EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 1, Number 4 (2010), 116 – 123

JUSTIFICATION OF THE DYNAMICAL SYSTEMS METHOD FOR GLOBAL HOMEOMORPHISM

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Communicated by M. Otelbaev

Key words: the dynamical systems method (DSM), surjectivity, global homeomorphisms, monotone operators.

AMS Mathematics Subject Classification: 47J35.

Abstract. The dynamical systems method (DSM) is justified for solving operator equations $F(u) = f$, where F is a nonlinear operator in a Hilbert space H. It is assumed that F is a global homeomorphism of H onto H, that $F \in C^1_{loc}$, that is, it has the Fréchet derivative $F'(u)$ continuous with respect to u, that the operator $[F'(u)]^{-1}$ exists for all $u \in H$ and is bounded, $||[F'(u)]^{-1}|| \le m(u)$, where $m(u) > 0$ depends on u, and is not necessarily uniformly bounded with respect to u. It is proved under these assumptions that the continuous analogue of the Newton's method

$$\dot{u} = -[F'(u)]^{-1}(F(u) - f), \quad u(0) = u_0, \tag{$*$}$$

converges strongly to the solution of the equation $F(u) = f$ for any $f \in H$ and any $u_0 \in H$. The global (and even local) existence of the solution to the Cauchy problem $(*)$ was not established earlier without assuming that $F'(u)$ is Lipschitz-continuous. The case when F is not a global homeomorphism but a monotone operator in H is also considered.

1 Introduction

Consider an operator equation:

$$F(u) = f, \tag{1.1}$$

where F is a nonlinear operator in a Hilbert space H.

We assume in this Section that F is a *global homeomorphism.*

For instance, F may be a hemicontinuous monotone operator such that a coercivity condition is satisfied, for example, the following condition:

$$\lim_{||u||\to\infty} \frac{(F(u), u)}{||u||} = \infty, \tag{1.2}$$

where $(\cdot, \cdot)$ denotes the inner product in H (see [1]). We assume that $F \in C^1_{loc}$, i.e., the Fréchet derivative of F, $F'(u)$, exists for every u and depends continuously on u.

Furthermore, we assume that $[F'(u)]^{-1}$ exists and is bounded for all $u \in H$,

$$||[F'(u)]^{-1}|| \leq m(u), \tag{1.3}$$

where $m(u) > 0$ depends on u and *is not necessarily uniformly bounded with respect to u.*

This assumption implies that F is a *local* homeomorphism, but it does not imply, in general, that F is a *global* homeomorphism. If $m(u) < m$, where $m > 0$ is a constant independent of u, then it was proved in [6] that F is a global homeomorphism.

While our main result in Section 1, Theorem 1, does not require the monotonicity of F, the result in Section 2, Theorem 2, will use the monotonicity of F.

We assume in Section 2 that F is monotone:

$$F'(u) \geq 0 \qquad \forall u \in H. \tag{1.4}$$

This means that $(F'(u)v, v) \geq 0$ for all $v \in H$.

In Remark 2, at the end of the paper, the following condition is mentioned:

$$\|F(u)\| < c \Rightarrow \|u\| < c_1, \qquad c, c_1 = const > 0, \tag{1.5}$$

which means that the preimages of bounded sets under the map F are bounded sets. This condition does not hold for the operator $F(u) := e^u$, $u \in \mathbb{R}$, $H = \mathbb{R}$, and that is why this monotone operator F is not surjective: equation $e^u = 0$ does not have a solution in H.

By $c > 0$ we denote in this paper various constants.

Our first main result, Theorem 1, says that if $F \in C^1_{loc}$ is a global homeomorphism and condition (1.3) holds, then a continuous analogue of the Newton's method (see equation (1.6) below) converges globally, that is, it converges for any initial approximation $u_0 \in H$ and any right-hand side $f \in H$.

One of the novel features of our result is the absence of any smoothness assumptions on $F'(u)$: only the continuity of $F'(u)$ with respect to u is assumed.

In the earlier work (see [5], [6]- [11], [3], and references therein, except for [2] and [10], [4]) it was often assumed that $F'(u)$ is Lipschitz continuous, or, at least, Hölder-continuous.

Our approach can be generalized to the case when F is a local homeomorphism, if one uses the results in [12].

In this paper for the first time no assumptions on the smoothness of $F'(u)$ are made, only the continuity of $F'(u)$ is assumed in a proof of the global existence of the solution to the Cauchy problem (6), see Theorem 1 below. The author does not know any way to prove even the local existence of the solution to (6) without using the novel idea and new method of the proof, given in the proof of Theorem 1. The known methods do not seem to give any results even on the local existence of the solution to problem (6) if $F'(u)$ is assumed to be only continuous. Recall that the known Peano theorem fails in infinite-dimensional Banach spaces. The standard assumption, that guarantees the local existence of the unique solution to the Cauchy problem (6) in an infinite-dimensional Banach space, is the Lipschitz condition for the operator $[F'(u)]^{-1}(F(u) - f)$, which holds, in general, only if $F'(u)$ is Lipschitz-continuous.

