_{i_r}) \neq \emptyset$ whenever $1 \leq i_1 \leq \dots i_k \leq n$, we have $\cap_{i=1}^n B(a_i, r_i) \neq \emptyset$. In recent years there has been a considerable interest in the study of ($n.k$.I.P)-spaces in connection with problems of minimal filling of subsets of Banach spaces and optimal networks (P. A. Borodin, B. B. Bednov, N. P. Strelkova, A. Yu. Eremin, A. O. Ivanov, A. A. Tuzhilin, Z. N. Ovsyannikov, O. V. Rubleva; see, for example, [10]).

By classical Helly's theorem if $\dim X = n$, then $X \in ((n+2).(n+1).I.P)$; as a result, any one- or two-dimensional space lies in $(4.3.I.P)$. Hence, $\oplus_\infty$-sums of such spaces also have the $(4.3.I.P)$ property. The converse result is due to Lima [21]. This being so,

$$X \in (4.3.I.P) \iff X = X_1 \oplus_\infty \cdots \oplus_\infty X_r, \quad \text{where} \quad \dim X_i \leq 2 \tag{4.2}$$

(the equality on the right is understood in the sense of isometrical isometry).

The following result now follows from (4.1).

Proposition 4.1. *The following statements are equivalent in the class of finite-dimensional polyhedral spaces X:*

a) $X \in (BM)$;

b) $X \in (4.3.I.P)$.

The condition of polyhedrality here is essential. Being smooth, the Euclidean space $\mathbb{R}^3$ is a (BM)-space, but in view of (4.2) it fails to have the 4.3 intersection property (the corresponding counterexample is simple: 4 balls centred at the vertices of a regular tetrahedron).

Note that if $X \in (4.3.I.P)$, $\dim X < \infty$, then $X \in (\infty.3.I.P)$ (see, for example, [12], § VIII, Theorem 5). Consequently, if $X \in (BM)$ is finite-dimensional and polyhedral, then $X \in (\infty.3.I.P)$.

It is also worth mentioning that the space $X = \ell^\infty(n)$ lies in the class (BM) and satisfies the more strong $(4.2.I.P)$ intersection property (as any L^1-predual space is characterized by the 4.2 intersection property).

We point out another property of finite-dimensional polyhedral (BM)-spaces.

Recall that a bounded closed convex set M is a *Mazur set* (see [17], [22]) if the following separation property holds: given any hyperplane H with positive distance from M, there is a ball B' such that $M \subset B'$ and $H \cap B' = \emptyset$. Spaces in which the class of Mazur sets coincides with the class $\mathcal{M}_X$ of intersections of closed balls are called *Mazur spaces* (see [17], [22]). For example, $C(Q)$, is a Mazur space if and only if Q is extremally disconnected; every two-dimensional normed linear space is a Mazur space. Mazur spaces appear naturally in the context of stability of intersections of convex subsets of Banach spaces.

Proposition 4.2. *The following statements are equivalent in the class of finite-dimensional polyhedral spaces X:*

a) $X \in (BM)$;

b) *X is a Mazur space.*

Proof. of Proposition 4.2. By [22, Corollary 4.2] a finite-dimensional polyhedral space X is a Mazur space if and only if the family $\mathcal{M}_X$ of intersections of closed balls in X is stable—this means by definition that $\overline{C + D} \in \mathcal{M}_X$ (the closure of the vector sum of sets C and D) whenever $C, D \in \mathcal{M}_X$. Next (see [22, Theorem 3.2]), in a finite-dimensional polyhedral space the family $\mathcal{M}_X$ is stable if and only of X has representation (4.1). To conclude the proof it remains to recall that finite-dimensional polyhedral (BM)-spaces are characterized by the property (4.1). □

Since the space $\ell^1(n)$, $n \geq 3$, is not a (BM)-space, we have in view of Proposition 4.2 the following result, which is in full agreement with the result of Granero, Moreno, and Phelps [17] to the effect that the only Mazur sets of the spaces $\ell^1(n)$, $n \geq 3$, are points and closed balls.