In our second result, in Theorem 2 in Section 2, the operator F is not assumed to be a global homeomorphism, and it is not assumed to be invertible (injective), but it is assumed to be a monotone operator, and it is assumed that equation (1.1) has a solution, possibly non-unique.

We give a Dynamical Systems Method (DSM) version for constructing the (unique) minimal-norm solution to equation (1.1) with monotone operator F. This DSM version is a regularized continuous analogue of the Newton's method. We make no smoothness assumptions about $F'(u)$, and assume only the continuity of $F'(u)$ with respect to u.

Since we do not assume in Section 2 that the operator $F'(u)$ is invertible in any sense, the problem, studied in this Section can be considered an ill-posed one.

Our proof of Theorem 2 contains new ideas and uses the ideas from the proof of Theorem 1.

Let us formulate our first result:

Theorem 1. *If $F \in C^1_{loc}$ is a global homeomorphism and condition* (1.3) *holds, then the problem*

$$\dot{u} = -[F'(u)]^{-1}(F(u) - f), \quad u(0) = u_0; \ \dot{u} = \frac{du}{dt}, \tag{1.6}$$

is globally solvable for any f and u_0 in H, there exists the limit $u(\infty) = \lim_{t\to\infty} u(t)$, and $F(u(\infty)) = f$.

Proof. Denote

$$v := F(u(t)) - f. \tag{1.7}$$

Then

$$\dot{v} = F'(u(t))\dot{u} = -v.$$

Thus, problem (1.6) is reduced to the following problem:

$$\dot{v} = -v, \quad v(0) = F(u_0) - f. \tag{1.8}$$

Problem (1.8) obviously has a unique global solution:

$$v(t) = (F(u_0) - f)e^{-t}, \qquad \lim_{t\to\infty} v(t) := v(\infty) = 0. \tag{1.9}$$

Therefore, problem (1.6) has a unique global solution.

Let us explain the above statement in detail. Consider an interval $[0, T]$, where $T > 0$ is arbitrarily large. The equation

$$F(u(t)) - f = v(t) \qquad 0 \le t \le T, \tag{1.10}$$

is uniquely solvable for $u(t)$ for any $v(t)$ because F is a global homeomorphism. Assumption (1.3), the continuity of $F'(u)$ with respect to u, and the abstract inverse function theorem, imply that the solution $u(t)$ to equation (1.10) is continuously differentiable with respect to t, because v is continuously differentiable with respect to t and F is continuously Fréchet differentiable with respect to u.

Differentiating (1.10) and using relations (1.8) and (1.7), one gets the following equation:

$$F'(u(t))\dot{u} = \dot{v} = -v = -(F(u(t)) - f). \tag{1.11}$$

Using assumption (1.3), one concludes from (1.11) that $u = u(t)$ solves (1.6) in the interval $t \in [0, T]$. Since $T > 0$ is arbitrary, $u = u(t)$ is a global solution to (1.6).

Since $\lim_{t\to\infty} v(t) := v(\infty)$ exists, and F is a global homeomorphism, one concludes that the limit $\lim_{t\to\infty} u(t) := u(\infty)$ does exist.

Since $v(\infty) = 0$, it follows that $F(u(\infty)) = f$.

Theorem 1 is proved. □

Remark 1. *Theorem 1 implies that any equation* (1.1) *with F being a global homeomorphism and $F \in C^1_{loc}$, such that condition* (1.3) *holds, can be solved by the DSM method* (1.6)*, which is a continuous analogue of the Newton's method.*

2 Finding the minimal-norm solution

Assumptions: In this Section we assume that $F \in C^1_{loc}$ is monotone, that is, $F'(u) \geq 0$, and assumptions (1.3)*-* (1.5) *hold, but F is not a global homeomorphism, so that equation* (1.1) *may have many solutions. We assume that* (1.1) *has a solution.*

Since F is monotone and continuous, and the set of of solutions to (1.1) is nonempty, this set is closed and convex, so it has a *unique element with minimal norm* (see [5]). This element is called the minimal-norm solution to (1.1), and is denoted by y.

Our aim is to give a method for finding this element by a version of the DSM.