Corollary 4.1. *The space $\ell^1(n)$, $n \geq 3$, is not a Mazur space.*

5 The main result

The main result of the paper (Theorem 5.1) puts forward new structural and stability properties of suns in finite-dimensional (BM)-spaces. In Theorem 5.1 there is a deal of emphasis on the existence of a continuous ε-selection on suns for any $\varepsilon > 0$. Here it is again worth mentioning that Berens and Hetzelt [11] presented a simple example of a sun in $\ell^\infty(3)$ which fails to have a continuous selection (i.e., 0-selection) from the metric projection. Accordingly, a few words on the stability of the approximation are appropriate here. The best approximation operator is known to be poorly stable even for Chebyshev subspaces, not to speak of nonlinear sets. For example, the (single-valued) metric projection operator onto the subspace of polynomials of degree $\le n$ in $C[0,1]$ is not uniformly continuous on the unit ball. Moreover, it has long been known that the metric projection onto a Chebyshev subspace may be discontinuous; in nondegenerate cases the metric projection onto the set of rational fractions in $C[0,1]$ always has points of discontinuity. In order to improve the situation and enhance the stability of best approximation it was proposed to appropriately associate one of the almost best approximations to an element being approximated. This has led to the concept of ε-selection.

There are many interesting results on the stability of the nearly best approximation operator for many classical objects in approximation theory (classical and generalized rational functions, splines with free knots, exponential sums), we mention the works of R. Wegmann, S. V.Konyagin, G. Nürnberger, I. G. Tsar'kov, H. Berens, A. V. Marinov, D. Repovš, P. V. Semenov, K. S. Ryutin, E. D. Livshtis, and others.

Theorem 5.1. *Let $X_n \in (BM)$. Then the following equivalent conditions hold for any sun $M \subset X_n$:*

a) *M is (extremally) monotone path-connected;*

b) *M is extremally contractible (in particular, M is B-contractible);*

c) *M is extremally sunny (that is, the intersection of M with any bar and, in particular, with any closed ball, is a sun or empty);*

d) *there exists a continuous multiplicative (additive) ε-selection on M for any $\varepsilon > 0$.*

e) *for any bar $\Pi \subset X_n$ there exists a continuous multiplicative (additive) ε-selection on the set $M \cap \Pi$ for any $\varepsilon > 0$.*

Remark 5.1. One may easily construct a non-(BM)-space, even two-dimensional, in which any sun is m-connected (and even monotone path-connected). Indeed, if B_ℓ is any nontrivial intersection of two Euclidean balls ('a planar lens'), then the space X_ℓ with the unit ball B_ℓ is not a (BM)-space (see [14, Theorem 5.5]). On the other hand, a sun in any X_2 is m-connected [14, Theorem 4.1], and hence is monotone path-connected [1].

Theorem 5.2. *In a polyhedral (BM)-space X_n a set M is a sun if and only if any of the conditions* a)–e) *of Theorem* 5.1 *is satisfied.*

This result follows from Theorem 5.1 and the aforementioned result of Brown [13] stating, in particular, that the polyhedral finite-dimensional (BM)-spaces X_n are characterized by the property that any sun in X_n is m-connected.

We shall repeatedly use the following celebrated general result by Tsar'kov [24], which characterizes the closed subsets of Banach spaces for which for any $\varepsilon > 0$ there exists a continuous ε-selection.

Theorem A. *Let X be a Banach space and let $M \subset X$ be nonempty and closed. Then following conditions are equivalent:*

a) $\mathring{P}^{\delta}_{M}x := \mathring{B}(x, \rho(x, M) + \delta) \cap M$ *is a retract of the ball* $B(x, \rho(x, M) + \delta)$ *for any* $x \in X$ *and* $\delta > 0$*;*
b) $\mathring{P}^{\delta}_{M}x$ *is contractible in itself to a point for any* $x \in X$ *and* $\delta > 0$*;*
c) M *is* $\mathring{B}$*-infinitely connected;*
d) M *is* $\mathring{B}$*-contractible;*
e) *for any* $\varepsilon > 0$ *there exists a continuous additive* ε*-selection for* M*;*
f) *for any positive lower semicontinuous function* $\psi : X \to (0, +\infty)$, $\psi(x) > \rho(x, M)$, $x \in X$, *there exists a mapping* $\varphi \in C(X, M)$ *such that* $\|\varphi(x) - x\| < \psi(x)$ *for all* $x \in X$;
g) *for any lower semicontinuous function* $\theta : X \to (1, +\infty)$ *there exists a mapping* $\varphi \in C(X, M)$ *such that* $\|\varphi(x) - x\| \leq \theta(x)\rho(x, M)$ *for all* $x \in X$.

This result shows that the existence of a continuous ε-selection for all $\varepsilon > 0$ is fairly restrictive in terms of the structure of a set.

Remark 5.2. Any assertion in Theorem 5.1 involving an additive (multiplicative) selection can be replaced by the corresponding assertion involving a selection φ from Theorem A.

Proof of Theorem 5.1 A sun in a (*BM*)-space is m-connected (Brown [14]). That a boundedly compact m-connected set is (extremally) monotone path-connected is established in [6, Theorem 2]. This proves assertion a). The implication a)⇒b) is secured by Theorem 4.1 of [1].

The implication b)⇒a). If M is B-contractible, then M is B-acyclic (relative to any continuous theory of (co)homologies)) and hence, by Vlasov's theorem [25, Theorem 4.4] is a sun. Finally, any sun in $X_n \in (BM)$ is known to be (extremally) monotone path-connected [6, Theorem 2].

The implication a)⇒c) is contained in [6, Theorem 2]. The converse assertion is again secured by the fact that any sun in $X_n \in (BM)$ is (extremally) monotone path-connected [1].

To prove d) we first employ one result of Tsar'kov [24] to the effect that, for a closed set M, the existence of a continuous additive ε-selection on M for any $\varepsilon > 0$ is equivalent to the existence of a continuous multiplicative ε-selection on M for any $\varepsilon > 0$. Now d) follows from Theorem A, because by assertion a) the set M is monotone path-connected, and hence is B-contractible [1], which in turn implies that M is $\mathring{B}$-contractible [9, Theorem 6.6], and hence, $\mathring{B}$-infinitely connected. To prove the implication d)⇒a) we shall use the following result of Tsar'kov [23]: Suppose that X is a Banach space, $M \subset X$, x is a point of approximative compactness of M, $K := \operatorname{cone}\{x, P_M x\}$, and for any $\varepsilon > 0$ there exists an upper semicontinuous acyclic ε-selection from the nearly best approximation operator $\mathring{P}^{\varepsilon}_{M}x := \mathring{B}(x, \rho(x, M) + \varepsilon) \cap M$ onto the set M with respect to K, then the set $P_M x$ is acyclic. In our setting the selection is single-valued and hence acyclic. So, if d) holds, then $P_M x$ is acyclic for any x. By the aforementioned Vlasov's theorem [25, Theorem 4.4], M is a sun, and hence ([6, Theorem 2]) is (extreme) monotone path-connected.

Assertion e) follows from d) in view of the fact that a sun in a (*BM*)-space is extremally sunny (assertion c)). □

Remark 5.3. As was noted above (see also [6, Remark 3]), for any $n \geq 3$ there is an example of a space $\hat{X}_n$ which contains an non-monotone path-connected Chebyshev set M' (a Chebyshev sun). Note that in any X_2 any sun is monotone path-connected. Clearly, such an M' is B-retract (and hence B-contractible). In view of Theorem 5.1 any such a space $\hat{X}_n$ is not a (BM)-space. The problem of characterization of spaces X_n in which any (bounded) Chebyshev set (sun, strict sun) is monotone path-connected (or, equally, m-connected) remains open.

Acknowledgments

This research was carried out with the financial support of the Russian Foundation for Basic Research (project 16-01-00295).

References

[1] A.R. Alimov, *Monotone path-connectedness and solarity of Menger-connected sets in Banach spaces,* Izvestiya: Mathematics 78 (2014), no. 4, 641–655.

[2] A.R. Alimov, *The Rainwater–Simons weak convergence theorem for the Brown associated norm,* Eurasian Math. J. 5 (2014), no. 2, 126–131.

[3] A.R. Alimov, *Monotone path-connectedness of R-weakly convex sets in spaces with linear ball embedding,* Eurasian Math. J. 3 (2012), no. 2, 21–30.