Consider the problem

$$\dot{u} = -[F'(u) + a(t)I]^{-1}[F(u) + a(t)u - f], \quad u(0) = u_0, \tag{2.1}$$

where $a \in C^1([0, \infty))$, $\dot{a} < 0$,

$$a(t) > 0 \ \forall t \geq 0, \quad \lim_{t\to\infty} \frac{\dot{a}}{a} = 0, \quad \lim_{t\to\infty} a(t) = 0. \tag{2.2}$$

The assumptions of Theorem 1 do not hold for the operator $F(\cdot) + a(t)I$ in the sense that the quantity $\frac{1}{a(t)}$ in the estimate (14), see below, tends to infinity as $t \to \infty$. Let us explain this statement.

Under our *Assumptions*, the operator $F(\cdot) + a(t)I$ for every $t > 0$ is a global homeomorphism because F is a monotone continuous operator and $a(t) > 0$. One has

$$\|[F'(u) + a(t)]^{-1}\| \leq \frac{1}{a(t)}. \tag{2.3}$$

Therefore, the constant $m(u)$ for the operator $F(u) + a(t)u$ is $\frac{1}{a(t)}$. As $t \to \infty$, this constant tends to infinity because $\lim_{t\to\infty} a(t) = 0$.

Let us state our result:

Theorem 2. *Assume that $F \in C^1_{loc}$ is a monotone operator, equation* (1.1) *has a solution for the given f, and conditions* (2.2) *hold. Then problem* (2.1) *has a unique global solution $u(t)$, there exists $u(\infty)$, and $u(\infty) = y$, where y is the minimal-norm solution to* (1.1).

Proof. Let

$$v(t) = F(u(t)) + a(t)u(t) - f. \tag{2.4}$$

Then

$$\dot{v} = -v + \dot{a}(t)u(t), \qquad v(0) = F(u_0) + a(0)u_0 - f. \tag{2.5}$$

The map $u = G(v)$, where

$$v(t) = G^{-1}(u) := F(u) + a(t)u - f,$$

is a local diffeomorphism for any $t \geq 0$, because $a(t) > 0$ $\forall t \geq 0$ and (2.3) holds.

As in the proof of Theorem 1 one concludes that the solution to (2.1) exists locally because the solution $v = v(t)$ to (2.5) exists locally.

The solution to (2.5) exists locally by the standard result, because the map $u = G(v)$ is C^1_{loc} local diffeomorphism. The solution to (2.5) exists globally (see, e.g., [5], p. 248) if

$$\sup_{t \geq 0} \|v(t)\| < c, \tag{2.6}$$

where $c > 0$ here and below denote various estimation constants.

Let us briefly recall the proof of this statement.

Assume that inequality (2.6) holds, but the maximal interval of the existence of v is finite, say, $[0, T)$, $T < \infty$. The length ℓ of the interval of the local existence of the solution to the Cauchy problem (2.5) depends only on the Lipschitz constant of $G(v)$ and on the norm of the right hand side of (2.5). Both these quantities depend only on the constant c. One solves the Cauchy problem for equation (2.5) with the initial data $v(T - 0.5\ell)$ at the initial point $t = T - 0.5\ell$. The unique solution to this problem exists on the interval $[T - 0.5\ell, T - 0.5\ell + \ell)$. Consequently, v exists on the interval $[0, T + 0.5\ell)$ greater than $[0, T)$. This is a contradiction which proves that $T = \infty$.

The map $u = G(v)$ is C^1_{loc} because it is inverse to the C^1_{loc} map $v = F(u) + a(t)u - f := G^{-1}(u) := Q(u)$, and $\|[Q'(u)]^{-1}\| \leq \frac{1}{a(t)} < \infty$ for every $t \geq 0$.

Therefore, the estimate $\sup_{t \geq 0} \|v(t)\| < c$ holds if and only if

$$\sup_{t \geq 0} \|u(t)\| < c, \tag{2.7}$$

where $c > 0$ stands for various constants.

Thus, to prove that $u(t)$ exists globally it is sufficient to prove inequality (2.7).

We prove this inequality, the existence of $u(\infty)$, and the relation $u(\infty) = y$, by establishing two facts:

a) the following inequality:

$$\|u(t) - w(t)\| \leq \frac{\|v(t)\|}{a(t)}, \tag{2.8}$$

and

b) the limiting relation:

$$\lim_{t \to \infty} \frac{\|v(t)\|}{a(t)} = 0. \tag{2.9}$$

In formula (2.8) $w(t)$ solves the problem

$$F(w) + a(t)w - f = 0, \tag{2.10}$$

and $a(t)$ satisfies (2.2). It is proved in [5] that if F is a monotone hemicontinuous operator and equation (1.1) has a solution, then equation (2.10) has a unique solution for any f if $a(t) > 0$, the limit $w(\infty)$ exists, and $w(\infty) = y$. This, (2.8), and (2.9) imply the existence of $u(\infty)$ and the relation $u(\infty) = y$.

Let us prove inequality (2.8). Since F is monotone, one has

$$(F(u) - F(w), u - w) \geq 0,$$

so

$$(v, u - w) = (F(u) - F(w) + a(t)(u - w), u - w) \geq a(t)\|u - w\|^2. \tag{2.11}$$

Applying the Cauchy inequality to the left side of (2.11), one gets (2.8).

Let us prove (2.9). Denote

$$h(t) := \|v(t)\|. \tag{2.12}$$

Multiply equation (2.5) by v and get

$$h\dot{h} \leq -h^2 + |\dot{a}|\|u(t)\|h. \tag{2.13}$$

If $h(t) > 0$, one obtains from (2.13) the following inequality

$$\dot{h}(t) \leq -h(t) + |\dot{a}(t)|(\|u(t) - w(t)\| + \|w(t)\|). \tag{2.14}$$

If $h(t) = 0$ on some interval $t \in (a, b)$, then $\dot{h} = 0$ on this interval, and the above inequality holds trivially. If $h(t) = 0$ at an isolated point $t = s$, i.e., $h(s) = 0$, then (2.14) holds by continuity at $s+0$. The existence of the derivative $\dot{h}(s+0)$ at the point s at which $h(s) = 0$ can be checked using the definition of the one-sided derivative:

$$\dot{h}(s + 0) = \lim_{\tau \to +0}[h(s + \tau) - h(s)]/\tau = \lim_{\tau \to +0} h(s + \tau)/\tau. \tag{2.15}$$

Since $v(t)$ is continuously differentiable, one has $h(s+\tau) := ||v(s+\tau)|| = ||\tau\dot{v}(s)+o(\tau)||$. Therefore the limit in (2.15) exists and is equal to $||\dot{v}(s)||$. This limit is denoted $\dot{h}(s)$. Thus, inequality (2.14) holds for all $t \geq 0$.

Since $w(\infty)$ exists, one has

$$\sup_{t \geq 0} \|w(t)\| < c. \tag{2.16}$$

Using (2.16) and (2.8), one gets from (2.14) the inequality

$$\dot{h} \leq -h + \frac{|\dot{a}(t)|}{a(t)} h(t) + |\dot{a}(t)|c. \tag{2.17}$$

Let us derive from inequality (2.17) the desired conclusion (2.9).

Fix an arbitrary small $\delta > 0$. The first assumption (2.2) implies that

$$\frac{|\dot{a}(t)|}{a(t)} \leq \delta \quad for\ t \geq t_\delta. \tag{2.18}$$

Using the well-known Gronwall inequality, one obtains from (2.17) the following inequality

$$h(t) \leq h(t_\delta)e^{-(1-\delta)(t-t_\delta)} + c\int_{t_\delta}^{t} e^{-(1-\delta)(t-s)}|\dot{a}(s)|ds. \tag{2.19}$$

Let us divide both sides of (2.19) by $a(t)$ and prove that the following two relations hold:

$$\lim_{t\to\infty} \frac{e^{-(1-\delta)t}}{a(t)} = 0, \tag{2.20}$$

and

$$\lim_{t\to\infty} \frac{\int_{t_\delta}^{t} e^{(1-\delta)s}|\dot{a}(s)|ds}{e^{(1-\delta)t}a(t)} = 0. \tag{2.21}$$

This will complete the proof of Theorem 2.

From inequality (2.18) one gets

$$ce^{-\delta t} \leq a(t). \tag{2.22}$$

This implies relation (2.20) if $\delta < \frac{1}{2}$.

Applying the L'Hospital rule one proves relation (2.21) because

$$\lim_{t\to\infty} \frac{|\dot{a}(t)|}{(1-\delta)a(t) + \dot{a}(t)} = 0,$$

as follows from the second assumption (2.2) provided that $\delta < 1$.
Theorem 2 is proved. □

Remark 2. *The equation* $e^u = 0$, $u \in \mathbb{R}$, $H = \mathbb{R}$, *does not have a solution, although* $F(u) = e^u$ *is monotone,* $F'(u) = e^u > 0$ *is boundedly invertible for every* $u \in \mathbb{R}$ *and* $\|[e^u]^{-1}\| = e^{-u} \leq m(u) < \infty$ *for every* $u \in \mathbb{R}$. *Assumption* (1.5) *is not satisfied in this example, and this is the reason for the unsolvability of the equation* $e^x = 0$. *Note that* $e^x \leq c$ *as* $x \to -\infty$, *so assumption* (1.5) *does not hold.*

In recent papers [13] and [14] some nonlinear differential inequalities are derived and used for a study of the large time behavior of solutions to evolution problems.

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Received: 19.07.2010