[4] A.R. Alimov, *Connectedness of suns in the space* c_0, Izvestiya: Mathematics 69 (2005), no. 4, 3–18.

[5] A.R. Alimov, *Convexity of Chebyshev sets contained in a subspace,* Math. Notes 78 (2005), no. 1, 3–15.

[6] A.R. Alimov, *Local solarity of suns in normed linear spaces,* Fundam. Prikl. Mat. 17 (2011/2012), no. 7, 3–14.

[7] A.R. Alimov, *Monotone path-connectedness of Chebyshev sets in the space* $C(Q)$, Sbornik: Mathematics 197 (2006), no. 9, 3–18.

[8] A.R. Alimov, V.Yu. Protasov, *Separation of convex sets by extreme hyperplanes,* Fundam. Prikl. Mat. 17 (2011/2012), no. 4, 3–12.

[9] A.R. Alimov, I.G. Tsar'kov, *Connectedness and other geometric properties of suns and Chebyshev sets,* Fundam. Prikl. Matem. 19 (2014), no. 4, 21–91.

[10] B.B. Bednov, P.A. Borodin, *Banach spaces that realize minimal fillings,* Sbornik: Mathematics 205 (2014), no. 4, 459–475.

[11] H. Berens, L. Hetzelt, *Die metrische Struktur der Sonnen in* $\ell^\infty(n)$, Aequat. Math. 27 (1984), 274–287.

[12] V. Boltyanski, H. Martini, P.S. Soltan, *Excursions into combinatorial geometry*, Berlin, Springer (1997).

[13] A.L. Brown, *Suns in polyhedral spaces*, in Seminar of Mathem. Analysis, Proceedings, D.G. Álvarez, G. Lopez Acedo and R. V. Caro, Eds., Univ. Malaga and Seville (Spain), Sept. 2002 – Feb. 2003, Universidad de Sevilla, Sevilla, 139–146. (2003).

[14] A.L Brown, *Suns in normed linear spaces which are finite-dimensional*, Math. Ann. 279 (1987), 87–101.

[15] A.L. Brown, *On the connectedness properties of suns in finite dimensional spaces*, Proc. Cent. Math. Anal. Aust. Natl. Univ. 20 (1988), 1–15.

[16] A.L. Brown, *On the problem of characterising suns in finite dimensional spaces*, Rend. Circ. Math. Palermo Ser. II 68 (2002), 315–328.

[17] A.S. Granero, J.P. Moreno, R.R. Phelps, *Mazur sets in normed spaces*, Discrete Comput Geom. 31 (2004), 411–420.

[18] C. Franchetti, S. Roversi, *Suns, M-connected sets and P-acyclic sets in Banach spaces*, Preprint No. 50139, Instituto di Matematica Applicata "G. Sansone", 1–29 (1988).

[19] L. Hetzelt, *On suns and cosuns in finite-dimensional normed real vector spaces*, Acta Math. Hungar. 45 (1985), no. 1–2, 53–68.

[20] V.A. Koshcheev, *The connectivity and approximative properties of sets in linear normed spaces*, Math. Notes 17 (1975), no. 2, 114–119.

[21] Å. Lima, *Banach spaces with the* 4.3 *intersection property*, Proc. Amer. Math. Soc. 80 (1980), 431–434.

[22] J.P. Moreno and R. Schneider, *Intersection properties of polyhedral norms*, Adv. Geom. 7 (2007), no. 3, 391–402.

[23] I.G. Tsar'kov, *Nonunique solvability of certain differential equations and their connection with geometric approximation theory*, Math. Notes 75 (2004), no. 2, 287–301.

[24] I.G. Tsar'kov, *Continuous ε-selection*, Mat. Sb. 207 (2016), no. 2, 123–142 (to be translated in Sbornik: Mathematics).

[25] L.P. Vlasov, *Approximative properties of sets in normed linear spaces*, Russian Math. Surveys 28 (1973), no. 6, 3–66.

Alexey Rostislavovich Alimov
Department of Mechanics and Mathematics
M.V. Lomonosov Moscow State University
1 Leninskie gory
Moscow 119991 Russia
E-mail: alexey.alimov-msu@yandex.ru

Received: 06.09.2015